DEMAZURE CONSTRUCTION FOR $\mathbb{Z}^n$-GRADED KRULL DOMAINS

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Dedicated to Professor Kei-ichi Watanabe on the occasion of his 74th birthday.

Abstract. For a Mori dream space X, the Cox ring $\mathrm{Cox}(X)$ is a Noetherian $\mathbb{Z}^n$-graded normal domain for some $n > 0$. Let $C(\mathrm{Cox}(X))$ be the cone (in $\mathbb{R}^n$) which is spanned by the vectors $\boldsymbol{a} \in \mathbb{Z}^n$ such that $\mathrm{Cox}(X)_{\boldsymbol{a}} \neq 0$. Then $C(\mathrm{Cox}(X))$ is decomposed into a union of chambers. Berchtold and Hausen [2] proved the existence of such decompositions for affine integral domains over an algebraically closed field. We shall give an elementary algebraic proof to this result in the case where the homogeneous component of degree $\mathbf{0}$ is a field.

Using such decompositions, we develop the Demazure construction for $\mathbb{Z}^n$-graded Krull domains. That is, under an assumption, we show that a $\mathbb{Z}^n$-graded Krull domain is isomorphic to the multi-section ring $R(X; D_1, \ldots, D_n)$ for certain normal projective variety X and $\mathbb{Q}$-divisors $D_1, \ldots, D_n$ on X.

1. Introduction

Let $A = \oplus_{m \geq 0} A_m$ be a Noetherian normal graded domain over a field $k = A_0$. Then, by the *Demazure construction* [4], [5], [17], [19] [1], there exists a normal projective variety X over k and a $\mathbb{Q}$-divisor D such that dD is an ample Cartier divisor on X for some $d > 0$, and A is isomorphic to

$$\bigoplus_{r \geq 0} H^0(X, \mathcal{O}_X(rD))$$

as a graded ring. When $X = \mathrm{Spec}(k)$, we think that 0 is the unique $\mathbb{Q}$-divisor, and it is an ample Cartier divisor.

One of our aims in this paper is to develop the Demazure construction for $\mathbb{Z}^n$-graded Krull domains (Theorem 8.6 and 8.9). For a normal projective variety X and $\mathbb{Q}$-divisors $D_1, \ldots, D_n$, we put

$$R(X; D_1, \ldots, D_n) = \bigoplus_{r_1, \ldots, r_n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(\sum_{i=1}^n r_i D_i)).$$

It has a $\mathbb{Z}^n$-graded ring structure, and is called the *multi-section ring* with respect to X and $D_1, \ldots, D_n$. Under an assumption, we show that a $\mathbb{Z}^n$-graded Krull domain is isomorphic to the multi-section ring $R(X; D_1, \ldots, D_n)$ for certain normal projective variety X and $\mathbb{Q}$-divisors $D_1, \ldots, D_n$ on X. In the case of $\mathbb{Z}^n$-graded

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[1] Some people call it the *Dolgachev-Pinkham-Demazure construction*.

affine integral domain over an algebraically closed field of characteristic 0, Altmann and Hausen developed another generalization of the Demazure construction using proper polyhedral divisors in Theorem 3.4 in [1]. In their construction, the given ring itself is not equal to a multi-section ring. However, using proper polyhedral divisors, they patch together multi-section rings to obtain the given ring.

Even if $D_1, \ldots, D_n$ are ample Cartier divisors, $R(X; D_1, \ldots, D_n)$ is not necessary a Noetherian ring. We remark that $R(X; D_1, \ldots, D_n)$ is a Noetherin ring if and only if it is finitely generated over $R(X; D_1, \ldots, D_n)_{\mathbf{0}}$ as a ring.

Finite generation of $R(X; D_1, \ldots, D_n)$ is deeply related to birational geometry. Let X be a normal $\mathbb{Q}$-factorial projective variety over a field k, and assume that $D_1, \ldots, D_n$ are free bases of the divisor class group $\mathrm{Cl}(X)$. Then the ring $R(X; D_1, \ldots, D_n)$ is called the *Cox ring* of X, and denoted by $\mathrm{Cox}(X)$. We say that X is a MDS (Mori dream space) if $\mathrm{Cox}(X)$ is Noetherian [8]. Finite generation of Cox rings is a very important problem both in algebraic geometry and in commutative ring theory. Let X be a blow-up of $\mathbb{P}^2_{\mathbb{C}}$ at finite closed points $p_1, \ldots, p_s$. Nagata [15] proved that $\mathrm{Cox}(X)$ is the invariant subring of a polynomial ring over $\mathbb{C}$ with some linear action under some weak assumption. It is known that $\mathrm{Cox}(X)$ is not finitely generated if $s \geq 9$ and points $p_1, \ldots, p_s$ are very general. Therefore it gives a counterexample to Hilbert's 14th problem.

For a MDS X, consider the $\mathbb{Z}^n$-graded ring $\mathrm{Cox}(X)$. We define the cone

$$C(\mathrm{Cox}(X)) = \sum_{\boldsymbol{a} \in \mathbb{Z}^n,\ \mathrm{Cox}(X)_{\boldsymbol{a}} \neq 0} \mathbb{R}_{\geq 0}\boldsymbol{a} \subset \mathbb{R}^n.$$

The cone $C(\mathrm{Cox}(X))$ is divided into some chambers and each chamber corresponds to a projective variety which is birational to X (cf. Hu-Keel [8], Okawa [16], Laface-Velasco [13]). In order to define a chamber, Laface-Velasco studied the ideal generated by elements with degree in a given ray, i.e., for $\boldsymbol{a} \in \mathbb{Z}^n$,

$$J_{\boldsymbol{a}}(\mathrm{Cox}(X)) = \sqrt{\text{the ideal of } \mathrm{Cox}(X) \text{ generated by } \cup_{r>0}\mathrm{Cox}(X)_{r\boldsymbol{a}}}.$$

For $\boldsymbol{a} \in \mathbb{Z}^n$, we put

$$X_{\boldsymbol{a}} = \mathrm{Proj}(\oplus_{r\geq 0}\mathrm{Cox}(X)_{r\boldsymbol{a}}).$$

It is easy to prove that, if

$$J_{\boldsymbol{b}}(\mathrm{Cox}(X)) \supset J_{\boldsymbol{a}}(\mathrm{Cox}(X)),$$

then we have a morphism

$$X_{\boldsymbol{a}} \longrightarrow X_{\boldsymbol{b}}$$

as in Lemma 8.3. We obtain important birational morphisms analyzing ideals of the form $J_{\boldsymbol{a}}(\mathrm{Cox}(X))$. One of our aims is to study such ideals for (not necessary Noetherian) $\mathbb{Z}^n$-graded rings.

In section 2, for a $\mathbb{Z}^n$-graded ring A, we define the cone $C(A)$, the ray ideal $J_{\boldsymbol{a}}(A)$ and ray ideal cones which are the generalization of chambers. We study basic properties of them (cf. Proposition 2.4). In Example 2.2, we know that $C(A)$ is a union of finitely many chambers for a Noetherian $\mathbb{Z}^2$-graded domain A such that $A_{\mathbf{0}}$ is a field.

In section 3, for a Noetherian $\mathbb{Z}^n$-graded domain A such that $A_{\mathbf{0}}$ is a field, we shall prove that if $J_{\boldsymbol{a}}(A) = J_{\boldsymbol{b}}(A)$, then $J_{\boldsymbol{c}}(A) = J_{\boldsymbol{a}}(A)$ for any $\boldsymbol{c}$ on the line segment between $\boldsymbol{a}$ and $\boldsymbol{b}$ (cf. Theorem 3.1 and Corollary 3.3). This result follows from Theorem 2.11 in [2]. If we remove the assumption that A is a Noetherian, Theorem 3.1 is false. We give a counter example which is a Cox ring of a normal projective rational surface in Example 3.4.

In section 4, we study Noetherian $\mathbb{Z}^n$-graded domains with only one chamber. For a $\mathbb{N}_0^n$-graded Noetherian ring A, we give a necessary and sufficient conditions for A to be integral over the subring generated by elements with degree in the coordinate axes (Theorem 4.1).

In section 5, for a Noetherian $\mathbb{Z}^n$-graded domain A such that $A_{\mathbf{0}}$ is a field, we decompose $C(A)$ into a union of chambers (cf. Theorem 5.1).

In section 6, we refine arguments in the previous section. Considering maximal ray ideal cones in stead of chambers, we give a structure of a fan to the set of maximal ray ideal cones (cf. Theorem 6.2). It is already known by Theorem 2.11 in [2]. For the non-Noetherian symbolic Rees ring in Example 3.4, the set of maximal ray ideal cones do not form a fan.

Theorems 3.1, 5.1, 6.2 follow from Theorem 2.11 in [2]. However, we give proofs since we need these arguments in the later sections and our proofs are elementary and algebraic.

In section 7, we study basic properties of $R(X; D_1, \ldots, D_n)$ for a normal projective variety X and $\mathbb{Q}$-divisors $D_1, \ldots, D_n$ (cf. Theorem 7.1). It is a generalization of results in [6], [12], [19]. We shall prove that $R(X; D_1, \ldots, D_n)$ is a Krull domain and determine the divisor class group of it.

Using these results, we study the Demazure construction for $\mathbb{Z}^n$-graded Krull domains in Section 8 (cf. Theorems 8.6 and 8.9).

2. Notation and basic properties

Throughout of this paper, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}_0$, and $\mathbb{N}$ denote the set of complex numbers, real numbers, rational numbers, integers, non-negative integers, and positive integers, respectively. Furthermore, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ denote the set of non-negative real numbers and positive real numbers, respectively.

Let

$$A = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^n} A_{\boldsymbol{a}}$$

be a $\mathbb{Z}^n$-graded ring, that is, each $A_{\boldsymbol{a}}$ is an additive subgroup such that $A_{\boldsymbol{a}} A_{\boldsymbol{b}} \subset A_{\boldsymbol{a}+\boldsymbol{b}}$ for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}^n$. For $\boldsymbol{a} \in \mathbb{R}^n$, $|\boldsymbol{a}|$ denotes the length of the vector $\boldsymbol{a}$. For $0 \neq x \in A_{\boldsymbol{a}}$, we say that the degree of x is $\boldsymbol{a}$, and denote $\deg(x) = \boldsymbol{a}$. We put

$$C(A) = \sum_{A_{\boldsymbol{a}} \neq 0} \mathbb{R}_{\geq 0} \boldsymbol{a} \subset \mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^n.$$

A subset σ in $\mathbb{R}^n$ is called a *cone* if the following two conditions are satisfied; (i) $\boldsymbol{a} + \boldsymbol{b} \in \sigma$ if $\boldsymbol{a}, \boldsymbol{b} \in \sigma$, (ii) $\mathbb{R}_{\geq 0} \boldsymbol{a} \subset \sigma$ if $\boldsymbol{a} \in \sigma$.

For a cone σ, $\dim\sigma$ denotes the dimension of $\sigma-\sigma$ as an $\mathbb{R}$-vector space. We say that σ is a *rational polyhedral cone* if σ is spanned by finitely many elements in $\mathbb{Q}^n$.

If A is a finitely generated $\mathbb{Z}^n$-graded ring over $A_{\mathbf{0}}$, then $C(A)$ is a rational polyhedral cone.

For $\boldsymbol{a}\in\mathbb{Q}^n$, we define $I_{\boldsymbol{a}}(A)$ to be the ideal of A generated by

$$\bigcup_{\boldsymbol{b}\in\mathbb{R}_{>0}\boldsymbol{a}\cap\mathbb{Z}^n} A_{\boldsymbol{b}}.$$

We put

$$J_{\boldsymbol{a}}(A)=\sqrt{I_{\boldsymbol{a}}(A)},$$

and call it the *ray ideal* of A at $\boldsymbol{a}$. In the case where A is a $\mathbb{Z}^n$-graded domain, for $\boldsymbol{a}\in\mathbb{Q}^n$, $I_{\boldsymbol{a}}(A)\neq 0$ if and only if $\boldsymbol{a}\in C(A)$.

Let σ be a cone in $\mathbb{R}^n$ such that $\sigma-\sigma$ is an $\mathbb{R}$-vector subspace spanned by finitely many elements in $\mathbb{Q}^n$. We say that σ is a *ray ideal cone* of A if $J_{\boldsymbol{a}}(A)=J_{\boldsymbol{b}}(A)\neq 0$ for any $\boldsymbol{a},\boldsymbol{b}\in\mathrm{rel.int}(\sigma)\cap\mathbb{Q}^n$, where $\mathrm{rel.int}(\sigma)$ denotes the relative interior of σ. For a ray ideal cone σ of A, $J_\sigma(A)$ denotes $J_{\boldsymbol{a}}(A)$ for $\boldsymbol{a}\in\mathrm{rel.int}(\sigma)\cap\mathbb{Q}^n$, and call it the *ray ideal* of the ray ideal cone σ. We sometimes denote $J_\sigma(A)$ simply by J_σ if no confusion is possible. A ray ideal cone σ is called a *chamber* of A if $\dim\sigma=\dim C(A)$.

Let T be an additive subsemigroup of $\mathbb{R}^n$ containing $\mathbf{0}$. For example, T is a subgroup of $\mathbb{Z}^n$ or a cone in $\mathbb{R}^n$. We put

$$A_T=\bigoplus_{\boldsymbol{a}\in T\cap\mathbb{Z}^n} A_{\boldsymbol{a}}.$$

Remark that it is a subring of A. We regard A_T as a $\mathbb{Z}^n$-graded subring of A unless otherwise specified.

Example 2.1. Let $A=k[x,y,z]$ be a $\mathbb{Z}^2$-graded polynomial ring over a field k with $\deg(x)=(1,0)$, $\deg(y)=(1,1)$, $\deg(z)=(0,1)$, and $A_{\mathbf{0}}=k$. Then the set of non-zero ray ideals of A consists of

$$A,\ (x),\ (xz,y),\ (z),\ (xy,xz),\ (xz,yz).$$

The following are the maximal ray ideal cones of the above ray ideals respectively.

$$\{0\},\ \mathbb{R}_{\geq 0}(1,0),\ \mathbb{R}_{\geq 0}(1,1),\ \mathbb{R}_{\geq 0}(0,1),\ \mathbb{R}_{\geq 0}(1,0)+\mathbb{R}_{\geq 0}(1,1),\ \mathbb{R}_{\geq 0}(1,1)+\mathbb{R}_{\geq 0}(0,1)$$

Example 2.2. Let $A=k[x_1,x_2,\ldots,x_t]$ be a $\mathbb{Z}^2$-graded domain over a field $A_{\mathbf{0}}=k$, where x_i is a non-zero homogeneous element with $\deg(x_i)=(\alpha_i,\beta_i)$ for $i=1,\ldots,t$. Here, we do not have to assume that $x_1,\ x_2,\ \ldots,\ x_t$ are algebraically independent over k. Then $C(A)$ is the cone spanned by $\{(\alpha_i,\beta_i)\mid i=1,\ldots,t\}$. For $\boldsymbol{a}\in\mathbb{Q}^n$, $J_{\boldsymbol{a}}(A)\neq 0$ if and only if $\boldsymbol{a}\in C(A)$ since A is a domain.

Consider the following three cases:

(I) Assume that $C(A)$ is *strongly convex*, that is, $-\boldsymbol{a}\notin C(A)$ for any $\mathbf{0}\neq\boldsymbol{a}\in C(A)$. Changing coordinates, we may assume $\alpha_i>0$ for all i and

$$-\infty<\frac{\beta_1}{\alpha_1}\leq\frac{\beta_2}{\alpha_2}\leq\cdots\leq\frac{\beta_t}{\alpha_t}<\infty.$$

Assume

$$\frac{\beta_i}{\alpha_i} < \frac{\beta_{i+1}}{\alpha_{i+1}} = \cdots = \frac{\beta_{i+s}}{\alpha_{i+s}} < \frac{\beta_{i+s+1}}{\alpha_{i+s+1}}.$$

Then $\mathbb{R}_{\geq 0}(\alpha_i, \beta_i) + \mathbb{R}_{\geq 0}(\alpha_{i+1}, \beta_{i+1})$ is a chamber with ray ideal

$$\sqrt{(x_1, x_2, \dots, x_i) \cap (x_{i+1}, x_{i+2}, \dots, x_t)}. \tag{2.1}$$

Furthermore, $\mathbb{R}_{\geq 0}(\alpha_{i+1}, \beta_{i+1})$ is a ray ideal cone with ray ideal

$$\sqrt{(x_1, x_2, \dots, x_{i+s}) \cap (x_{i+1}, x_{i+2}, \dots, x_t)}. \tag{2.2}$$

Here, remark that the ideal (2.1) is contained in the ideal (2.2). If

$$\frac{\beta_1}{\alpha_1} = \frac{\beta_2}{\alpha_2} = \cdots = \frac{\beta_i}{\alpha_i} < \frac{\beta_{i+1}}{\alpha_{i+1}},$$

then $\mathbb{R}_{\geq 0}(\alpha_1, \beta_1)$ is a ray ideal cone with ray ideal

$$\sqrt{(x_1, x_2, \dots, x_i)}. \tag{2.3}$$

Thus, in this case, $C(A)$ is a union of some chambers $\sigma_1, \sigma_2, \dots, \sigma_s$. Furthermore, for $i = 1, 2, \dots, s$, the ray ideal of σ_i is contained in the ray ideals of the faces of σ_i.

