On Affine structures coming from Berkovich Geometry

Keita Goto

Ph.D student (D3), Department of Mathematics, Kyoto University
JSPS, Research Fellowship for Young Scientists (DC1)

October 29, 2022
The pan-industry social event for young mathematicians
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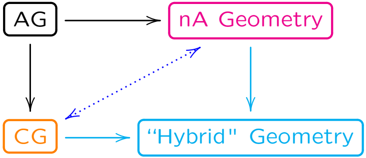
Abstract
I'm studying Berkovich geometric analogs of SYZ fibrations appearing in the context of what is called SYZ mirror symmetry. In particular, I'm interested in affine structures induced by them. Today, I introduce my works related to them. This work is supported by JSPS KAKENHI Grant Number JP20J23401.
Key words: Calabi-Yau varieties, SYZ fibrations, Affine structures, Berkovich Geometry.



$$\begin{cases} \text{AG} := \text{Algebraic Geometry} \\ \text{CG} := \text{Complex Geometry} \\ \text{nA} := \text{non-Archimedean} \end{cases}$$

Berkovich Geometry includes nA Geometry and Hybrid Geometry!

The SYZ mirror symmetry

The SYZ mirror symmetry

X : (polarized) Calabi-Yau Variety/$\mathbb{C}$ ⇢ $\check{X}$: A mirror pair of X

↓ SYZ fibr'n ↑ Inv. problem of SYZ fibr'n

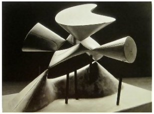
$(B, \nabla_A, \nabla_B, g)$ → Legendre transform → $(\check{B}, \check{\nabla}_A, \check{\nabla}_B, g)$

Def : (polarized) Calabi-Yau variety

sm. proj. var. X (w/ some ample l.b. L) s.t. $\bigwedge^{\dim_{\mathbb{C}} X} \Omega^1_X \cong \mathcal{O}_X$.

Examples

- Elliptic curves
- K3 surfaces ⊃ (singular) Kummer surfaces
- Abelian varieties
- K-triv fqav's



singular Kummer surface

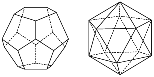
Def : SYZ fibration

(X, L) : pol. CY, B: top. sp.
Ω : non-triv. gl. sec. of $\bigwedge^{\dim_{\mathbb{C}} X} \Omega^1_X$,
ω : sympl. form corr. to L.
Then $f : X \to B$: SYZ fibration if

- $\omega|_{X_b} = 0$ and $\mathrm{Im}\Omega|_{X_b} = 0$,
- $X_b \cong (S^1)^{\dim_{\mathbb{C}} X}$,

for $^\forall b \in B^{sm} \subset B$, $\mathrm{codim}(B \setminus B^{sm}) \geq 2$.

$f : X \to B$ gives two affine structures ∇_A, ∇_B and one metric g to the base B.



Affine structures

Berkovich Geometry

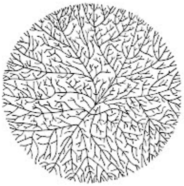
Berkovich Geometry is a geometry over Banach rings !

Table: Comparison of 3 kinds of geometries

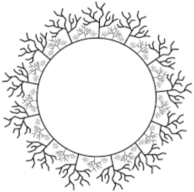
	AG	CG	BG
points	prime ideals	points in $\mathbb{C}^n$	semi-val'ns
pieces	affine schemes	zeros of hol. fcn's on $\mathbb{C}^n$	Berkovich spectra
topologies	◯	◎	◎
str sheaves	◯	◯	(◯)
base field	any	only $\mathbb{C}$	any

Berkovich Analytification

$(k, ||\cdot||)$: commutative unital Banach ring.
X: loc. alg. sch. / k
$\mapsto X^{\mathrm{An}}$: Berkovich analytification
Locally, it is given by
$U = \mathrm{Spec} A$
$\mapsto U^{\mathrm{An}} := \{$all semi-val'ns $|\cdot|_x$ on A
s.t. $|a|_x \leq ||a||, \ ^\forall a \in k\}$.
If k : nAF, then denote $X^{\mathrm{an}} := X^{\mathrm{An}}$.
X^{an} has many properties that hold in CG-world, as well.



$(\mathbb{P}^1_k)^{\mathrm{an}}$,
where
$\mathbb{P}^1_k$: proj. line/k



E^{an},
where
E : ell. curve/k

non-Archimedean SYZ fibration

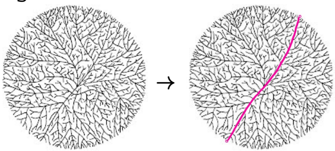
Tropicalization map

$k = \mathbb{C}((t))$: DVF
$\mathbb{G}_m^n := \mathrm{Spec}\left(k[X_1^{\pm}, \ldots, X_n^{\pm}]\right)$

Then tropicalization map
$\mathrm{Trop} : \mathbb{G}_m^{n,\mathrm{an}} \to \mathbb{R}^n$ defined by
$|\cdot|_x \mapsto$
$(-\log|X_1|_x, \ldots, -\log|X_n|_x)$.

Actually, $\mathrm{Trop} : \mathbb{G}_m^{n,\mathrm{an}} \to \mathbb{R}^n$ gives an affine structure to the base $\mathbb{R}^n$.

e.g.