(II) Assume that $C(A) = \mathbb{R}_{\geq 0} \times \mathbb{R}$. Put $\boldsymbol{a} = (\alpha, \beta) \in \mathbb{Q}^2$. If $\alpha > 0$, then

$$J_{\boldsymbol{a}}(A) = J_{(1,0)}(A) \subsetneq A.$$

For any $\beta \in \mathbb{Q}$,

$$J_{(0,\beta)}(A) = A.$$

If $\alpha < 0$, then $J_{\boldsymbol{a}}(A) = 0$. In this case, A has two maximal ray ideal cones $\mathbb{R}_{\geq 0} \times \mathbb{R}$ and $\{0\} \times \mathbb{R}$.

(III) Assume that $C(A) = \mathbb{R}^2$. For any $\boldsymbol{a} \in \mathbb{Q}^2$, $J_{\boldsymbol{a}}(A) = A$ in this case.

Remark 2.3. Let

$$A = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^n} A_{\boldsymbol{a}}$$

be a $\mathbb{Z}^n$-graded ring.

(1) It is well known that A is Noetherian if and only if
- $A_{\mathbf{0}}$ is Noetherian, and
- A is finitely generated over $A_{\mathbf{0}}$ as a ring.

(2) Let $\boldsymbol{b} \in \mathbb{Q}^n$. If A is Noetherian, then so is

$$\bigoplus_{(\boldsymbol{a},\boldsymbol{b}) \geq 0} A_{\boldsymbol{a}},$$

where $(\boldsymbol{a}, \boldsymbol{b})$ is the inner product of $\boldsymbol{a}$ and $\boldsymbol{b}$. Using it, we can prove that if A is Noetherian, then so is A_σ for any rational polyhedral cone σ in $\mathbb{R}^n$, where $A_\sigma = \oplus_{\boldsymbol{a} \in \sigma \cap \mathbb{Z}^n} A_{\boldsymbol{a}}$.

(3) Suppose that T is a subgroup or a finitely generated sub-monoid of $\mathbb{Z}^n$. If A is Noetherian, then so is A_T.

Proposition 2.4. *Let A be a $\mathbb{Z}^n$-graded ring.*

(1) *Suppose that* S *is a* $\mathbb{Z}^n$*-graded subring of* A. *Let* $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Q}^n$. *Assume that* $J_{\boldsymbol{a}}(S) \subset J_{\boldsymbol{b}}(S)$, *and* $\sqrt{I_{\boldsymbol{a}}(S)A} = J_{\boldsymbol{a}}(A)$. *Then* $J_{\boldsymbol{a}}(A) \subset J_{\boldsymbol{b}}(A)$ *holds.*

(2) *Let* T *be a subgroup of* $\mathbb{Z}^n$. *Let* $\boldsymbol{a}, \boldsymbol{b} \in T$. *Then*
$$J_{\boldsymbol{a}}(A_T) \subset J_{\boldsymbol{b}}(A_T) \Longleftrightarrow J_{\boldsymbol{a}}(A) \subset J_{\boldsymbol{b}}(A).$$

(3) *Let* σ *be a cone in* $\mathbb{R}^n$. *Then, for* $\boldsymbol{a} \in \mathrm{rel.int}(\sigma) \cap \mathbb{Q}^n$ *and* $\boldsymbol{b} \in \sigma \cap \mathbb{Q}^n$,
$$J_{\boldsymbol{a}}(A_\sigma) \subset J_{\boldsymbol{b}}(A_\sigma) \Longleftrightarrow J_{\boldsymbol{a}}(A) \subset J_{\boldsymbol{b}}(A).$$
In particular, if σ *is a ray ideal cone of* A, *then* σ *itself is a ray ideal cone of* A_σ.

(4) *Let* S *be a* $\mathbb{Z}^n$*-graded subring of* A. *Assume that* A *is integral over* S. *Then, for any* $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Q}^n$,
$$J_{\boldsymbol{a}}(S) \subset J_{\boldsymbol{b}}(S) \Longleftrightarrow J_{\boldsymbol{a}}(A) \subset J_{\boldsymbol{b}}(A).$$

Proof. The assertion (1) follows from
$$J_{\boldsymbol{a}}(A) = \sqrt{I_{\boldsymbol{a}}(S)A} = \sqrt{J_{\boldsymbol{a}}(S)A} \subset \sqrt{J_{\boldsymbol{b}}(S)A} \subset J_{\boldsymbol{b}}(A).$$

We shall prove (2). Assume $\boldsymbol{a}, \boldsymbol{b} \in T$. For a homogeneous element w of A with degree on the ray $\mathbb{R}_{>0}\boldsymbol{a}$, some power of w is contained in $I_{\boldsymbol{a}}(A_T)$. Therefore we have $\sqrt{I_{\boldsymbol{a}}(A_T)A} = J_{\boldsymbol{a}}(A)$. The implication ($\Rightarrow$) follows from (1) as above. Next, we shall prove ($\Leftarrow$). For $x \in A_{\boldsymbol{c}}$ where $\boldsymbol{c} \in \mathbb{R}_{>0}\boldsymbol{a} \cap T$, we want to show $x^m \in I_{\boldsymbol{b}}(A_T)$ for some m. Since $x \in A_{\boldsymbol{c}} \subset J_{\boldsymbol{a}}(A) \subset J_{\boldsymbol{b}}(A) = \sqrt{I_{\boldsymbol{b}}(A_T)A}$, there exists homogeneous elements $y_1, \ldots, y_t, z_1, \ldots, z_t$ of A such that $x^m = \sum_i y_i z_i$ for some m, where $\deg(y_1), \ldots, \deg(y_t)$ are in $\mathbb{R}_{>0}\boldsymbol{b} \cap T$. We may assume that $m\boldsymbol{c} = \deg(y_i) + \deg(z_i)$ for each i. Then, since $\deg(z_i) = m\boldsymbol{c} - \deg(y_i) \in T$, z_i is in A_T for each i. Thus x^m is in $I_{\boldsymbol{b}}(A_T)$.

We shall prove (3). The implication ($\Rightarrow$) follows from (1) as above. Next, we shall prove ($\Leftarrow$). It is enough to show $A_{\boldsymbol{c}} \subset J_{\boldsymbol{b}}(A_\sigma)$ for $\boldsymbol{c} \in \mathbb{R}_{>0}\boldsymbol{a} \cap \mathbb{Z}^n$. Let x be an element of $A_{\boldsymbol{c}}$. By the assumption, $x \in J_{\boldsymbol{a}}(A) \subset J_{\boldsymbol{b}}(A) = \sqrt{I_{\boldsymbol{b}}(A)}$. Then there exists homogeneous elements $y_1, \ldots, y_t, z_1, \ldots, z_t$ of A such that $x^m = \sum_i y_i z_i$ for some m, where $\deg(y_1), \ldots, \deg(y_t)$ are in $\mathbb{R}_{>0}\boldsymbol{b}$. We may assume that $m\boldsymbol{c} = \deg(y_i) + \deg(z_i)$ for each i. Let s be a positive integer. We have
$$x^{m+s} = \sum_i y_i(x^s z_i).$$
Since $\deg(z_i) = m\boldsymbol{c} - \deg(y_i)$, $\deg(z_i)$ is contained in $\sigma - \sigma$. Since $\deg(x)$ is in $\mathrm{rel.int}(\sigma)$, $\deg(x^s z_i)$ is in σ for $s \gg 0$. Since $x^{m+s} \in I_{\boldsymbol{b}}(A_\sigma)$ for $s \gg 0$, we have $x \in J_{\boldsymbol{b}}(A_\sigma)$.

We shall prove (4). It is enough to show that

(i) $\sqrt{I_{\boldsymbol{a}}(S)A} = J_{\boldsymbol{a}}(A)$, and

(ii) $J_{\boldsymbol{a}}(S) = J_{\boldsymbol{a}}(A) \cap S$

for any $\boldsymbol{a} \in \mathbb{Q}^n$.

First, we shall prove (i). It is easy to see that the left-hand-side is contained in the right one. In order to show the opposite inclusion, we shall prove $A_{\boldsymbol{c}} \subset \sqrt{I_{\boldsymbol{a}}(S)A}$

for $\boldsymbol{c} \in \mathbb{R}_{>0}\boldsymbol{a} \cap \mathbb{Z}^n$. Take $x \in A_{\boldsymbol{c}}$. Since A is integral over S, we have an integral equation

$$x^m + a_1 x^{m-1} + \cdots + a_m = 0$$

for some m and some $a_1, a_2, \ldots, a_m \in S$. Here we may assume that a_i is contained in $S_{i\boldsymbol{c}}$ for each i. Therefore x^m is in $I_{\boldsymbol{a}}(S)A$.

Next, we shall prove (ii). It is easy to see that the left-hand-side is contained in the right one. We shall prove the opposite inclusion. It suffices to show that, if a prime ideal P of S contains the left-hand-side, then P contains the right one. Since $S \hookrightarrow A$ is an integral extension, there exists a prime ideal Q of A such that $Q \cap S = P$. Take $x \in A_{\boldsymbol{c}}$ for some $\boldsymbol{c} \in \mathbb{R}_{>0}\boldsymbol{a} \cap \mathbb{Z}^n$. Take an integral equation

$$x^m + a_1 x^{m-1} + \cdots + a_m = 0$$

for some m and some $a_1, a_2, \ldots, a_m \in S$. We may assume that a_i is contained in $S_{i\boldsymbol{c}}$ for each i. Then $a_i \in J_{\boldsymbol{a}}(S) \subset P \subset Q$. By the above integral equation, we know $x \in Q$. Thus Q contains $J_{\boldsymbol{a}}(A)$. Hence P contains $J_{\boldsymbol{a}}(A) \cap S$. $\square$

Example 2.5. Let $A = k[x, y, z, w]/(xw - yz)$ be a $\mathbb{Z}^2$-graded ring with $\deg(x) = (3, 0)$, $\deg(y) = (2, 1)$, $\deg(z) = (1, 2)$, $\deg(w) = (0, 3)$ and $A_{\boldsymbol{0}} = k$. As in Example 2.1, we have mutually distinct non-zero ray ideals of A as

$$\begin{aligned} &A, \\ &\sqrt{(x)},\ \sqrt{(y, xz, xw)},\ \sqrt{(z, xw, yw)},\ \sqrt{(w)}, \\ &\sqrt{(x) \cap (y, z, w)},\ \sqrt{(x, y) \cap (z, w)},\ \sqrt{(x, y, z) \cap (w)}. \end{aligned}$$

Now, suppose that there exists homogeneous elements a, b, c in A such that the inclusion

$$S = k[a, b, c] \to A$$

is finite. Then, by Example 2.2 (I), S has at most 6 non-zero ray ideals. Then, by Proposition 2.4 (4), A also has at most 6 non-zero ray ideals. It is a contradiction. Therefore A never have a $\mathbb{Z}^2$-graded Noether normalization.

3. Ray ideals for $\mathbb{Z}^n$-graded Noetherian rings

The aim of this section is to prove the following theorem. This result follows from Theorem 2.11 in [2]. If we remove the assumption that A is Noetherian, it is not true as in Example 3.4.

Theorem 3.1. *Let A be a Noetherian $\mathbb{Z}^n$-graded domain such that $A_{\boldsymbol{0}}$ is a field* [2]*. Suppose $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{Q}^n \subset \mathbb{R}^n$. Assume that $\boldsymbol{c}$ is on the line segment between $\boldsymbol{a}$ and $\boldsymbol{b}$. Assume $0 \neq J_{\boldsymbol{a}}(A) \subset J_{\boldsymbol{b}}(A)$. Then $J_{\boldsymbol{c}}(A) = J_{\boldsymbol{a}}(A)$ if $\boldsymbol{c} \neq \boldsymbol{b}$.*

[2] Instead of assuming that A is Noetherian, it is sufficient to assume that $A_{\mathbb{R}_{\geq 0}\boldsymbol{a}}$ is Noetherian. In fact, if $A_{\mathbb{R}_{\geq 0}\boldsymbol{a}}$ is Noetherian, we can find a Noetherian subring of A with the same situation as A.

Proof. If $\boldsymbol{c} = \boldsymbol{a}$, then it is obvious. Assume $\boldsymbol{c} \neq \boldsymbol{a}, \boldsymbol{b}$ in the rest of this section.

First, we shall prove $I_{\boldsymbol{a}}(A)I_{\boldsymbol{b}}(A) \subset J_{\boldsymbol{c}}(A)$. Let $x \in A_{\boldsymbol{d}}$ and $y \in A_{\boldsymbol{e}}$ for $\boldsymbol{d} = u\boldsymbol{a} \in \mathbb{Z}^n$ and $\boldsymbol{e} = v\boldsymbol{b} \in \mathbb{Z}^n$, where $u, v \in \mathbb{Q}_{>0}$. By the assumption, we have $\boldsymbol{c} = s\boldsymbol{a} + (1-s)\boldsymbol{b}$ for a rational number s such that $0 < s < 1$. Choose an integer $m \geq 0$ such that $ms = uu'$ and $m(1-s) = vv'$ for some $u', v' \in \mathbb{N}$. Then $x^{u'}y^{v'} \in A_{m\boldsymbol{c}} \subset I_{\boldsymbol{c}}(A)$. Thus $xy \in J_{\boldsymbol{c}}(A)$.

Since $I_{\boldsymbol{a}}(A)I_{\boldsymbol{b}}(A) \subset J_{\boldsymbol{c}}(A)$, we have

$$J_{\boldsymbol{a}}(A) \subset J_{\boldsymbol{c}}(A).$$

In the rest of this section, we shall show the opposite inclusion.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are not linearly independent over $\mathbb{R}$, it is easy to prove the assertion.

Assume that $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent over $\mathbb{R}$. Put $T = (\mathbb{R}\boldsymbol{a} + \mathbb{R}\boldsymbol{b}) \cap \mathbb{Z}^n$. Replacing A by A_T, we may assume that A is $\mathbb{Z}^2$-graded by Proposition 2.4 (2). Consider the normalization $\tilde{A}$ of A. It is well known that $\tilde{A}$ also has a structure of a $\mathbb{Z}^2$-graded ring. Replacing A by $\tilde{A}$, we may assume that A is a $\mathbb{Z}^2$-graded normal domain by Proposition 2.4 (4).

In this case, $C(A)$ is a 2-dimensional cone such that $\boldsymbol{a}, \boldsymbol{b} \in C(A) \subset \mathbb{R}^2$. Let σ be a strongly convex cone in $\mathbb{R}^2$ such that $\boldsymbol{a}, \boldsymbol{b} \in \mathrm{int}(\sigma)$. By Proposition 2.4 (3), we may replace A by A_σ. In the rest, we assume that $C(A)$ is strongly convex.

As in Example 2.2 (I), one of the following two cases occurs:

(i) There exists a chamber τ such that $\boldsymbol{a} \in \mathrm{int}(\tau)$.
(ii) There exists a chamber τ which is contained in $\mathbb{R}_{\geq 0}\boldsymbol{a} + \mathbb{R}_{\geq 0}\boldsymbol{b}$ such that $\boldsymbol{a}$ is in the boundary of τ.

Assume that (ii) occurs. Then $J_{\boldsymbol{a}}(A)$ contains $J_\tau(A)$ by Example 2.2. Then there exists a rational point $\boldsymbol{a}'$ (near $\boldsymbol{a}$) on the line segment between $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{a}'$ is in the interior of τ. Then we have $J_{\boldsymbol{a}'}(A) \subset J_{\boldsymbol{a}}(A)$. It is enough to show $J_{\boldsymbol{a}'}(A) \supset J_{\boldsymbol{c}}(A)$. Therefore, replacing $\boldsymbol{a}$ by $\boldsymbol{a}'$, we may assume that (i) as above occurs.

In the rest, let A be a $\mathbb{Z}^2$-graded normal domain and we assume that $\boldsymbol{a} = (\alpha, 0)$ and $\boldsymbol{b} = (\gamma, \delta)$, where α, γ, δ are positive rational numbers. Put $\boldsymbol{c} = s\boldsymbol{a} + (1-s)\boldsymbol{b} = (s\alpha + (1-s)\gamma, (1-s)\delta)$, where s is a rational number such that $0 < s < 1$. Here, remark that

$$0 < \frac{(1-s)\delta}{s\alpha + (1-s)\gamma} < \frac{\delta}{\gamma}. \tag{3.1}$$

Since $\boldsymbol{a}$ is in the interior of the chamber τ, there exists $\alpha_1, \beta_1 \in \mathbb{Q}_{>0}$ such that

$$(\mathbb{R}_{\geq 0}\boldsymbol{a} + \mathbb{R}_{\geq 0}(\alpha_1, \beta_1)) \setminus \{\boldsymbol{0}\} \subset \mathrm{int}(\tau). \tag{3.2}$$

Put

$$\rho = \mathbb{R}_{\geq 0}\boldsymbol{a} + \mathbb{R}_{\geq 0}(\alpha_1, \beta_1).$$

Since A_ρ is Noetherian, there exists homogeneous elements $y_1, \ldots, y_t$ such that

$$A_\rho = A_{\boldsymbol{0}}[y_1, y_2, \ldots, y_t].$$

Put $\deg(y_i) = (\gamma_i, \delta_i) \in \rho \setminus \{\mathbf{0}\}$ for $i = 1, 2, \ldots, t$. By definition, we have $\gamma_i > 0$ and $\delta_i \geq 0$ for $i = 1, 2, \ldots, t$. Remark that

$$0 \leq \frac{\delta_i}{\gamma_i} \leq \frac{\beta_1}{\alpha_1}$$

for each i.

Let x be a homogeneous element of A such that $\deg(x) \in \mathbb{R}_{>0}\boldsymbol{c} \cap \mathbb{Z}^2$. It is enough to show $x \in J_{\boldsymbol{a}}(A)$. Since τ is a chamber,

$$y_1, y_2, \ldots, y_t \in J_{\boldsymbol{a}}(A) \subset J_{\boldsymbol{b}}(A)$$

by (3.2). Therefore there exist homogeneous elements b_1, b_2, $\ldots$, b_h such that

(3.3)
(i) $y_1, y_2, \ldots, y_t \in \sqrt{(b_1, b_2, \ldots, b_h)A}$,
(ii) $\deg(b_1) = \deg(b_2) = \cdots = \deg(b_h) = e\boldsymbol{b} = (e\gamma, e\delta)$ for some $e \in \mathbb{N}$,
(iii) $e\gamma$ is strictly bigger than the first component of the vector $\deg(x)$.

Hence y_1^w, y_2^w, $\ldots$, y_t^w are in $(b_1, b_2, \ldots, b_h)A$ for some $w \in \mathbb{N}$. We put

$$\varphi = \min\{\delta_i - (\beta_1/\alpha_1)\,\gamma_i \mid i = 1, 2, \ldots, t\}$$

and

$$\Omega = \left\{(\xi, \eta) \in \mathbb{R}^2 \ \mid 0 < \xi,\ 0 \leq \eta \leq (\beta_1/\alpha_1)\,\xi + wt\varphi \right\}.$$

By definition, we know $\varphi < 0$. Hence we have $\Omega \subset \rho$.

Here, we shall prove the following claim:

Claim 3.2. *If $\boldsymbol{d} \in \Omega \cap \mathbb{Z}^2$, then $A_{\boldsymbol{d}} \subset (b_1, b_2, \ldots, b_h)A$.*

Put $\boldsymbol{d} = (\xi_1, \eta_1) \in \Omega \cap \mathbb{Z}^2$. Remark that $A_{\boldsymbol{d}} = (A_\rho)_{\boldsymbol{d}}$. Put

$$M = y_1^{u_1} y_2^{u_2} \cdots y_t^{u_t} \in A_{\boldsymbol{d}}.$$

Here, $(\xi_1, \eta_1) = (u_1\gamma_1 + u_2\gamma_2 + \cdots + u_t\gamma_t, u_1\delta_1 + u_2\delta_2 + \cdots + u_t\delta_t)$. If M is not in $(b_1, b_2, \ldots, b_h)A$, then $u_i < w$ for $i = 1, 2, \ldots, t$. Remark that

$$0 \geq \delta_i - \frac{\beta_1}{\alpha_1}\gamma_i \geq \varphi$$

for each i. Then

$$\begin{aligned}
\eta_1 - \frac{\beta_1}{\alpha_1}\xi_1 &= (u_1\delta_1 + u_2\delta_2 + \cdots + u_t\delta_t) - \frac{\beta_1}{\alpha_1}(u_1\gamma_1 + u_2\gamma_2 + \cdots + u_t\gamma_t) \\
&= u_1\left(\delta_1 - \frac{\beta_1}{\alpha_1}\gamma_1\right) + u_2\left(\delta_2 - \frac{\beta_1}{\alpha_1}\gamma_2\right) + \cdots + u_t\left(\delta_t - \frac{\beta_1}{\alpha_1}\gamma_t\right) \\
&\geq (u_1 + u_2 + \cdots + u_t)\varphi > wt\varphi.
\end{aligned}$$

It contradicts $\boldsymbol{d} \in \Omega$. We have completed the proof of Claim 3.2.

Consider the subring

$$A_{\mathbb{R}_{\geq 0}\boldsymbol{a}} = \bigoplus_{\boldsymbol{g} \in \mathbb{R}_{\geq 0}\boldsymbol{a} \cap \mathbb{Z}^n} A_{\boldsymbol{g}}$$

of A. Remark that $A_{\mathbb{R}_{\geq 0}\boldsymbol{a}}$ is Noetherian by Remark 2.3. We think $A_{\mathbb{R}_{\geq 0}\boldsymbol{a}}$ as a $\mathbb{Z}^2$-graded subring of A. Let z_1, z_2, ..., z_p be homogeneous elements satisfying $\deg(z_i) \in \mathbb{R}_{>0}\boldsymbol{a} \cap \mathbb{Z}^2$ for $i = 1, 2, \ldots, p$ and

$$A_{\mathbb{R}_{\geq 0}\boldsymbol{a}} = A_{\boldsymbol{0}}[z_1, z_2, \ldots, z_p].$$

Then there exist positive integers k_1, k_2, ..., k_p such that

(i) $\deg(z_1^{k_1}) = \deg(z_2^{k_2}) = \cdots = \deg(z_p^{k_p})$, and
(ii) $\deg(xz_j^{k_j}) \in \Omega$ for $j = 1, 2, \ldots, p$.

By Claim 3.2, we have

$$xz_j^{k_j} = \sum_{u=1}^{h} b_u c_{ju}$$

where $c_{ju} = 0$ or c_{ju} is a homogeneous element of A with $\deg(xz_j^{k_j}) = \deg(b_u c_{ju})$ for each u. Assume $c_{ju} \neq 0$. Then the second component of $\deg(c_{ju})$ is negative by (3.1) and the definition of b_1, b_2, ..., b_h (cf. (3.3) (iii)). Let ζ be the length of the vector $e\boldsymbol{b} = \deg(b_u)$, which is independent of u. Let L be the line segment between $\deg(xz_j^{k_j})$ and $\deg(c_{ju})$. Let $\boldsymbol{f}$ be the intersection of L and the coordinate axis $\mathbb{R}\boldsymbol{a}$. Let $\nu\zeta$ be the distance between $\boldsymbol{f}$ and $\deg(xz_j^{k_j})$. Then ν is a rational number such that $0 < \nu < 1$. Put

$$\phi = |\boldsymbol{f} - \deg(z_j^{k_j})|,$$

where $|\ |$ denotes the length of a vector. Let μ be a positive integer such that

$$\text{(3.4)} \qquad \begin{array}{ll} \text{(i)} & \mu\nu \text{ is a positive integer, and} \\ \text{(ii)} & |\deg(z_1^{k_1} z_2^{k_2} \cdots z_p^{k_p})| < \mu\phi. \end{array}$$

Since $\mu > \mu\nu > 0$, we can describe as

$$(xz_j^{k_j})^{\mu} = \left(\sum_{u=1}^{h} b_u c_{ju} \right)^{\mu} = \sum_{w_1+w_2+\cdots+w_h=\mu\nu} b_1^{w_1} b_2^{w_2} \cdots b_h^{w_h} C_{j,w_1,w_2,\ldots,w_h},$$

where $C_{j,w_1,w_2,\ldots,w_h}$ is a homogeneous element of A with

$$\text{(3.5)} \qquad \deg(C_{j,w_1,w_2,\ldots,w_h}) = \mu \deg(xz_j^{k_j}) - \mu\nu e\boldsymbol{b} = \mu\boldsymbol{f}.$$

Therefore

$$\text{(3.6)} \qquad |\deg(C_{j,w_1,w_2,\ldots,w_h})| = \mu|\deg(z_j^{k_j})| + \mu\phi.$$

On the other hand, by (3.5), $C_{j,w_1,w_2,\ldots,w_h}$ is an $A_{\boldsymbol{0}}$-linear combination of monomials $z_1^{v_1} z_2^{v_2} \cdots z_p^{v_p}$ with $\deg(z_1^{v_1} z_2^{v_2} \cdots z_p^{v_p}) = \mu\boldsymbol{f}$. We put $v_j = r_j k_j + q_j$ for $r_j, q_j \in \mathbb{Z}$ such that $0 \leq q_j < k_j$ for $j = 1, 2, \ldots, p$. Then we have

$$|\deg(z_1^{q_1} z_2^{q_2} \cdots z_p^{q_p})| < |\deg(z_1^{k_1} z_2^{k_2} \cdots z_p^{k_p})| < \mu\phi$$

by (3.4). Therefore

$$\begin{aligned} |\deg(C_{j,w_1,w_2,\ldots,w_h})| &= |\deg(z_1^{v_1}z_2^{v_2}\cdots z_p^{v_p})| \\ &= |\deg(z_1^{r_1k_1}z_2^{r_2k_2}\cdots z_p^{r_pk_p})| + |\deg(z_1^{q_1}z_2^{q_2}\cdots z_p^{q_p})| \\ &< |\deg(z_1^{r_1k_1}z_2^{r_2k_2}\cdots z_p^{r_pk_p})| + \mu\phi. \end{aligned}$$

Comparing the above inequality with the equation (3.6), we have

$$\mu|\deg(z_j^{k_j})| < |\deg(z_1^{r_1k_1}z_2^{r_2k_2}\cdots z_p^{r_pk_p})| = (r_1+r_2+\cdots+r_p)|\deg(z_j^{k_j})|.$$

Therefore we obtain

$$\mu < r_1+r_2+\cdots+r_p.$$

Then

$$C_{j,w_1,w_2,\ldots,w_h} \in (z_1^{k_1}, z_2^{k_2}, \ldots, z_p^{k_p})^{\mu+1}A.$$

Let V be a discrete valuation ring containing A. Let v be the valuation of V. Put

$$v_0 = v(z_j^{k_j}) = \min\{v(z_1^{k_1}), v(z_2^{k_2}), \ldots, v(z_p^{k_p})\}.$$

Since $(xz_j^{k_j})^\mu \in (z_1^{k_1}, z_2^{k_2}, \ldots, z_p^{k_p})^{\mu+1}A$, we know that

$$v(x^\mu) \geq v_0.$$

Therefore x^μ is contained in the integral closure of $(z_1^{k_1}, z_2^{k_2}, \ldots, z_p^{k_p})A$ (cf. (6.8.2) in [10]). In particular,

$$x \in \sqrt{(z_1^{k_1}, z_2^{k_2}, \ldots, z_p^{k_p})A} = J_{\boldsymbol{a}}(A).$$

We have completed the proof of Theorem 3.1. □

Corollary 3.3. *Let A be a Noetherian $\mathbb{Z}^n$-graded domain such that $A_{\mathbf{0}}$ is a field. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be linearly independent vectors in $\mathbb{Q}^n$ such that $J_{\boldsymbol{a}}(A) = J_{\boldsymbol{b}}(A) \neq 0$. Consider the line L passing through $\boldsymbol{a}$ and $\boldsymbol{b}$.*

Then there exists a sufficiently small ϵ such that $J_{\boldsymbol{c}}(A)$ coincides with $J_{\boldsymbol{b}}(A)$ for any $\boldsymbol{c} \in L \cap U_\epsilon(\boldsymbol{b}) \cap \mathbb{Q}^n$, where $U_\epsilon(\boldsymbol{b}) = \{\boldsymbol{d} \in \mathbb{R}^n \mid |\boldsymbol{d}-\boldsymbol{b}| < \epsilon\}$.

Proof. Let $T = (\mathbb{R}\boldsymbol{a} + \mathbb{R}\boldsymbol{b}) \cap \mathbb{Z}^2$. Replacing A by A_T, we may assume that A is a Noetherian $\mathbb{Z}^2$-graded domain. Take $\boldsymbol{c} \in L \cap U_\epsilon(\boldsymbol{b}) \cap \mathbb{Q}^n$ for a sufficiently small ϵ. If $\boldsymbol{c}$ is on the line segment between $\boldsymbol{a}$ and $\boldsymbol{b}$, then $J_{\boldsymbol{c}}(A) = J_{\boldsymbol{b}}(A)$ by Theorem 3.1. Assume that $\boldsymbol{c}$ is not on the line segment between $\boldsymbol{a}$ and $\boldsymbol{b}$. As in Example 2.2, one of the following three cases occurs:

(i) $\boldsymbol{b}$ is in the interior of a chamber τ.
(ii) $\boldsymbol{b}$ is in the boundary of two chambers σ_1 and σ_2.
(iii) $\boldsymbol{b}$ is in the boundary of $C(A)$.

If (i) occurs, then we may assume that $\boldsymbol{c}$ is in the interior of the chamber τ. Hence $J_{\boldsymbol{b}}(A) = J_{\boldsymbol{c}}(A)$. If (ii) occurs, then

$$0 \neq J_{\boldsymbol{c}}(A) \subset J_{\boldsymbol{b}}(A) = J_{\boldsymbol{a}}(A).$$

By Theorem 3.1, we have $J_{\boldsymbol{c}}(A) = J_{\boldsymbol{b}}(A)$. Assume that (iii) occurs. Let x be a non-zero homogeneous element such that $\deg(x) = s\boldsymbol{b}$ for some $s \in \mathbb{N}$. Then any power of x is not in $I_{\boldsymbol{a}}(A)$. It contradicts $J_{\boldsymbol{a}}(A) = J_{\boldsymbol{b}}(A)$. $\square$

Theorem 3.1 is false if we remove the assumption that A is Noetherian.

Example 3.4. Let a, b, c be pairwise coprime positive integers. Put $S = k[x, y, z]$, where k is a field. Let P be the kernel of the k-algebra homomorphism

$$\phi : S \longrightarrow k[T]$$

defined by $\phi(x) = T^a$, $\phi(y) = T^b$ and $\phi(z) = T^c$. For $n \in \mathbb{N}$, put

$$P^{(n)} = P^n S_P \cap S$$

and call it the *nth symbolic power* of P. Put

$$A = S[t^{-1}, Pt, P^{(2)}t^2, P^{(3)}t^3, \ldots] \subset S[t, t^{-1}].$$

Here, we think that $S[t, t^{-1}]$ is a $\mathbb{Z}^2$-graded ring with $\deg(x) = (0, a)$, $\deg(y) = (0, b)$, $\deg(z) = (0, c)$, $\deg(t) = (1, 0)$. Then A is a $\mathbb{Z}^2$-graded subring of $S[t, t^{-1}]$.

In this situation, A is Noetherian if and only if it satisfies Huneke's criterion [9], that is, there exist positive integers r and s, and elements $f \in P^{(r)}$ and $g \in P^{(s)}$ such that

$$\ell(S/(f, g, x)) = rsa. \tag{3.7}$$

One can prove that, once A is Noetherian, we can choose homogeneous elements f and g satisfying (3.7) as in Cutkosky [3]. Assume that A is Noetherian and homogeneous elements f and g satisfy Huneke's criterion. Put $\deg(f) = (0, d_1)$ and $\deg(g) = (0, d_2)$. In this case, we can prove that

$$B = k[t^{-1}, x, ft^r, gt^s] \subset A$$

is a finite map. Then, by Proposition 2.4 (4), ray ideal cones of A are the same as those of B. Remark that t^{-1}, x, ft^r, gt^s are algebraically independent over k. Degrees of them are $(-1, 0)$, $(0, a)$, (r, d_1), (s, d_2), respectively. Here assume that $d_1/r < d_2/s$ is satisfied [3]. Then, by Example 2.2, the non-zero ray ideals of B are

B,

$(t^{-1})B$, $(t^{-1}, x)B \cap (x, gt^s, ft^r)B$, $(t^{-1}, x, gt^s)B \cap (gt^s, ft^r)$, $(ft^r)B$,

$(t^{-1})B \cap (x, gt^s, ft^r)B$, $(t^{-1}, x)B \cap (gt^s, ft^r)B$, $(t^{-1}, x, gt^s)B \cap (ft^r)B$.

The corresponding maximal ray ideal cones are

$\{\mathbf{0}\}$,

$\mathbb{R}_{\geq 0}(-1, 0)$, $\mathbb{R}_{\geq 0}(0, 1)$, $\mathbb{R}_{\geq 0}(s, d_2)$, $\mathbb{R}_{\geq 0}(r, d_1)$,

$\mathbb{R}_{\geq 0}(-1, 0) + \mathbb{R}_{\geq 0}(0, 1)$, $\mathbb{R}_{\geq 0}(0, 1) + \mathbb{R}_{\geq 0}(s, d_2)$, $\mathbb{R}_{\geq 0}(s, d_2) + \mathbb{R}_{\geq 0}(r, d_1)$.

[3] In many cases, d_1/r does not equal d_2/s when there exist homogeneous elements f, g satisfying (3.7). In the case $(a, b, c) = (1, 1, 1)$, $d_1/r = d_2/s = 1$ holds. The authors do not know any other examples satisfying $d_1/r = d_2/s$.

One can see that Theorem 3.1 is satisfied in this case.

Next, we assume that $(a, b, c) = (25, 29, 72)$. Then the symbolic Rees ring A is not Notherian by Goto-Nishida-Watanabe [7]. There exist a homogeneous element $ft \in A_{(1,216)}$ and an ideal J of A such that, for positive rational numbers α and β, we have

$$J_{(\alpha,\beta)}(A) = \begin{cases} J & \text{if } \beta/\alpha > 25\cdot 29/3, \\ (ft)A \cap J & \text{if } 25\cdot 29/3 \geq \beta/\alpha > 216, \\ (ft)A & \text{if } \beta/\alpha = 216, \\ 0 & \text{if } 216 > \beta/\alpha. \end{cases}$$

Here remark that

$$J \supsetneq (ft)A \cap J \subsetneq (ft)A.$$

Thus Theorem 3.1 is not true if we remove the assumption that A is Noetherian. Since the height of J is bigger than 1, this example satisfies the condition (I) in Theorem 8.6.

4. When is $(\mathbb{R}_{\geq 0})^n$ a chamber?

The purpose of this section is to prove the following theorem:

Theorem 4.1. *Let A be a Noetherian $\mathbb{Z}^n$-graded domain such that $A_{\mathbf{0}}$ is a field. Assume that $C(A) = (\mathbb{R}_{\geq 0})^n$. Let $\boldsymbol{e}_i$ be the i-th unit vector in $\mathbb{R}^n$. Let B be the subring of A generated by*

$$\bigcup_{i=1}^{n} \left[\bigcup_{m>0} A_{m\boldsymbol{e}_i} \right]$$

over $A_{\mathbf{0}}$.