The magenta line in $(\mathbb{P}_k^1)^{\mathrm{an}}$ corresponds to the base $\mathbb{R}$ of $\mathrm{Trop} : \mathbb{G}_m^{1,\mathrm{an}} \to \mathbb{R}^1$.
Note:
· The tropicalization map looks like a retraction.
· $\mathbb{G}_m^{1,\mathrm{an}} \subset (\mathbb{P}_k^1)^{\mathrm{an}}$ and its complement consists of the end points of the magenta line.

non-Archimedean SYZ fibration

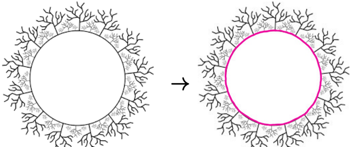
X: sm proj CY var$/k = \mathbb{C}((t))$ of $\dim = n$, B : top. sp.
$f : X \to B$ non-Archimedean SYZ fibration if, for $^{\forall} b \in B^{\mathrm{sm}} \subset B$, where $\mathrm{codim}(B \setminus B^{\mathrm{sm}}) \geq 2$, $^{\exists} U \subset B^{\mathrm{sm}}$: nbd of b

s.t.

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow[\cong]{\text{isom.}} & \mathrm{Trop}^{-1}(V) \\ \downarrow f & & \downarrow \mathrm{Trop} \\ U & \xrightarrow[\cong]{\text{homeo.}} & ^{\exists} V \end{array}$$

Actually, $f : X \to B$ gives an affine structure to the base B^{sm}, as well.
Note:
· Such an f is obtained each time you take any "minimal model" $\mathcal{X}/R = \mathbb{C}[[t]]$ of X.
· Each fiber over $b \in B^{\mathrm{sm}}$ is what is called an affinoid torus and has similar properties to an usual torus $(S^1)^n$.

e.g.



Hybrid SYZ fibration

Hybrid Geometry

$\mathbb{C}$ has the hybrid norm defined by $|z|_{\text{hyb}} := \max\{|z|_\infty, |z|_0\}$.

$^\forall \epsilon \ll 1$, $^\exists \mathscr{A}_\epsilon$: Banach ring $/\mathbb{C}^{\text{hyb}} := (\mathbb{C}, |\cdot|_{\text{hyb}})$ s.t. $\mathscr{M}(\mathscr{A}_\epsilon) \cong \mathbb{D}_\epsilon$, where $\mathbb{D}_\epsilon := \{t \in \mathbb{C} \mid |t|_\infty \leq \epsilon\}$.

Now consider
$\pi : \mathcal{X} \to \mathbb{D}_\epsilon$: maximal degeneration of pol. CY vareties.
Then $\mathcal{X}$: var/ the Banach ring $\mathscr{A}_\epsilon$.
$\rightsquigarrow$ Denote $\mathcal{X}^{\text{hyb}} := \mathcal{X}^{\text{An}}$ (wrt $\mathscr{A}_\epsilon$).
We call it a hybrid analytification of $\mathcal{X}$.

Properties of hybrid analytification

- π induces a continuous map $\pi^{\text{hyb}} : \mathcal{X}^{\text{hyb}} \to \mathscr{M}(\mathscr{A}_\epsilon) \cong \mathbb{D}_\epsilon$.
- $\mathcal{X}^{\text{hyb}} \setminus (\pi^{\text{hyb}})^{-1}(0) \cong \mathcal{X} \setminus \pi^{-1}(0)$.
- $(\pi^{\text{hyb}})^{-1}(0) \cong (\mathcal{X} \times_{\mathscr{A}_\epsilon} k)^{\text{an}}$, where $k = \mathbb{C}((t))$

Kontsevich-Soibelman Conjecture

$\pi : \mathcal{X}^* \to \mathbb{D}_\epsilon \setminus \{0\}$: max. degen. fam. of pol. CY varieties ($\epsilon \ll 1$).

· Does an SYZ fibr'n $f_t : \mathcal{X}_t \to {}^\exists B_t$ exist for any $t \in \mathbb{D}_\epsilon \setminus \{0\}$?
· Do such f_t's connect to $^\exists$ nA SYZ fibr'n $f_0 : (\mathcal{X}^* \times_{\mathscr{A}_\epsilon} k)^{\text{an}} \to B_0$ continuously?

Main Theorem (G.-Odaka,'22)

$\pi : \mathcal{X}^* \to \mathbb{D}_\epsilon \setminus \{0\}$: max. degen. fam. of pol. av's (resp. K-triv. fqav's) ($\epsilon \ll 1$). Set $X := \mathcal{X}^* \times_{\mathscr{A}_\epsilon} k$.

Then $^\exists f^{\text{hyb}} : {}^\exists \mathcal{X}^{\text{hyb}} \to {}^\exists \mathcal{B}/\mathbb{D}_\epsilon$ (that we call hybrid SYZ fibr'n) is a continuous map s.t.

$$\begin{cases} f_t : \mathcal{X}_t \to \mathcal{B}_t \text{ SYZ fibr'n (if } t \neq 0), \\ f_0 : X^{\text{an}} \to \mathcal{B}_0 \text{ nA SYZ fibr'n.} \end{cases}$$

Further, ∇_B-affine structures induced by f_t ($^\forall t \in \mathbb{D}_\epsilon \setminus \{0\}$) "coverges" to the affine structure induced by f_0.