Then the following conditions are equivalent:

(1) The inclusion $B \to A$ is a finite extension.

(2) Any face of $(\mathbb{R}_{\geq 0})^n$ is a ray ideal cone of A.

(3) $C(A)$ itself is a ray ideal cone of A.

Proof. First, we shall prove (1) $\Rightarrow$ (2). By Proposition 2.4 (4), we may assume that $B = A$. Restricting $\mathbb{Z}^n$ to a subgroup generated by a subset of $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ (cf. Proposition 2.4 (2)), we have only to show that $C(A)$ itself is a ray ideal cone of A. Take

$$\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + \cdots + a_n\boldsymbol{e}_n \in \operatorname{int}(C(A)) \cap \mathbb{Z}^n,$$

where a_i's are positive integers. Take $x_i \in A_{b_i\boldsymbol{e}_i}$ for $i = 1, 2, \ldots, n$, where b_i's are positive integers. Put $b = b_1 b_2 \cdots b_n$. Then,

$$\deg(x_1^{(ba_1/b_1)} x_2^{(ba_2/b_2)} \cdots x_n^{(ba_n/b_n)}) = b\boldsymbol{a}.$$

Therefore we know $J_{\boldsymbol{a}}(A) \supset J_{\boldsymbol{e}_1}(A) J_{\boldsymbol{e}_2}(A) \cdots J_{\boldsymbol{e}_n}(A)$. On the other hand, since A is generated by homogeneous elements with degree on coordinate axes, we have $I_{\boldsymbol{a}}(A) \subset I_{\boldsymbol{e}_1}(A) I_{\boldsymbol{e}_2}(A) \cdots I_{\boldsymbol{e}_n}(A)$ immediately. Thus we obtain

$$J_{\boldsymbol{a}}(A) = \sqrt{J_{\boldsymbol{e}_1}(A) J_{\boldsymbol{e}_2}(A) \cdots J_{\boldsymbol{e}_n}(A)}.$$

Therefore $C(A)$ itself is a ray ideal cone of A.

The implication (2) ⇒ (3) is trivial.

In the rest of this section, we shall prove (3) ⇒ (1). There exists a positive integer c such that $A_{c\boldsymbol{e}_i} \neq 0$ for $i = 1, 2, , \ldots, n$. Replacing A by $A_{c\mathbb{Z}^n}$, we may assume $A_{\boldsymbol{e}_i} \neq 0$ for $i = 1, 2, , \ldots, n$. Let $\tilde{A}$ be the normalization of A. Since A is finitely generated over the field $A_{\boldsymbol{0}}$, $\tilde{A}$ is also a finitely generated $\mathbb{Z}^n$-graded ring over the field $\tilde{A}_{\boldsymbol{0}}$. Then $C(\tilde{A}) = C(A) = (\mathbb{R}_{\geq 0})^n$, and $(\mathbb{R}_{\geq 0})^n$ is a chamber of $\tilde{A}$ by Proposition 2.4 (4). Let B' be the subring of $\tilde{A}$ generated by

$$\bigcup_{i=1}^{n} \left[\bigcup_{m>0} \tilde{A}_{m\boldsymbol{e}_i} \right]$$

over $\tilde{A}_{\boldsymbol{0}}$. Consider the following diagram:

$$\begin{array}{ccc} B & \longrightarrow & A \\ \downarrow & & \downarrow \\ B' & \longrightarrow & \tilde{A} \end{array}$$

It is easy to see that $B \to B'$ is a finite extension. If $B' \to \tilde{A}$ is a finite extension, then so is $B \to A$. Replacing A with $\tilde{A}$, we may assume that A is a normal domain.

We denote the field of fractions by $Q(\ )$.

Let $\overline{B}$ be the integral closure of B in A. Since $Q(A)$ is finitely generated over $Q(B)$ as a field, $\overline{B}$ is finite over A. We shall prove $A = \overline{B}$.

Let us prove the following claim:

Claim 4.2. *The ray ideal* $J_{C(A)}$ *is* $\sqrt{J_{\boldsymbol{e}_1}(A) J_{\boldsymbol{e}_2}(A) \cdots J_{\boldsymbol{e}_n}(A)}$.

It is easy to see that $J_{C(A)}$ contains $\sqrt{J_{\boldsymbol{e}_1}(A) J_{\boldsymbol{e}_2}(A) \cdots J_{\boldsymbol{e}_n}(A)}$ as in the proof of the implication (1) ⇒ (2). We shall prove the opposite inclusion. Let P be a minimal prime ideal of $\sqrt{J_{\boldsymbol{e}_1}(A) J_{\boldsymbol{e}_2}(A) \cdots J_{\boldsymbol{e}_n}(A)}$. It is well known that P is $\mathbb{Z}^n$-graded. We want to show that P contains $J_{C(A)}$. There exists i such that P contains $J_{\boldsymbol{e}_i}(A)$. Then $C(A/P)$ does not contain $\boldsymbol{e}_i$. Since A is Noetherian, both $C(A)$ and $C(A/P)$ are rational polyhedral cones. Since $C(A/P) \subsetneq C(A)$, there exists $\boldsymbol{b} \in \mathrm{int}(C(A)) \cap \mathbb{Q}^n$ such that $\boldsymbol{b} \notin C(A/P)$. Therefore P contains $J_{\boldsymbol{b}}(A) = J_{C(A)}$. We have completed the proof of Claim 4.2.

Next, we shall prove the following claim:

Claim 4.3. *For* $\boldsymbol{a} \in \mathbb{N}^n$, $A_{\boldsymbol{a}}$ *is contained in* $\overline{B}$.

There exists $b \in \mathbb{Z}$ such that $J_{\boldsymbol{e}_i}(A) = \sqrt{A_{b\boldsymbol{e}_i} A}$ for $i = 1, 2, \ldots, n$, where $A_{b\boldsymbol{e}_i} A$ denotes the ideal of A generated by $A_{b\boldsymbol{e}_i}$.

We put $\boldsymbol{a} = a_1 \boldsymbol{e}_1 + a_2 \boldsymbol{e}_2 + \cdots + a_n \boldsymbol{e}_n$, where $a_1, a_2, \ldots, a_n \in \mathbb{N}$. Then, by Claim 4.2, we have

$$\begin{aligned} J_{\boldsymbol{a}}(A) &= \sqrt{J_{\boldsymbol{e}_1}(A) J_{\boldsymbol{e}_2}(A) \cdots J_{\boldsymbol{e}_n}(A)} \\ &= \sqrt{A_{b\boldsymbol{e}_1} A_{b\boldsymbol{e}_2} \cdots A_{b\boldsymbol{e}_n} A} \\ &= \sqrt{A_{ba_1\boldsymbol{e}_1} A_{ba_2\boldsymbol{e}_2} \cdots A_{ba_n\boldsymbol{e}_n} A}, \end{aligned}$$

where $A_{ba_1\boldsymbol{e}_1}A_{ba_2\boldsymbol{e}_2}\cdots A_{ba_n\boldsymbol{e}_n}A$ denotes the ideal of A generated by

$$A_{ba_1\boldsymbol{e}_1}A_{ba_2\boldsymbol{e}_2}\cdots A_{ba_n\boldsymbol{e}_n} = \{x_1x_2\cdots x_n \mid x_1 \in A_{ba_1\boldsymbol{e}_1},\ x_2 \in A_{ba_2\boldsymbol{e}_2},\ \ldots,\ x_n \in A_{ba_n\boldsymbol{e}_n}\}.$$

Then there exists $m \in \mathbb{N}$ such that

$$J_{\boldsymbol{a}}(A)^m \subset A_{ba_1\boldsymbol{e}_1}A_{ba_2\boldsymbol{e}_2}\cdots A_{ba_n\boldsymbol{e}_n}A. \tag{4.1}$$

We put

$$T_\ell = A_{\ell b\boldsymbol{a}} \text{ and } T = \bigoplus_{\ell\in\mathbb{N}_0} T_\ell.$$

Since T is finitely generated over $T_0 = A_{\boldsymbol{0}}$ as a ring, we have

$$T_\ell \subset J_{\boldsymbol{a}}(A)^m \tag{4.2}$$

for $\ell \gg 0$. Here, recall that

$$A_{ba_1\boldsymbol{e}_1}A_{ba_2\boldsymbol{e}_2}\cdots A_{ba_n\boldsymbol{e}_n} \subset T_1.$$

Then, by (4.1) and (4.2),

$$T_\ell = A_{ba_1\boldsymbol{e}_1}A_{ba_2\boldsymbol{e}_2}\cdots A_{ba_n\boldsymbol{e}_n}T_{\ell-1}$$

for $\ell \gg 0$. Then we know that the inclusion

$$A_{\boldsymbol{0}}[A_{ba_1\boldsymbol{e}_1}A_{ba_2\boldsymbol{e}_2}\cdots A_{ba_n\boldsymbol{e}_n}] \longrightarrow T$$

is a finite morphism. Since B contains $A_{\boldsymbol{0}}[A_{ba_1\boldsymbol{e}_1}A_{ba_2\boldsymbol{e}_2}\cdots A_{ba_n\boldsymbol{e}_n}]$, $\overline{B}$ contains T. Since elements in $A_{\boldsymbol{a}}$ are integral over T, $A_{\boldsymbol{a}}$ is contained in $\overline{B}$. We have completed the proof of Claim 4.3.

Now, we start to prove $\overline{B} = A$. It is enough to prove $\overline{B}_P \supset A$ for a prime ideal P of $\overline{B}$ of height one since $\overline{B}$ is a normal domain. By Claim 4.3, $Q(\overline{B}) = Q(A)$. Let v_P be the valuation associated to the discrete valuation ring $\overline{B}_P$.

First, assume that, for each i, there exists $x_i \in A_{a_i\boldsymbol{e}_i}$ for some $a_i \in \mathbb{N}$ such that $v_P(x_i) = 0$. For any non-zero homogeneous element x in A,

$$\deg(xx_1x_2\cdots x_n) \in \mathbb{N}^n.$$

Then, by Claim 4.3, we have

$$xx_1x_2\cdots x_n \in \overline{B}.$$

Then we have $v_P(x) \geq 0$ immediately. Thus x is in $\overline{B}_P$ in this case.

Next, assume that there exists i such that P contains

$$\bigcup_{m\in\mathbb{N}} A_{m\boldsymbol{e}_i}.$$

For the simplicity, suppose that the above i is 1. Then $P \cap B$ contains all the homogeneous element such that the first component of the degree is positive. Since $\overline{B}$ is integral over B, P contains all the homogeneous element in $\overline{B}$ such that the first component of the degree is positive. Let $\overline{B}_{(0)}$ be the homogeneous localization of $\overline{B}$, that is

$$\overline{B}_{(0)} = \left\{ z \in Q(\overline{B}) \;\middle|\; yz \in \overline{B} \text{ for some non-zero homogeneous element } y \in \overline{B} \right\}.$$

It is still a $\mathbb{Z}^n$-graded ring. Then, by Claim 4.3, it is easy to see $A \subset \overline{B}_{(0)}$. Since $A_{\boldsymbol{e}_i} \neq 0$ for $i = 1, 2, \ldots, n$, we may assume that $\overline{B}_{(0)} = K[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$, where $\deg(t_i) = \boldsymbol{e}_i$ and K is a field such that $(\overline{B}_{(0)})_{\boldsymbol{0}} = K$. Then $P = t_1 K[t_1, t_2, \ldots, t_n] \cap \overline{B}$ since the height of P is 1. Hence $\overline{B}_P = K[t_1, t_2, \ldots, t_n]_{(t_1)}$ because $\overline{B}_P$ is a discrete valuation ring. Thus we have

$$A \subset K[t_1, t_2, \ldots, t_n] \subset K[t_1, t_2, \ldots, t_n]_{(t_1)} = \overline{B}_P.$$

We have completed the proof of Theorem 4.1. □

5. Chamber decomposition

The aim of this section is to prove the following theorem, which also follows from Theorem 2.11 in [2].

Theorem 5.1. *Let A be a Noetherian $\mathbb{Z}^n$-graded domain such that $A_{\boldsymbol{0}}$ is a field. Then there exists finitely many chambers $\sigma_1, \sigma_2, \ldots, \sigma_m$ such that*

$$C(A) = \bigcup_{i=1}^{m} \sigma_i.$$

Proof. Replacing $\mathbb{Z}^n$ by the subgroup generated by $\{\boldsymbol{a} \in \mathbb{Z}^n \mid A_{\boldsymbol{a}} \neq 0\}$, we may assume that $\dim C(A) = n$.

Let $x_1, x_2, \ldots, x_s$ be a set of non-zero homogeneous elements such that $A = A_{\boldsymbol{0}}[x_1, x_2, \ldots, x_s]$. Put $\boldsymbol{a}_i = \deg(x_i)$ for each i. Then,

$$C(A) = \sum_{i=1}^{s} \mathbb{R}_{\geq 0} \boldsymbol{a}_i.$$

Here, put

$$\{H_1, H_2, \ldots, H_\ell\} = \left\{ H \;\middle|\; \begin{array}{l} H \text{ is an } (n-1)\text{-dimensional linear subspace of } \mathbb{R}^n \\ \text{spanned by a subset of } \boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_s \end{array} \right\}.$$

Let $f_i : \mathbb{R}^n \to \mathbb{R}$ be a $\mathbb{R}$-linear map such that

$$H_i = \{\boldsymbol{a} \in \mathbb{R}^n \mid f_i(\boldsymbol{a}) = 0\}$$

for each i. For $\xi_1, \xi_2, \ldots, \xi_\ell \in \{-1, 1\}$, we put

$$C(\xi_1, \xi_2, \ldots, \xi_\ell) = \{\boldsymbol{a} \in \mathbb{R}^n \mid \xi_i f_i(\boldsymbol{a}) > 0 \text{ for } i = 1, 2, \ldots, \ell\}.$$

Here, we prove the following claim:

Claim 5.2. *Let $\overline{C(\xi_1, \xi_2, \ldots, \xi_\ell)}$ be the closure of $C(\xi_1, \xi_2, \ldots, \xi_\ell)$ in the classical topology of $\mathbb{R}^n$. Then*

$$C(A) = \bigcup_{C(A) \cap C(\xi_1, \xi_2, \ldots, \xi_\ell) \neq \emptyset} \overline{C(\xi_1, \xi_2, \ldots, \xi_\ell)}. \tag{5.1}$$

First, we prove that the right-hand-side is contained in the left one. Assume that $C(A) \cap C(\xi_1, \xi_2, \ldots, \xi_\ell) \neq \emptyset$. Take $\boldsymbol{x} \in C(A) \cap C(\xi_1, \xi_2, \ldots, \xi_\ell)$. Since

$$\boldsymbol{x} \in C(A) = \sum_i \mathbb{R}_{\geq 0} \boldsymbol{a}_i,$$

there exist linearly independent vectors $\boldsymbol{a}_{i_1}$, $\boldsymbol{a}_{i_2}$, $\ldots$, $\boldsymbol{a}_{i_n}$ such that

$$\boldsymbol{x} \in \mathbb{R}_{\geq 0} \boldsymbol{a}_{i_1} + \mathbb{R}_{\geq 0} \boldsymbol{a}_{i_2} + \cdots + \mathbb{R}_{\geq 0} \boldsymbol{a}_{i_n}$$

by Carathéodory's theorem. Since $\boldsymbol{x} \in C(\xi_1, \xi_2, \ldots, \xi_\ell)$, $\boldsymbol{x}$ is not contained in hypersurface spanned by a subset of $\boldsymbol{a}_{i_1}$, $\boldsymbol{a}_{i_2}$, $\ldots$, $\boldsymbol{a}_{i_n}$. Hence we know

$$\boldsymbol{x} \in \mathbb{R}_{>0} \boldsymbol{a}_{i_1} + \mathbb{R}_{>0} \boldsymbol{a}_{i_2} + \cdots + \mathbb{R}_{>0} \boldsymbol{a}_{i_n}. \tag{5.2}$$

Take any $\boldsymbol{y} \in C(\xi_1, \xi_2, \ldots, \xi_\ell)$. Since $\boldsymbol{a}_{i_1}$, $\boldsymbol{a}_{i_2}$, $\ldots$, $\boldsymbol{a}_{i_n}$ span $\mathbb{R}^n$, we can write

$$\boldsymbol{y} = r_1 \boldsymbol{a}_{i_1} + r_2 \boldsymbol{a}_{i_2} + \cdots + r_n \boldsymbol{a}_{i_n}, \tag{5.3}$$

where $r_1, r_2, \ldots, r_n \in \mathbb{R}$. Assume that $r_j \leq 0$ for some j. Let H_t be the hypersurface spanned by $\boldsymbol{a}_{i_1}$, $\ldots$, $\boldsymbol{a}_{i_{j-1}}$, $\boldsymbol{a}_{i_{j+1}}$, $\ldots$, $\boldsymbol{a}_{i_n}$. By (5.2), we obtain $\xi_t f_t(\boldsymbol{a}_{i_j}) > 0$. On the other hand, by (5.3), $0 < \xi_t f_t(\boldsymbol{y}) = r_j \xi_t f_t(\boldsymbol{a}_{i_j})$. It is a contradiction. Hence we obtain

$$C(\xi_1, \xi_2, \ldots, \xi_\ell) \subset \mathbb{R}_{>0} \boldsymbol{a}_{i_1} + \mathbb{R}_{>0} \boldsymbol{a}_{i_2} + \cdots + \mathbb{R}_{>0} \boldsymbol{a}_{i_n} \subset C(A). \tag{5.4}$$

Since $C(A)$ is a closed subset, the closure $\overline{C(\xi_1, \xi_2, \ldots, \xi_\ell)}$ is contain in $C(A)$.

Next, we shall show that the right-hand-side contains the left one in (5.1). It is enough to show that any point $\boldsymbol{z}$ in the interior of $C(A)$ is contained in the right-hand-side. Let $U_\epsilon(\boldsymbol{z})$ be an open ball of radius ϵ with center $\boldsymbol{z}$. If ϵ is small enough, then $U_\epsilon(\boldsymbol{z})$ is contained in $C(A)$. Since

$$U_\epsilon(\boldsymbol{z}) \not\subset H_1 \cup H_2 \cup \cdots \cup H_\ell,$$

$U_\epsilon(\boldsymbol{z})$ intersects the right-hand-side in (5.1). Since the right-hand-side is a closed set, $\boldsymbol{z}$ is contained in the right-hand-side.

We have completed the proof of Claim 5.2.

In order to prove Theorem 5.1, it is enough to show the following claim:

Claim 5.3. *Suppose* $C(A) \cap C(\xi_1, \xi_2, \ldots, \xi_\ell) \neq \emptyset$. *Then* $\overline{C(\xi_1, \xi_2, \ldots, \xi_\ell)}$ *is a chamber of* A.

Suppose that $C(A) \cap C(\xi_1, \xi_2, \ldots, \xi_\ell) \neq \emptyset$. Then it is easy to see that

$$\overline{C(\xi_1, \xi_2, \ldots, \xi_\ell)} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \forall i, \quad \xi_i f_i(\boldsymbol{x}) \geq 0\}$$

is a rational polyhedral cone. The interior of $\overline{C(\xi_1, \xi_2, \ldots, \xi_\ell)}$ is $C(\xi_1, \xi_2, \ldots, \xi_\ell)$. Suppose $\deg(x_1^{t_1} x_2^{t_2} \cdots x_s^{t_s}) \in C(\xi_1, \xi_2, \ldots, \xi_\ell)$. We shall prove that $x_1^{t_1} x_2^{t_2} \cdots x_s^{t_s} \in J_{\boldsymbol{a}}(A)$ for any $\boldsymbol{a} \in C(\xi_1, \xi_2, \ldots, \xi_\ell) \cap \mathbb{Q}^n$. Since

$$\deg(x_1^{t_1} x_2^{t_2} \cdots x_s^{t_s}) \in \sum_{t_i > 0} \mathbb{R}_{\geq 0} \boldsymbol{a}_i,$$

there exists i_1, i_2, $\ldots$, i_n such that

- $\boldsymbol{a}_{i_1}, \boldsymbol{a}_{i_2}, \ldots, \boldsymbol{a}_{i_n}$ are linearly independent,
- $t_{i_1} > 0, t_{i_2} > 0, \ldots, t_{i_n} > 0$, and
- $\deg(x_1^{t_1} x_2^{t_2} \cdots x_s^{t_s}) \in \mathbb{R}_{>0}\boldsymbol{a}_{i_1} + \mathbb{R}_{>0}\boldsymbol{a}_{i_2} + \cdots + \mathbb{R}_{>0}\boldsymbol{a}_{i_n}$

by the same argument just before (5.2). Then, by the argument that (5.2) implies (5.4), we obtain

$$\boldsymbol{a} \in C(\xi_1, \xi_2, \ldots, \xi_\ell) \subset \mathbb{R}_{>0}\boldsymbol{a}_{i_1} + \mathbb{R}_{>0}\boldsymbol{a}_{i_2} + \cdots + \mathbb{R}_{>0}\boldsymbol{a}_{i_n}.$$

Then it is easy to see that a power of $x_1^{t_1} x_2^{t_2} \cdots x_s^{t_s}$ is contained in $I_{\boldsymbol{a}}(A)$.

We have completed the proofs of Claim 5.3 and Theorem 5.1. $\square$

6. The fan structure of ray ideal cones

In this section, we shall prove that, for a Noetherian $\mathbb{Z}^n$-graded domain, there exists the unique maximal ray ideal cone σ_J for each non-zero ray ideal J. Furthermore, we shall prove that the set of maximal ray ideal cones forms a fan in Theorem 6.2, which follows from Theorem 2.11 in [2]. If A is not Notherian, Theorem 6.2 is not true. In fact, the set of maximal ray ideal cones of the non-Noetherian symbolic Rees ring in Example 3.4 do not form a fan.

Lemma 6.1. *Let A be a Noetherian $\mathbb{Z}^n$-graded ring and σ be a ray ideal cone of A. Take $\boldsymbol{b} \in \sigma \cap \mathbb{Q}^n$. Then $J_{\boldsymbol{b}}(A)$ contains the ray ideal J_σ of the ray ideal cone σ.*

Proof. Let P be a homogeneous prime ideal containing $J_{\boldsymbol{b}}(A)$. Remark that $C(A/P)$ is a closed subset of $\mathbb{R}^n$ since A is Noetherian. If $C(A/P)$ contains $\mathrm{rel.int}(\sigma) \cap \mathbb{Q}^n$, then $C(A/P)$ contains σ. However, $C(A/P)$ does not contain $\boldsymbol{b}$. Therefore there exists $\boldsymbol{a} \in \mathrm{rel.int}(\sigma) \cap \mathbb{Q}^n$ such that $P \supset J_{\boldsymbol{a}}(A) = J_\sigma$. Thus $J_{\boldsymbol{b}}(A)$ contains J_σ. $\square$

Theorem 6.2. *Let A be a Noetherian $\mathbb{Z}^n$-graded domain such that $A_{\boldsymbol{0}}$ is a field.*

(1) There exists only finitely many non-zero ray ideals.

(2) For each non-zero ray ideal J, there exists the unique maximal ray ideal cone σ_J of the ray ideal J. Furthermore, σ_J is a rational polyhedral cone. For any $\boldsymbol{a} \in \mathbb{Q}^n \setminus \sigma_J$, $J_{\boldsymbol{a}}(A)$ does not coincide with J.

(3) Suppose that a rational polyhedral cone σ is a ray ideal cone. Then any face of σ is also a ray ideal cone.

(4) Let J_1 and J_2 be non-zero ray ideals of A. Then J_1 contains J_2 if and only if σ_{J_1} is a face of σ_{J_2}.

(5) Assume that $A_{\boldsymbol{0}} \setminus \{0\}$ coincides with the set of units in A. Then

$$\{\sigma_J \mid J \text{ is a non-zero ray ideal}\}$$

forms a fan.

Proof. First, we shall prove (1). It is enough to show that there exist finitely many ray ideal cones $\rho_1, \rho_2, \ldots, \rho_\ell$ such that

$$C(A) = \bigcup_{i=1}^{\ell} \mathrm{rel.int}(\rho_i), \tag{6.1}$$

where each ρ_i is a rational polyhedral cone. (If it is satisfied, A has only finitely many non-zero ray ideals J_{ρ_1}, J_{ρ_2}, $\ldots$, J_{ρ_ℓ}.) We shall prove it by induction on the dimension of $C(A)$. If $\dim C(A) = 0$, it is trivial. Assume that $\dim C(A) > 0$. By Theorem 5.1, we have chambers σ_1, σ_2, $\ldots$, σ_m satisfying

$$C(A) = \bigcup_{i=1}^{m} \sigma_i.$$

It is sufficient to show (6.1) for A_{σ_1}, A_{σ_2}, $\ldots$, A_{σ_m} since any ray ideal cone of A_{σ_i} is a ray ideal cone of A by Proposition 2.4 (1). Replacing A by A_{σ_i}, we assume that $C(A)$ itself is a chamber of A and $C(A)$ is a rational polyhedral cone. Then the boundary of $C(A)$ is a union of finitely many faces of $C(A)$. Let τ be a face of $C(A)$. Since $\dim\tau < \dim C(A)$, (6.1) holds for A_τ. Therefore (6.1) is satisfied since any ray ideal cone of A_τ is a ray ideal cone of A by Proposition 2.4 (1).

Next, we shall prove (2). Let J be a non-zero ray ideal of A. Consider a decomposition of $C(A)$ as in (6.1). Suppose that the ray ideal of ρ_i is J if and only if $i = 1, 2, \ldots, s$. Then it is easy to see

$$\{\boldsymbol{a} \in \mathbb{Q}^n \mid J_{\boldsymbol{a}}(A) = J\} \subset \bigcup_{i=1}^{s} \mathrm{rel.int}(\rho_i) \subset \sum_{\boldsymbol{a}\in\mathbb{Q}^n,\ J_{\boldsymbol{a}}(A)=J} \mathbb{R}_{\geq 0}\boldsymbol{a}. \tag{6.2}$$

Take an element

$$\boldsymbol{x} = u_1\boldsymbol{a}_1 + u_2\boldsymbol{a}_2 + \cdots + u_t\boldsymbol{a}_t$$

in the right end of (6.2), where $u_i \in \mathbb{R}_{>0}$ and $J_{\boldsymbol{a}_i}(A) = J$ for $i = 1, 2, \ldots, t$. Let $\{u_{ij}\}_j$ be a sequence of positive rational numbers which converges to u_i. We put

$$\boldsymbol{x}_j = u_{1j}\boldsymbol{a}_1 + u_{2j}\boldsymbol{a}_2 + \cdots + u_{tj}\boldsymbol{a}_t$$

for each j. By Theorem 3.1, $J_{\boldsymbol{x}_j}(A) = J$ for all j. Since $\{\boldsymbol{x}_j\}_j$ converges to $\boldsymbol{x}$, $\boldsymbol{x}$ is contained in the closure of the left end in (6.2). Taking the closure of each set in (6.2), we obtain

$$\overline{\{\boldsymbol{a} \in \mathbb{Q}^n \mid J_{\boldsymbol{a}}(A) = J\}} = \bigcup_{i=1}^{s} \rho_i = \overline{\sum_{\boldsymbol{a}\in\mathbb{Q}^n,\ J_{\boldsymbol{a}}(A)=J} \mathbb{R}_{\geq 0}\boldsymbol{a}}. \tag{6.3}$$

The right end of (6.3) is a cone in $\mathbb{R}^n$. We denote this cone by σ_J. Since σ_J is a cone spanned by ρ_1, ρ_2, $\ldots$, ρ_s, it is a rational polyhedral cone. It is easy to see that σ_J is the unique maximal ray ideal cone with ray ideal J.

Next, we shall prove (3). If $\dim\sigma \leq 2$, then it is easy. Assume that $\dim\sigma \geq 3$. Assume that τ is a proper face of σ such that there exist $\boldsymbol{a}, \boldsymbol{b} \in \mathrm{rel.int}(\tau) \cap \mathbb{Q}^n$ satisfying $J_{\boldsymbol{a}}(A) \neq J_{\boldsymbol{b}}(A)$. We can take $\boldsymbol{a}', \boldsymbol{b}' \in \mathrm{rel.int}(\tau) \cap \mathbb{Q}^n$ such that $\boldsymbol{a}, \boldsymbol{b} \in \mathrm{rel.int}(\mathbb{R}_{\geq 0}\boldsymbol{a}' + \mathbb{R}_{\geq 0}\boldsymbol{b}')$. Take $\boldsymbol{c} \in \mathrm{rel.int}(\sigma) \cap \mathbb{Q}^n$. Put

$$T = \mathbb{R}_{\geq 0}\boldsymbol{a}' + \mathbb{R}_{\geq 0}\boldsymbol{b}' + \mathbb{R}_{\geq 0}\boldsymbol{c}.$$

By definition, $\dim T = 3$. The relative interior of T is contained in that of σ. By Proposition 2.4 (3), T itself is a ray ideal cone of A_T. Then, by Theorem 4.1, any face of T is a ray ideal cone of A_T. Since $\boldsymbol{a}, \boldsymbol{b} \in \mathrm{rel.int}(\mathbb{R}_{\geq 0}\boldsymbol{a}' + \mathbb{R}_{\geq 0}\boldsymbol{b}')$, we know $J_{\boldsymbol{a}}(A_T) = J_{\boldsymbol{b}}(A_T)$. By Proposition 2.4 (1), we have $J_{\boldsymbol{a}}(A) = J_{\boldsymbol{b}}(A)$. It is a contradiction.

Next, we shall prove (4). If σ_{J_1} is a face of σ_{J_2}, then J_1 contains J_2 by Lemma 6.1.

Conversely assume that $J_1 \supsetneq J_2$. Let $\boldsymbol{a} \in \text{rel.int}(\sigma_{J_2}) \cap \mathbb{Q}^n$ and $\boldsymbol{b} \in \text{rel.int}(\sigma_{J_1}) \cap \mathbb{Q}^n$. Let $\boldsymbol{c}$ be a rational point on the line segment between $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{c} \neq \boldsymbol{b}$. Then $J_{\boldsymbol{c}}(A) = J_{\boldsymbol{a}}(A)$ by Theorem 3.1. Thus one can see

$$\text{rel.int}(\sigma_{J_1}) \subset \overline{\text{rel.int}(\sigma_{J_2})}.$$

Hence

$$\sigma_{J_1} \subset \sigma_{J_2}.$$

Since σ_{J_1} is contained in the boundary of σ_{J_2}, there exists the minimal face τ of σ_{J_2} containing σ_{J_1}. By the minimality of τ, σ_{J_1} is not contained in the boundary of τ. Here, remark that

$$\sigma_{J_1} = \overline{\{\boldsymbol{a} \in \mathbb{Q}^n \mid J_{\boldsymbol{a}}(A) = J_1\}}$$

by (6.3). Therefore there exists $\boldsymbol{d} \in \text{rel.int}(\tau) \cap \mathbb{Q}^n$ such that $J_{\boldsymbol{d}}(A) = J_1$. By (3), τ is a ray ideal cone of the ray ideal J_1. Since σ_{J_1} is the unique maximal ray ideal cone with ray ideal J_1, σ_{J_1} coincides with τ.

Next, we shall prove (5). Put

$$\Delta(A) = \{\sigma_J \mid J \text{ is a non-zero ray ideal}\}.$$

Since $A_{\boldsymbol{0}} \setminus \{0\}$ coincides with the set of units in A, each σ_J in $\Delta(A)$ is a strongly convex rational polyhedral cone. In order to show that $\Delta(A)$ forms a fan, it is enough to show

(a) any face of a cone in $\Delta(A)$ is in $\Delta(A)$, and
(b) for $\sigma_1, \sigma_2 \in \Delta(A)$, $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

First, we shall prove (a). Take $\sigma_J \in \Delta(A)$. Let τ be a face of σ_J. By (3), τ is a ray ideal cone of A. Let J' be the ray ideal of the ray ideal cone τ. By Lemma 6.1, J' contains J. Then, by (4), $\sigma_{J'}$ is a face of σ_J. Since $\sigma_{J'}$ contains τ, τ is a face of $\sigma_{J'}$. We want to show $\tau = \sigma_{J'}$. Assume that $\tau \subsetneq \sigma_{J'}$. Take $\boldsymbol{a} \in \text{rel.int}(\sigma_{J'}) \cap \mathbb{Q}^n$ and $\boldsymbol{b} \in \text{rel.int}(\tau) \cap \mathbb{Q}^n$. Then we have $J_{\boldsymbol{a}}(A) = J_{\boldsymbol{b}}(A) \neq 0$. By Corollary 3.3, there exists $\boldsymbol{c} \in \mathbb{Q}^n$ such that $J_{\boldsymbol{c}}(A) = J_{\boldsymbol{b}}(A)$ and $\boldsymbol{c} \notin \sigma_{J'}$. It contradicts the maximality of $\sigma_{J'}$ in (2). Hence we have $\tau = \sigma_{J'}$. Therefore any face of σ_J is contained in $\Delta(A)$.

Next, we shall prove (b). Take $\sigma_{J_1}, \sigma_{J_2} \in \Delta(A)$. We shall prove that $\sigma_{J_1} \cap \sigma_{J_2}$ is a face of both σ_{J_1} and σ_{J_2}. If $J_1 \supset J_2$ or $J_1 \subset J_2$, then the assertion follows from (4). Assume $J_1 \not\supset J_2$ and $J_1 \not\subset J_2$. Since both σ_{J_1} and σ_{J_2} are rational polyhedral cones, so is $\sigma_{J_1} \cap \sigma_{J_2}$. By Lemma 6.1, $J_{\boldsymbol{a}}(A)$ contains both J_1 and J_2 for any $\boldsymbol{a} \in \sigma_{J_1} \cap \sigma_{J_2} \cap \mathbb{Q}^n$. Then $\sigma_{J_1} \cap \sigma_{J_2}$ is contained in the boundary of σ_{J_i} for $i = 1, 2$ by (4). Let τ_i be the minimal face of σ_{J_i} containing $\sigma_{J_1} \cap \sigma_{J_2}$ for $i = 1, 2$. There exists $\boldsymbol{b} \in \sigma_{J_1} \cap \sigma_{J_2} \cap \mathbb{Q}^n$ such that $\boldsymbol{b} \in \text{rel.int}(\tau_i)$ for $i = 1, 2$. Put $J = J_{\boldsymbol{b}}(A)$. Since $\tau_1, \tau_2 \in \Delta(A)$ by (a), we know $\tau_1 = \tau_2 = \sigma_J$. Since J contains J_1 and J_2, we have $\sigma_{J_1} \cap \sigma_{J_2} = \sigma_J$.

We have completed the proof of Theorem 6.2. □

Example 6.3. Let $\pi : X \to \mathbb{P}^2_{\mathbb{C}}$ be the blow-up at very general points $p_1, \ldots, p_t$. Let H be the hyperplane in $\mathbb{P}^2_{\mathbb{C}}$. We put $E_i = \pi^{-1}(p_i)$ for $i = 1, 2, \ldots, t$. Consider the multi-section ring (see Section 7)

$$R = R(X; -E_1 - E_2 - \cdots - E_t, \pi^{-1}(H)).$$

Nagata [15] conjectured that, if $t \geq 10$ and $d \leq \sqrt{t}m$, then $R_{(m,d)} = 0$. If Nagata's conjecture is true, then

$$\mathbb{R}_{\geq 0}(1, \sqrt{t}) + \mathbb{R}_{\geq 0}(0, 1)$$

is a maximal ray ideal cone of R. Remark that, if $\sqrt{t} \notin \mathbb{Q}$, then it is not a rational polyhedral cone.

7. Multi-section rings defined by $\mathbb{Q}$-divisors

Let X be a normal projective variety over a field k of $\dim X > 0$. Let K be the function field of X. Let $H_1(X)$ be the set of closed subvarieties of X of codimension 1. Let D_i be a $\mathbb{Q}$-Weil divisor for $i = 1, 2, \ldots, n$, that is a finite sum

$$D_i = \sum_{F \in H_1(X)} m_{i,F} F,$$

where $m_{i,F} \in \mathbb{Q}$.

We define a $\mathbb{Z}^n$-graded ring as

$$R(X; D_1, D_2, \ldots, D_n) = \bigoplus_{r_1, \ldots, r_n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(\sum_i r_i D_i)) t_1^{r_1} t_2^{r_2} \cdots t_n^{r_n} \subset K[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}],$$

where

$$H^0(X, \mathcal{O}_X(\sum_i r_i D_i)) = \{a \in K^\times \mid \mathrm{div}(a) + \sum_i r_i D_i \geq 0\} \cup \{0\} \subset K.$$

We sometimes denote $R(X; D_1, D_2, \ldots, D_n)$ simply by R. We put

$$m_{i,F} = \frac{p_{i,F}}{q_{i,F}},$$

where $p_{i,F} \in \mathbb{Z}$ and $q_{i,F} \in \mathbb{N}$ such that $\mathrm{GCD}(p_{i,F}, q_{i,F}) = 1$. Here, remark that $q_{i,F} = 1$ if $p_{i,F} = 0$. We put

$$q_F = \mathrm{LCM}(q_{1,F}, q_{2,F}, \ldots, q_{n,F}).$$

We define

$$P_F = \bigoplus_{r_1, \ldots, r_n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(\sum_i r_i D_i - \frac{1}{q_F} F)) t_1^{r_1} t_2^{r_2} \cdots t_n^{r_n} \subset R(X; D_1, D_2, \ldots, D_n).$$

In this section, we shall prove the following:

Theorem 7.1. *Let X be a normal projective variety over a field k with $\dim X > 0$. Let $D_1, D_2, \ldots, D_n$ be $\mathbb{Q}$-Weil divisors.*

(1) *The ring* $R(X; D_1, D_2, \ldots, D_n)$ *is a Krull domain. The set of height* 1 *homogeneous prime ideals of* $R(X; D_1, D_2, \ldots, D_n)$ *is contained in* $\{P_F \mid F \in H_1(X)\}$.

(2) *Assume that* $a_1D_1 + a_2D_2 + \cdots + a_nD_n$ *is an ample Cartier divisor for some* $a_1, a_2, \ldots, a_n \in \mathbb{Z}$. *Then the set of height* 1 *homogeneous prime ideals of* $R(X; D_1, D_2, \ldots, D_n)$ *coincides with* $\{P_F \mid F \in H_1(X)\}$. *Putting* $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$, *the ray ideal* $J_{\boldsymbol{a}}(R(X; D_1, D_2, \ldots, D_n))$ *is not contained in any height* 1 *prime ideal of* $R(X; D_1, D_2, \ldots, D_n)$.

(3) *Assume that* $a_1D_1 + a_2D_2 + \cdots + a_nD_n$ *is an ample Cartier divisor for some* $a_1, a_2, \ldots, a_n \in \mathbb{Z}$. *We put*

$$\{F_1, F_2, \ldots, F_\ell\} = \{F \in H_1(X) \mid m_{i,F} \neq 0 \text{ for some } i\}$$

and

$$M = \bigoplus_{j=1}^{\ell} \frac{1}{q_{F_j}}\mathbb{Z} \supset L = \sum_{i=1}^{n} \mathbb{Z}(m_{i,F_1}, m_{i,F_2}, \ldots, m_{i,F_\ell}).$$

Then we have an exact sequence

$$L \cap \mathbb{Z}^\ell \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(R(X; D_1, D_2, \ldots, D_n)) \longrightarrow \frac{M}{L + \mathbb{Z}^\ell} \longrightarrow 0, \tag{7.1}$$

where Cl( ) *denotes the divisor class group.*

By the above theorem, we can immediately prove the following corollary.

Corollary 7.2. *Assume that* $a_1D_1 + a_2D_2 + \cdots + a_nD_n$ *is an ample Cartier divisor for some* $a_1, a_2, \ldots, a_n \in \mathbb{Z}$. *Then* $R(X; D_1, D_2, \ldots, D_n)$ *is factorial if and only if*

- $M = L + \mathbb{Z}^\ell$, *and*
- $\left\{ \sum_{i=1}^{n} b_iD_i \;\middle|\; \begin{array}{l} b_1,\ b_2,\ \ldots,\ b_n \text{ are integers such that} \\ \sum_{i=1}^{n} b_i(m_{i,F_1}, m_{i,F_2}, \ldots, m_{i,F_\ell}) \in \mathbb{Z}^\ell \end{array} \right\}$ *generates* $\mathrm{Cl}(X)$.

Now, we start to prove Theorem 7.1.

First, we shall prove (1). Let v_F be the normalized valuation of the discrete valuation ring $\mathcal{O}_{X,F}$ for $F \in H_1(X)$. We put

$$\begin{aligned} R_F &= \bigoplus_{r_1,\ldots,r_n \in \mathbb{Z}} \{a \in K \mid v_F(a) + \sum_i r_im_{i,F} \geq 0\} t_1^{r_1}t_2^{r_2}\cdots t_n^{r_n} \\ S &= K[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]. \end{aligned}$$

By definition, we have

$$R = \left(\bigcap_{F \in H_1(X)} R_F \right) \bigcap S. \tag{7.2}$$

Remark that R_F satisfies $R \subset R_F \subset S$. Let α_F be a generator of the maximal ideal of the discrete valuation ring $\mathcal{O}_{X,F}$. Then we have

$$\{a \in K \mid v_F(a) + \sum_i r_im_{i,F} \geq 0\} = \alpha_F^{-\lfloor \sum_i r_im_{i,F} \rfloor}\mathcal{O}_{X,F},$$

where $\lfloor \sum_i r_i m_{i,F} \rfloor$ is the largest integer which is not bigger than $\sum_i r_i m_{i,F}$.

Here, we show that R_F is a Noetherian normal domain. If $m_{i,F}$ is an integer for all i and F, then R_F is a Noetherian normal domain since

$$\begin{aligned} R_F &= \bigoplus_{r_1,\ldots,r_n\in\mathbb{Z}} \alpha_F^{-\sum_i r_i m_{i,F}} \mathcal{O}_{X,F} t_1^{r_1} t_2^{r_2} \cdots t_n^{r_n} \\ &= \mathcal{O}_{X,F}[(\alpha_F^{-m_{1,F}} t_1)^{\pm 1}, (\alpha_F^{-m_{2,F}} t_2)^{\pm 1}, \ldots, (\alpha_F^{-m_{n,F}} t_n)^{\pm 1}]. \end{aligned}$$

Suppose that $m_{i,F}$'s are rational numbers. We put

$$R_F^{(q_F)} = \bigoplus_{r_1,\ldots,r_n\in\mathbb{Z}} \{a \in K \mid v_F(a) + \sum_i q_F r_i m_{i,F} \geq 0\} t_1^{q_F r_1} t_2^{q_F r_2} \cdots t_n^{q_F r_n}.$$

Since $q_F m_{i,F}$'s are integers, $R_F^{(q_F)}$ is a Noetherian normal domain. For $a \in K$,

$$\begin{aligned} & a t_1^{r_1} t_2^{r_2} \cdots t_n^{r_n} \text{ is integral over } R_F^{(q_F)} \\ \Longleftrightarrow\ & (a t_1^{r_1} t_2^{r_2} \cdots t_n^{r_n})^{q_F} \text{ is integral over } R_F^{(q_F)} \\ \Longleftrightarrow\ & (a t_1^{r_1} t_2^{r_2} \cdots t_n^{r_n})^{q_F} \in R_F^{(q_F)} \\ \Longleftrightarrow\ & a t_1^{r_1} t_2^{r_2} \cdots t_n^{r_n} \in R_F. \end{aligned}$$

Therefore R_F is the integral closure of $R_F^{(q_F)}$ in S. Hence R_F is a Noetherian normal domain.

Put

$$Q_F = \bigoplus_{r_1,\ldots,r_n\in\mathbb{Z}} \{a \in K \mid v_F(a) + \sum_i r_i m_{i,F} > 0\} t_1^{r_1} t_2^{r_2} \cdots t_n^{r_n}.$$

Then Q_F is a prime ideal of R_F satisfying

$$P_F = Q_F \cap R. \tag{7.3}$$

It is easy to see $Q_F = \sqrt{\alpha_F R_F}$. Remark that Q_F is the unique height 1 homogeneous prime ideal of R_F. Furthermore, we have

$$v_{Q_F}(\alpha_F) = q_F \quad \text{and} \quad v_{Q_F}(t_i) = q_F m_{i,F} \tag{7.4}$$

for $i = 1, 2, \ldots, n$, where v_{Q_F} is the normalized valuation of the discrete valuation ring $(R_F)_{Q_F}$. Let $\{Q_\lambda \mid \lambda \in \Lambda\}$ be the set of non-homogeneous height 1 prime ideals of R_F. Since R_F is a Krull domain,

$$R_F = (R_F)_{Q_F} \bigcap \left(\bigcap_{\lambda\in\Lambda} (R_F)_{Q_\lambda} \right). \tag{7.5}$$

It is easy to see

$$S = R_F[\alpha_F^{-1}] = \bigcap_{\lambda\in\Lambda} (R_F)_{Q_\lambda}. \tag{7.6}$$

Therefore we have

$$\begin{aligned} R &= \left(\bigcap_{F \in H_1(X)} R_F \right) \bigcap S = \left(\bigcap_{F \in H_1(X)} (R_F)_{Q_F} \right) \bigcap S \\ &= \left(\bigcap_{F \in H_1(X)} ((R_F)_{Q_F} \cap Q(R)) \right) \bigcap (S \cap Q(R)) \end{aligned}$$

by (7.2), (7.5) and (7.6). Since R is the intersection of discrete valuation rings with finiteness condition (e.g., Section 12 in Matsumura [14]), we know that R is a Krull domain. By Theorem 12.3 in [14], we know that the set of height one prime ideals of R is contained in

$$\{P_F \mid F \in H_1(X)\} \cup \{Q \cap R \mid Q \in \mathrm{Spec}(S),\ \mathrm{ht}\, Q = 1\}.$$

The second assertion of (1) follows from the fact that any height 1 prime ideal Q of S does not contain a non-zero homogeneous element of R.

Next, we shall prove (2). Remark that $Q(R)$ coincides with $Q(S)$ since $\sum_i a_i D_i$ is ample. We shall prove that R_{P_F} coincides with $(R_F)_{Q_F}$ for any $F \in H_1(X)$. We have $R_{P_F} \subset (R_F)_{Q_F}$ by (7.3). In order to prove the opposite inclusion, it is enough to show $R_F \subset R_{P_F}$. Let M_F be the set of homogeneous elements contained in $R \setminus P_F$. We want to show $R_F \subset R[M_F^{-1}]$. In the case where all the $m_{i,G}$'s are integers, it is proved in the latter half of the proof of Theorem 1.1 in [6]. Here, put

$$q = \mathrm{LCM}\{q_G \mid G \in H_1(X)\}.$$

Then $R[M_F^{-1}]$ contains $R_F^{(q)}$, where

$$R_F^{(q)} = \bigoplus_{r_1, \ldots, r_n \in \mathbb{Z}} \{a \in K \mid v_F(a) + \sum_i q r_i m_{i,F} \geq 0\} t_1^{qr_1} t_2^{qr_2} \cdots t_n^{qr_n}.$$

Since $R[M_F^{-1}]$ is integrally closed in S and R_F is integral over $R_F^{(q)}$, $R[M_F^{-1}]$ contains R_F. Thus we have

$$R_{P_F} = (R_F)_{Q_F}. \tag{7.7}$$

Hence the set of height 1 homogeneous prime ideal of $R(X; D_1, D_2, \ldots, D_n)$ coincides with $\{P_F \mid F \in H_1(X)\}$. The latter assertion immediately follows from the first half.

Next, we shall prove (3). Let $\mathrm{Div}(X)$ be the set of Weil divisors on X, that is,

$$\mathrm{Div}(X) = \left\{ \sum_{F \in H_1(X)} n_F F \;\middle|\; n_F \in \mathbb{Z} \right\}.$$

Let $\mathrm{HDiv}(R)$ be the set of homogeneous Weil divisors of R, that is,

$$\mathrm{HDiv}(R) = \left\{ \sum_{F \in H_1(X)} n_F P_F \;\middle|\; n_F \in \mathbb{Z} \right\}.$$

Let

$$\phi : \mathrm{Div}(X) \longrightarrow \mathrm{HDiv}(R)$$

be a map defined by

$$\phi(F) = q_F P_F. \tag{7.8}$$

Here, we have an exact sequence

$$0 \longrightarrow \mathrm{Div}(X) \xrightarrow{\phi} \mathrm{HDiv}(R) \xrightarrow{\varphi} M/\mathbb{Z}^\ell \longrightarrow 0, \tag{7.9}$$

where φ is defined by $\varphi(\sum_F n_F F) = \overline{(n_{F_1}/q_{F_1}, n_{F_2}/q_{F_2}, \ldots, n_{F_\ell}/q_{F_\ell})}$.

For $a \in K^\times$ and $b \in Q(R)^\times$, we define

$$\mathrm{div}_X(a) = \sum_{F \in H_1(X)} v_F(a)F, \quad \mathrm{div}_R(b) = \sum_{P \in H_1(R)} v_{R_P}(b)P,$$

where $H_1(R)$ is the set of prime ideals of height 1 and v_{R_P} is the normalized valuation of the discrete valuation ring R_P. Then we have

$$\mathrm{Cl}(X) = \mathrm{Div}(X)/\{\mathrm{div}_X(a) \mid a \in K^\times\}$$

$$\mathrm{Cl}(R) = \mathrm{HDiv}(R)/\{\mathrm{div}_R(b) \mid b \text{ is a non-zero homogeneous element of } S\}$$

by [18], [11]. Here, remark that, for a non-zero homogeneous element b of S, $v_{R_P}(b) = 0$ if P is a non-homogeneous prime ideal of R of height one. Put

$$P(X) = \{\mathrm{div}_X(a) \mid a \in K^\times\}.$$

By (7.4), (7.7) and (7.8), we have

$$\phi(\mathrm{div}_X(a)) = \mathrm{div}_R(a)$$

for any $a \in K^\times$. Thus we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(\xi) & \longrightarrow & \mathbb{Z}^n & \xrightarrow{\xi} & L & \longrightarrow & 0 \\ & & \downarrow \eta' & & \downarrow \eta & & \downarrow \zeta & & \\ 0 & \longrightarrow & \mathrm{Div}(X)/P(X) & \xrightarrow{\overline{\phi}} & \mathrm{HDiv}(R)/\phi(P(X)) & \xrightarrow{\overline{\varphi}} & M/\mathbb{Z}^\ell & \longrightarrow & 0, \end{array} \tag{7.10}$$

where the exact sequence in the second row is induced by (7.9), ζ is induced by the inclusion $L \hookrightarrow M$, η is defined by

$$\eta(\boldsymbol{e}_i) = \mathrm{div}_R(t_i) = \sum_{F \in H_1(X)} q_F m_{i,F} P_F,$$

ξ is defined by

$$\xi(\boldsymbol{e}_i) = (m_{i,F_1}, m_{i,F_2}, \ldots, m_{i,F_\ell}).$$

Then the induced map η' coincides with 0. Since the cokernel of η is $\mathrm{Cl}(R)$, we obtain the exact sequence as in (7.1). We have completed the proof of Theorem 7.1. □

Let us prove Corollary 7.2. The map

$$L \cap \mathbb{Z}^\ell \longrightarrow \mathrm{Cl}(X)$$

in (7.1) sends

$$\sum_{i=1}^{n} b_i(m_{i,F_1}, m_{i,F_2}, \ldots, m_{i,F_\ell}) \in L \cap \mathbb{Z}^\ell \quad (b_1, b_2, \ldots, b_n \in \mathbb{Z})$$

to

$$\sum_{i=1}^{n} b_i \left(\sum_{F \in H_1(X)} m_{i,F} F \right) = \sum_{i=1}^{n} b_i D_i.$$

Therefore Corollary 7.2 immediately follows from Theorem 7.1 (3). □

8. Demazure construction for $\mathbb{Z}^n$-graded Krull domains

The aim of this section is to develop the Demazure construction for $\mathbb{Z}^n$-graded Krull domains.

For a $\mathbb{Z}^n$-graded domain A, $A_{(0)}$ denotes the homogeneous localization at 0, that is, $A_{(0)} = A[M^{-1}]$, where M is the set consisting of all the non-zero homogeneous elements of A. Remark that $A_{(0)}$ also has a structure of a $\mathbb{Z}^n$-graded ring.

Lemma 8.1. *Let A be a $\mathbb{Z}^n$-graded domain such that $A_{\mathbf{0}}$ is a field. Take $\boldsymbol{a} \in \mathrm{rel.int}(C(A)) \cap \mathbb{Q}^n$. Let N be the set of non-zero homogeneous elements with degree on the half line $\mathbb{R}_{>0}\boldsymbol{a}$, that is,*

$$N = \left(\bigcup_{\boldsymbol{b} \in \mathbb{R}_{>0}\boldsymbol{a} \cap \mathbb{Z}^n} A_{\boldsymbol{b}} \right) \setminus \{0\}.$$

Then $A[N^{-1}]$ coincides with $A_{(0)}$.

Proof. Let M be the set consisting of all the non-zero homogeneous elements of A. For any $g \in M$, there exists $u \in A$ such that gu is a homogeneous element contained in N since $\boldsymbol{a}$ is in the relative interior of $C(A)$. Then the assertion immediately follows from $1/g = u/(gu)$. □

Definition 8.2. Let A be a $\mathbb{Z}^n$-graded domain such that $A_{\mathbf{0}}$ is a field. Take $\boldsymbol{a} \in C(A) \cap \mathbb{Q}^n$. Assume that the ring

$$A_{\mathbb{R}_{\geq 0}\boldsymbol{a}} = \bigoplus_{\boldsymbol{b} \in \mathbb{R}_{\geq 0}\boldsymbol{a} \cap \mathbb{Z}^n} A_{\boldsymbol{b}}$$

is Noetherian. We think that $A_{\mathbb{R}_{\geq 0}\boldsymbol{a}}$ is a $\mathbb{N}_0$-graded ring. Then $\mathrm{Proj}(A_{\mathbb{R}_{\geq 0}\boldsymbol{a}})$ is called the *projective variety on the ray* $\mathbb{R}_{\geq 0}\boldsymbol{a}$, and denoted by $X_{\boldsymbol{a}}$.

If $\boldsymbol{a} \in \mathrm{rel.int}(C(A)) \cap \mathbb{Q}^n$, the function field of $X_{\boldsymbol{a}}$ coincides with $(A_{(0)})_{\mathbf{0}}$ by Lemma 8.1. For any $\boldsymbol{b} \in C(A) \cap \mathbb{Q}^n$, the function field of $X_{\boldsymbol{b}}$ is contained in $(A_{(0)})_{\mathbf{0}}$. Therefore, for $\boldsymbol{a} \in \mathrm{rel.int}(C(A)) \cap \mathbb{Q}^n$ and $\boldsymbol{b} \in C(A) \cap \mathbb{Q}^n$, there is a unique rational map from $X_{\boldsymbol{a}}$ to $X_{\boldsymbol{b}}$ if the coordinate rings are Noetherian.

Lemma 8.3. *Let A be a $\mathbb{Z}^n$-graded domain such that $A_{\mathbf{0}}$ is a field. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be in $C(A) \cap \mathbb{Q}^n$ such that $A_{\mathbb{R}_{\geq 0}\boldsymbol{a}}$ and $A_{\mathbb{R}_{\geq 0}\boldsymbol{b}}$ are Noetherian. Assume that $J_{\boldsymbol{b}}(A)$ contains $J_{\boldsymbol{a}}(A)$.*

(1) *The function field of* $X_{\boldsymbol{b}}$ *is naturally contained in that of* $X_{\boldsymbol{a}}$.
(2) *The unique rational map from* $X_{\boldsymbol{a}}$ *to* $X_{\boldsymbol{b}}$ *is a morphism of schemes.*

Proof. Let M be the set consisting of all the non-zero homogeneous elements of A. Let $N_{\boldsymbol{a}}$ (resp. $N_{\boldsymbol{b}}$) be the set consisting of all the non-zero homogeneous elements of A with degree in $\mathbb{R}_{>0}\boldsymbol{a}$ (resp. $\mathbb{R}_{>0}\boldsymbol{b}$). Then the function field of $X_{\boldsymbol{a}}$ (resp. $X_{\boldsymbol{b}}$) is $(A[N_{\boldsymbol{a}}^{-1}])_{\boldsymbol{0}}$ (resp. $(A[N_{\boldsymbol{b}}^{-1}])_{\boldsymbol{0}}$). Here, remark that both $(A[N_{\boldsymbol{a}}^{-1}])_{\boldsymbol{0}}$ and $(A[N_{\boldsymbol{b}}^{-1}])_{\boldsymbol{0}}$ are contained in the field $(A[M^{-1}])_{\boldsymbol{0}}$.

We may assume that $I_{\boldsymbol{a}}(A)^m \subset I_{\boldsymbol{b}}(A)$ for some $m \in \mathbb{N}$ since $A_{\mathbb{R}_{\geq 0}\boldsymbol{a}}$ is Noetherian. Therefore there exist non-zero homogeneous elements b_1, b_2, …, b_s with degree in $\mathbb{R}_{>0}\boldsymbol{b}$, and non-zero homogeneous elements c_1, c_2, …, c_s such that

- $\deg(c_ib_i)$ is in $\mathbb{R}_{>0}\boldsymbol{a}$ for $i = 1, 2, \ldots, s$, and
- c_1b_1, c_2b_2, …, c_sb_s generate an ideal of A containing $I_{\boldsymbol{a}}(A)^m$.

Then $c_1b_1, c_2b_2, \ldots, c_sb_s$ generate an ideal of $A_{\mathbb{R}_{\geq 0}\boldsymbol{a}}$ which contains $(A_{\mathbb{R}_{>0}\boldsymbol{a}})^m$. Therefore we have an affine covering

$$X_{\boldsymbol{a}} = \bigcup_{i=1}^{s} \operatorname{Spec}\left((A_{\mathbb{R}_{\geq 0}\boldsymbol{a}}[(c_ib_i)^{-1}])_{\boldsymbol{0}}\right).$$

Since

$$(A_{\mathbb{R}_{\geq 0}\boldsymbol{b}}[b_i^{-1}])_{\boldsymbol{0}} = (A[b_i^{-1}])_{\boldsymbol{0}} \subset (A[(c_ib_i)^{-1}])_{\boldsymbol{0}} = (A_{\mathbb{R}_{\geq 0}\boldsymbol{a}}[(c_ib_i)^{-1}])_{\boldsymbol{0}},$$

we have

$$(A[N_{\boldsymbol{b}}^{-1}])_{\boldsymbol{0}} = Q((A_{\mathbb{R}_{\geq 0}\boldsymbol{b}}[b_i^{-1}])_{\boldsymbol{0}}) \subset Q((A_{\mathbb{R}_{\geq 0}\boldsymbol{a}}[(c_ib_i)^{-1}])_{\boldsymbol{0}}) = (A[N_{\boldsymbol{a}}^{-1}])_{\boldsymbol{0}}.$$

The assertion (1) has been proved.

We have a morphism

$$\operatorname{Spec}\left((A_{\mathbb{R}_{\geq 0}\boldsymbol{a}}[(c_ib_i)^{-1}])_{\boldsymbol{0}}\right) \longrightarrow \operatorname{Spec}\left((A_{\mathbb{R}_{\geq 0}\boldsymbol{b}}[b_i^{-1}])_{\boldsymbol{0}}\right) \subset X_{\boldsymbol{b}}.$$

Patching together these morphisms, we obtain a morphism $X_{\boldsymbol{a}} \to X_{\boldsymbol{b}}$. □

Definition 8.4. Let A be a $\mathbb{Z}^n$-graded domain such that $A_{\boldsymbol{0}}$ is a field. Let σ be a ray ideal cone of A such that A_σ is Noetherian. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be in $\operatorname{rel.int}(\sigma) \cap \mathbb{Q}^n$. Then, by Lemma 8.3, we have $X_{\boldsymbol{a}} = X_{\boldsymbol{b}}$. We denote this variety by X_σ, and call it the *variety of the ray ideal cone* σ.

Example 8.5. Let k be a field.

(1) Let $A = k[x, y, z]$ be a $\mathbb{Z}^2$-graded polynomial ring with $\deg(x) = (-1, 0)$, $\deg(y) = (0, 1)$, $\deg(z) = (1, 1)$ and $A_{\boldsymbol{0}} = k$. Then $J_{(0,1)}(A) \supsetneq J_{(-1,1)}(A)$. However, the morphism $X_{(-1,1)} \to X_{(0,1)}$ is an isomorphism.
(2) Let $A = k[x, y, z, w, u]$ be a $\mathbb{Z}^2$-graded polynomial ring with $\deg(x) = \deg(y) = (1, 0)$, $\deg(z) = \deg(w) = (0, 1)$, $\deg(u) = (1, 1)$ and $A_{\boldsymbol{0}} = k$. There exist just two maximal chambers $\mathbb{R}_{\geq 0}(1, 0) + \mathbb{R}_{\geq 0}(1, 1)$ and $\mathbb{R}_{\geq 0}(1, 1) + \mathbb{R}_{\geq 0}(0, 1)$. Put $\boldsymbol{a} = (2, 1)$, $\boldsymbol{b} = (1, 1)$ and $\boldsymbol{c} = (1, 2)$. Since $J_{\boldsymbol{b}}(A) \supset J_{\boldsymbol{a}}(A)$ and $J_{\boldsymbol{b}}(A) \supset J_{\boldsymbol{c}}(A)$, we have birational morphisms $X_{\boldsymbol{a}} \to X_{\boldsymbol{b}}$ and $X_{\boldsymbol{c}} \to X_{\boldsymbol{b}}$. We know $X_{\boldsymbol{b}} = \operatorname{Proj}(k[xz, xw, yz, yw, u])$. Then $X_{\boldsymbol{a}} \to X_{\boldsymbol{b}}$ is the blow-up along $V_+(xz, yz)$, and $X_{\boldsymbol{c}} \to X_{\boldsymbol{b}}$ is the blow-up along $V_+(xz, xw)$.

We shall prove the following theorem:

Theorem 8.6. *Let A be a $\mathbb{Z}^n$-graded domain such that $A_{\mathbf{0}}$ is a field. Suppose that $\dim C(A) = n$. Then the following three conditions are equivalent:*

- (I) *The ring A is a Krull domain such that $(A_{(0)})_{\boldsymbol{e}_i} \neq 0$ for $i = 1, 2, \ldots, n$. Furthermore, there exists a chamber σ of A satisfying the following two conditions:*
  - (a) *The ray ideal J_σ has height bigger than 1.*
  - (b) *The ring A_σ is Noetherian.*
- (II) *There exist a normal projective variety X over $A_{\mathbf{0}}$, $\mathbb{Q}$-divisors D_1, D_2, ..., D_n and $\boldsymbol{c} = (c_1, c_2, \ldots, c_n) \in \mathbb{Z}^n$ satisfying the following three conditions:*
  - (a) *$\sum_i c_i D_i$ is an ample Cartier divisor.*
  - (b) *There exists a n-dimensional rational polyhedral cone ρ such that $\boldsymbol{c} \in \mathrm{int}(\rho)$ and A_ρ is Noetherian.*
  - (c) *The ring A is isomorphic to $R(X; D_1, D_2, \ldots, D_n)$ as a $\mathbb{Z}^n$-graded ring.*
- (III) *There exist a normal projective variety X over $A_{\mathbf{0}}$, $\mathbb{Q}$-divisors D_1, D_2, ..., D_n and $\boldsymbol{c} = (c_1, c_2, \ldots, c_n) \in \mathbb{Z}^n$ satisfying the following three conditions:*
  - (a) *$\sum_i c_i D_i$ is an ample Cartier divisor.*
  - (b) *There exists a positive integer b such that bD_1, bD_2, ..., bD_n are Cartier divisors.*
  - (c) *The ring A is isomorphic to $R(X; D_1, D_2, \ldots, D_n)$ as a $\mathbb{Z}^n$-graded ring.*

Remark 8.7. The variety X is not determined uniquely in Theorem 8.6. Let A be the ring in Example 8.5 (2). Then A is equal to the multi-section ring of $X_{\boldsymbol{a}}$, $X_{\boldsymbol{b}}$ and $X_{\boldsymbol{c}}$.

In the case where A is Noetherian, obviously we may remove the conditions (b) in (I), and (b) in (II).

When we show (I) $\Rightarrow$ (III), we choose $\boldsymbol{c}$ which is contained in $\mathrm{int}(\sigma)$.

When we show (II) $\Rightarrow$ (I), we choose σ such that σ contains $\boldsymbol{c}$, but $\mathrm{int}(\sigma)$ may not contain $\boldsymbol{c}$.

When we show (II) $\Rightarrow$ (III), we sometimes need to change the variety X.

If $\dim X = 0$, we think that any $\mathbb{Q}$-divisor is 0, and 0 is an ample Cartier divisor of X.

Using Lemma 8.1, it is easy to see that the following four conditions are equivalent:

- $A = A_{\mathbf{0}}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$ with $\deg(t_i) = \boldsymbol{e}_i$ for $i = 1, 2, \ldots, n$,
- the above condition (I) is satisfied with $J_\sigma = A$,
- the above condition (II) is satisfied with $\dim X = 0$,
- the above condition (III) is satisfied with $\dim X = 0$.

Let us start to prove Theorem 8.6. Assume the condition (III). We shall prove (II). By (b) in (III), there exists an n-dimensional rational polyhedral simplicial cone ρ such that $\boldsymbol{c} \in \mathrm{int}(\rho)$ and $\sum_i bc'_i D_i$ is ample Cartier divisor for any $(c'_1, c'_2, \ldots, c'_n) \in \rho \cap \mathbb{Z}^n$. Then, by Zariski's theorem (Lemma 2.8 in Hu-Keel [8]), we know that A_ρ is Noeherian.

Next, we shall prove (II) $\Rightarrow$ (I). When $\dim X = 0$, this implication follows as in Remark 8.7. From now on, we assume $\dim X > 0$. By Theorem 7.1 (1), A is a Krull domain. By (a) in (II), we know $(A_{(0)})_{\boldsymbol{e}_i} \neq 0$ for $i = 1, 2, \ldots, n$ immediately. Choose a chamber σ of A_ρ such that

$$\boldsymbol{c} \in \sigma \subset \rho.$$

Here, $\mathrm{int}(\sigma)$ need not contain $\boldsymbol{c}$. Since $A_\sigma = (A_\rho)_\sigma$, A_σ is Noetherian by Remark 2.3 (3). By Proposition 2.4 (1), σ is also a chamber of A. Suppose that a height one homogeneous prime ideal P of A contains J_σ. By Theorem 7.1 (2), there exists $F \in H_1(X)$ such that $P = P_F$. Take $\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in \mathrm{int}(\sigma) \cap \mathbb{Z}^n$ such that $\sum_{i=1}^n a_i m_{i,G}$ is an integer for any $G \in H_1(X)$. Since $\boldsymbol{c} \in \sigma$ and $\boldsymbol{a} \in \mathrm{int}(\sigma)$, $\boldsymbol{a} + m\boldsymbol{c} \in \mathrm{int}(\sigma)$ for any $m \in \mathbb{N}$. Therefore $A_{\boldsymbol{a}+m\boldsymbol{c}} \subset P_F$. On the other hand, since $\sum_i c_i D_i$ is ample and $\sum_{i=1}^n a_i D_i$ is a Weil divisor with integer coefficients, it is easy to see that $A_{\boldsymbol{a}+m\boldsymbol{c}} \not\subset P_F$ for $m \gg 0$. Contradiction.

Next, we shall prove (I) $\Rightarrow$ (III). We may assume $J_\sigma \neq A$ by Remark 8.7. Changing a coordinate, we may assume that the interior of the chamber σ contains the unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\ldots$, $\boldsymbol{e}_n$. By Lemma 8.3, we have

$$X_{\boldsymbol{e}_1} = X_{\boldsymbol{e}_2} = \cdots = X_{\boldsymbol{e}_n}.$$

We denote this scheme by X. Then X is a normal projective variety of dimension positive with function field $(A_{(0)})_{\boldsymbol{0}}$ by Lemma 8.1. (If $\dim X = 0$, then $A_{\mathbb{R}_{\geq 0}\boldsymbol{e}_1}$ is one dimensional $\mathbb{N}_0$-graded normal domain. It is easy to see that $A_{\mathbb{R}_{\geq 0}\boldsymbol{e}_1}$ is a polynomial ring over the field $A_{\boldsymbol{0}}$ with one variable. Hence $I_{\boldsymbol{e}_1}(A)$ is a principal ideal. In this case, $J_{\boldsymbol{e}_1}(A)$ is a unit ideal if the height of $J_{\boldsymbol{e}_1}(A)$ is bigger than 1.) Here, we can choose $0 \neq t_i \in (A_{(0)})_{\boldsymbol{e}_i}$ for $i = 1, 2, \ldots, n$ by (I). Then, by the Demazure construction [4], [19], there exists $\mathbb{Q}$-divisors D_1, D_2, $\ldots$, D_n of X such that

- there exists a positive integer b such that bD_1, bD_2, $\ldots$, bD_n are ample Cartier divisors on X,
- for $x \in (A_{(0)})_{\boldsymbol{0}}$, $m \in \mathbb{N}$ and $i = 1, 2, \ldots, n$,

$$x t_i^m \in A_{m\boldsymbol{e}_i} \Longleftrightarrow x \in H^0(X, \mathcal{O}_X(mD_i)).$$

Both A and $R(X; D_1, D_2, \ldots, D_n)$ are $\mathbb{Z}^n$-graded subrings of $A_{(0)} = (A_{(0)})_{\boldsymbol{0}}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$. It is enough to show $A = R(X; D_1, D_2, \ldots, D_n)$ as a subset of $A_{(0)}$. We denote $R(X; D_1, D_2, \ldots, D_n)$ simply by R.

By Proposition 2.4 (3), $(\mathbb{R}_{\geq 0})^n$ is a chamber of the Noetherian $\mathbb{Z}^n$-graded ring $A_{(\mathbb{R}_{\geq 0})^n}$. Let B be the subring of $A_{(\mathbb{R}_{\geq 0})^n}$ generated by

$$\bigcup_{i=1}^n \left[\bigcup_{m>0} A_{m\boldsymbol{e}_i} \right]$$

over $A_{\boldsymbol{0}}$. Then the inclusion $B \to A_{(\mathbb{R}_{\geq 0})^n}$ is finite by Theorem 4.1. Therefore $A_{(\mathbb{R}_{\geq 0})^n}$ is the integral closure of B in $A_{(0)}$. On the other hand, R is integrally closed in $A_{(0)}$ by Theorem 7.1, and $R_{(\mathbb{R}_{\geq 0})^n}$ is integral over B by Zariski's theorem (Lemma 2.8 in

Hu-Keel [8]). Therefore we have

$$A_{(\mathbb{R}_{\geq 0})^n} = R_{(\mathbb{R}_{\geq 0})^n}.$$

Let y_1, y_2, ..., y_t be a homogeneous generating system of the homogeneous maximal ideal of $A_{\mathbb{R}_{\geq 0}(\boldsymbol{e}_1+\boldsymbol{e}_2+\cdots+\boldsymbol{e}_n)}$. Here, for a subring T of $A_{(0)}$, we define the ideal transform to be

$$D_{\underline{y}}(T) = \{x \in A_{(0)} \mid \ y_1^m x, \ y_2^m x, \ldots, \ y_t^m x \in T \text{ for } m \gg 0 \ \}.$$

By definition, $D_{\underline{y}}(A_{(\mathbb{R}_{\geq 0})^n})$ contains A. Therefore we obtain $D_{\underline{y}}(A_{(\mathbb{R}_{\geq 0})^n}) = D_{\underline{y}}(A)$. In the same way, we obtain $D_{\underline{y}}(R_{(\mathbb{R}_{\geq 0})^n}) = D_{\underline{y}}(R)$. Thus we have

$$D_{\underline{y}}(A) = D_{\underline{y}}(A_{(\mathbb{R}_{\geq 0})^n}) = D_{\underline{y}}(R_{(\mathbb{R}_{\geq 0})^n}) = D_{\underline{y}}(R).$$

By (I) (a), the ideal $(y_1, y_2, \ldots, y_t)A$ is of height bigger than 1. Since A is a Krull domain, we obtain $D_{\underline{y}}(A) = A$. By Theorem 7.1 (2), $(y_1, y_2, \ldots, y_t)R$ is of height bigger than 1. Since R is a Krull domain by Theorem 7.1 (1), we obtain $D_{\underline{y}}(R) = R$.

We have completed the proof of Theorem 8.6.

Example 8.8. Let $A = k[x, y, z, w]$ be a $\mathbb{Z}^2$-graded polynomial ring over a field $A_{\boldsymbol{0}} = k$ with $\deg(x) = (1, 0)$, $\deg(y) = (2, 0)$, $\deg(z) = (0, 1)$ and $\deg(w) = (0, 2)$. Let σ be a 2-dimensional cone contained in

$$\mathbb{R}_{\geq 0}(1, 0) + \mathbb{R}_{\geq 0}(0, 1).$$

Then A and σ satisfy the condition (I) in Theorem 8.6. In this case, $X_\sigma = \mathbb{P}^1_k \times \mathbb{P}^1_k$. Letting

$$D_1 = \frac{1}{2}\left(\{(1:0)\} \times \mathbb{P}^1_k\right) \quad \text{and}$$
$$D_1 = \frac{1}{2}\left(\mathbb{P}^1_k \times \{(1:0)\}\right),$$

A is isomorphic to $R(X_\sigma; D_1, D_2)$.

Let $B = k[x, y, z, u, v]$ be a $\mathbb{Z}^2$-graded polynomial ring over a field k such that $\deg(x) = (1, 0)$, $\deg(y) = (2, 1)$, $\deg(z) = (1, 1)$, $\deg(u) = (1, 2)$ and $\deg(v) = (0, 1)$. We put

$$\sigma_1 = \mathbb{R}_{\geq 0}(2, 1) + \mathbb{R}_{\geq 0}(1, 1), \quad \sigma_2 = \mathbb{R}_{\geq 0}(1, 1) + \mathbb{R}_{\geq 0}(1, 2).$$

Then both σ_1 and σ_2 satisfy the condition (I) in Theorem 8.6.

Next, we shall prove the following theorem:

Theorem 8.9. *Let A be a $\mathbb{Z}^n$-graded domain such that $A_{\boldsymbol{0}}$ is a field. Suppose that $\dim C(A) = n$. Then the following three conditions are equivalent:*

(I) *The ring A is a Krull domain such that $(A_{(0)})_{\boldsymbol{e}_i} \neq 0$ for $i = 1, 2, \ldots, n$. Furthermore there exists a chamber σ of A satisfying the following two conditions:*
   (a) *For any height 1 prime ideal P containing J_σ, P contains $J_{\boldsymbol{a}}(A)$ for any $\boldsymbol{a} \in \mathrm{int}(C(A)) \cap \mathbb{Q}^n$.*
   (b) *The ring A_σ is Noetherian.*

(II) *There exist a normal projective variety X over $A_{\mathbf{0}}$, $\mathbb{Q}$-divisors D_1, D_2, ..., D_n and $\boldsymbol{c} = (c_1, c_2, \ldots, c_n) \in \mathbb{Z}^n$ satisfying the following three conditions:*
  (a) *$\boldsymbol{c}$ is in* $\mathrm{int}(C(A))$ *and $\sum_i c_i D_i$ is an ample Cartier divisor.*
  (b) *There exists an n-dimensional rational polyhedral cone ρ such that $\boldsymbol{c} \in$* $\mathrm{int}(\rho)$ *and A_ρ is Noetherian.*
  (c) *There exists a rational polyhedral cone τ such that the ring A is isomorphic to $R(X; D_1, D_2, \ldots, D_n)_\tau$ as a $\mathbb{Z}^n$-graded ring.*

(III) *There exist a normal projective variety X over $A_{\mathbf{0}}$, $\mathbb{Q}$-divisors D_1, D_2, ..., D_n and $\boldsymbol{c} = (c_1, c_2, \ldots, c_n) \in \mathbb{Z}^n$ satisfying the following three conditions:*
  (a) *$\boldsymbol{c}$ is in $C(A)$ and $\sum_i c_i D_i$ is an ample Cartier divisor.*
  (b) *There exists a positive integer b such that bD_1, bD_2, ..., bD_n are Cartier divisors.*
  (c) *There exists a rational polyhedral cone τ such that the ring A is isomorphic to $R(X; D_1, D_2, \ldots, D_n)_\tau$ as a $\mathbb{Z}^n$-graded ring.*

Proof. First, we shall prove (III) $\Rightarrow$ (II). Assume that (III) is satisfied. Then it is easy to see that there exists $\boldsymbol{c}' = (c'_1, c'_2, \ldots, c'_n) \in \mathbb{Z}^n$ such that $\boldsymbol{c}'$ is in $\mathrm{int}(C(A))$ and $\sum_{i=1}^n c'_i D_i$ is an ample Cartier divisor. Then this implication will be proved in the same way as in the proof of Theorem 8.6.

Next, we shall prove (II) $\Rightarrow$ (I). Put $R = R(X; D_1, D_2, \ldots, D_n)$. By Theorem 8.6, R satisfies the condition (I) in Theorem 8.6. Therefore there exists a chamber σ of R such that the height of $J_\sigma(R)$ is bigger than 1 and R_σ is Noetherian. We may assume $\boldsymbol{c} = (c_1, c_2, \ldots, c_n) \in \sigma$ (cf. Remark 8.7). By (II) (a), $\boldsymbol{c}$ is in $\mathrm{int}(C(A))$. Replacing σ by $\sigma \cap C(A)$, we may assume that

$$\sigma \subset C(A) \subset \tau.$$

Since $A = R_\tau$, we have

$$A = R \cap (R_{(0)})_\tau.$$

In order to show that A is a Krull domain, it is enough to show that $(R_{(0)})_\tau$ is a Krull domain. (The intersection of two Krull domains are Krull again.) Since τ is an n-dimensional rational polyhedral cone, there exist a finite number of vectors $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, ..., $\boldsymbol{b}_m$ in $\mathbb{Q}^n$ such that

$$\tau = \bigcap_{i=1}^{m} \{\boldsymbol{a} \in \mathbb{R}^n \mid (\boldsymbol{a}, \boldsymbol{b}_i) \geq 0\}.$$

We may assume that $\tau \cap \{\boldsymbol{a} \in \mathbb{R}^n \mid (\boldsymbol{a}, \boldsymbol{b}_i) = 0\}$ is a face of τ of dimension $n-1$ for $i = 1, 2, \ldots, m$. Here, we put

$$T_i = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^n,\ (\boldsymbol{a}, \boldsymbol{b}_i) \geq 0} (R_{(0)})_{\boldsymbol{a}}$$

and

$$Q_i = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^n,\ (\boldsymbol{a}, \boldsymbol{b}_i) > 0} (R_{(0)})_{\boldsymbol{a}}.$$

Then Q_i is a prime ideal of the ring T_i. Put $V_i = (T_i)_{Q_i}$. Then

$$T_i = R_{(0)} \cap V_i \ \text{ and } \ (R_{(0)})_\tau = R_{(0)} \cap \left(\bigcap_{i=1}^{m} T_i \right) = R_{(0)} \cap \left(\bigcap_{i=1}^{m} V_i \right).$$

Since $R_{(0)}$ is Noetherian normal domain and V_i's are discrete valuation rings, $(R_{(0)})_\tau$ is a Krull domain. Hence A is a Krull domain. Here, we have

$$A = R \cap \left(\bigcap_{i=1}^{m} V_i \right).$$

Let P be a height one prime ideal of A containing J_σ. Height one prime ideals which come from R does not contain J_σ by the definition of σ. Thus there exists i such that $P = A \cap Q_i V_i$. Then

$$P = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^n, \ (\boldsymbol{a}, \boldsymbol{b}_i) > 0} A_{\boldsymbol{a}}.$$

Thus P satisfies the condition (I) (a).

Next, we shall prove (I) $\Rightarrow$ (III). If the height of J_σ is bigger than 1, this implication immediately follows from Theorem 8.6.

Assume hat $\mathrm{ht}\, J_\sigma = 1$. Let P be a height one prime ideal containing J_σ. Then, by the condition (I) (a), P contains $I_{\boldsymbol{a}}(A)$ for any $\boldsymbol{a} \in \mathrm{int}(C(A)) \cap \mathbb{Q}^n$. Since P contains a non-zero homogeneous element and the height of P is one, P is a homogeneous prime ideal. Put $\nu = C(A/P)$. By the assumption, we have $\nu \cap \mathrm{int}(C(A)) = \emptyset$. Then we obtain

$$(\nu - \nu) \cap \mathrm{int}(C(A)) = \emptyset. \tag{8.1}$$

We shall prove the following claim:

Claim 8.10. *(1) The dimension of ν is $n-1$.*
(2) There exists $\mathbf{0} \neq \boldsymbol{b} \in \mathbb{Q}^n$ such that

$$A = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^n, \ (\boldsymbol{a}, \boldsymbol{b}) \geq 0} A_{\boldsymbol{a}} \quad \textit{and} \quad P = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^n, \ (\boldsymbol{a}, \boldsymbol{b}) > 0} A_{\boldsymbol{a}}.$$

First, we shall prove (1). Let s be the dimension of ν. By (8.1), we have $s < n$. Assume that $s < n-1$. Changing the basis of $\mathbb{Z}^n$, we may assume

$$\nu - \nu = \mathbb{R}\boldsymbol{e}_1 + \mathbb{R}\boldsymbol{e}_2 + \cdots + \mathbb{R}\boldsymbol{e}_s.$$

Let

$$\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-s}$$

be the projection defined by

$$\phi((d_1, d_2, \ldots, d_n)) = (d_{s+1}, d_{s+2}, \ldots, d_n).$$

By (8.1), we know that $\phi(C(A)) \neq \mathbb{R}^{n-s}$. Then there exists a non-zero vector [4] $\boldsymbol{b}' \in \mathbb{R}^{n-s}$ such that $(\boldsymbol{a}', \boldsymbol{b}') \geq 0$ for any $\boldsymbol{a}' \in \phi(C(A))$. Here, put

$$\Omega = \{\phi(\boldsymbol{a}) \mid \boldsymbol{a} \in \mathbb{Z}^n,\ A_{\boldsymbol{a}} \neq 0\}$$
$$\Omega_P = \{\phi(\boldsymbol{a}) \mid \boldsymbol{a} \in \mathbb{Z}^n,\ 0 \neq A_{\boldsymbol{a}} \subset P\}.$$

If $A_{\boldsymbol{a}} \not\subset P$, then $\boldsymbol{a} \in \nu$ and $\phi(\boldsymbol{a}) = \boldsymbol{0}$. Thus we have $\Omega = \Omega_P \cup \{\boldsymbol{0}\}$. Remark that

$$\Omega_P \text{ generates } \phi(C(A)) \text{ as a cone} \tag{8.2}$$

and

$$\Omega_P \subset \Omega \subset \phi(C(A)) \subset \{\boldsymbol{a}' \in \mathbb{R}^{n-s} \mid (\boldsymbol{a}', \boldsymbol{b}') \geq 0\}.$$

If $\phi(\boldsymbol{a}) \neq \boldsymbol{0}$, then $A_{\boldsymbol{a}} \subset P$. Thus we have

$$P \supset \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^n,\ (\phi(\boldsymbol{a}), \boldsymbol{b}') > 0} A_{\boldsymbol{a}}.$$

Remark that the right-hand-side is a non-zero prime ideal of A. Since the height of P is one, we know that P coincides with the right-hand-side. That is, for $0 \neq q \in A_{\boldsymbol{a}}$,

$$q \in P \Longrightarrow (\phi(\boldsymbol{a}), \boldsymbol{b}') > 0 \Longrightarrow \phi(\boldsymbol{a}) \neq \boldsymbol{0} \Longrightarrow q \in P. \tag{8.3}$$

Let $x = x_1 + x_2 + \cdots + x_\ell$ be a non-zero element of A, where $0 \neq x_i \in A_{\boldsymbol{c}_i}$ for $i = 1, 2, \ldots, \ell$, and assume that $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_\ell$ are mutually distinct. Here, we define

$$\mathrm{ord}(x) = \min\{(\phi(\boldsymbol{c}_i), \boldsymbol{b}') \mid i = 1, 2, \ldots, \ell\} \geq 0$$
$$\mathrm{in}(x) = \sum_{(\phi(\boldsymbol{c}_i), \boldsymbol{b}') = \mathrm{ord}(x)} x_i.$$

Then it is easy to see that, for non-zero elements $x, y \in A$, we have

$$\mathrm{ord}(x) + \mathrm{ord}(y) = \mathrm{ord}(xy),$$
$$\mathrm{in}(x) \cdot \mathrm{in}(y) = \mathrm{in}(xy).$$

Let $\boldsymbol{c}'$ be an element in $\mathbb{Z}^{n-s}$. We say that x is ϕ-homogeneous with ϕ-deg$(x) = \boldsymbol{c}'$ if $\phi(\boldsymbol{c}_i) = \boldsymbol{c}'$ for $i = 1, 2, \ldots, \ell$.

For $0 \neq q \in A$, we have

$$q \notin P \Longleftrightarrow \mathrm{ord}(q) = 0 \Longleftrightarrow \mathrm{in}(q) \text{ is } \phi\text{-homogeneous with } \phi\text{-deg}(\mathrm{in}(q)) = \boldsymbol{0} \tag{8.4}$$

by (8.3).

Since A_P is a discrete valuation ring, there exists $\pi \in P$ such that $PA_P = \pi A_P$. Take $0 \neq y \in A_{\boldsymbol{a}} \subset P$. Since $y \in PA_P = \pi A_P$, there exists $r \in \mathbb{N}$ and $z, w \in A \setminus P$ such that

$$z\pi^r = wy. \tag{8.5}$$

Then we know that

$$\mathrm{in}(z) \cdot \mathrm{in}(\pi)^r = \mathrm{in}(w) \cdot y. \tag{8.6}$$

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[4] The difficulty in this proof lies in that we have to find such a vector $\boldsymbol{b}'$ contained in $\mathbb{Q}^{n-s}$.

By (8.4), $\mathrm{in}(z)$ and $\mathrm{in}(w)$ are ϕ-homogeneous with $\phi\text{-deg}(\mathrm{in}(z)) = \phi\text{-deg}(\mathrm{in}(w)) = \mathbf{0}$. Therefore $\mathrm{in}(\pi)^r$ is ϕ-homogeneous with $\phi\text{-deg}(\mathrm{in}(\pi)^r) = \phi(\boldsymbol{a})$. Then it is easy to see that $\mathrm{in}(\pi)$ is ϕ-homogeneous and

$$\phi\text{-deg}(y) = r \cdot \phi\text{-deg}(\mathrm{in}(\pi)).$$

Thus we have

$$\Omega_P = \{r \cdot \phi\text{-deg}(\mathrm{in}(\pi)) \mid r \in \mathbb{N}\}.$$

It contradicts that $\dim \phi(C(A)) = n - s \geq 2$ (cf. (8.2)).

Next we shall prove (2) of Claim 8.10. Since $s = n-1$, we may assume that the vector $\boldsymbol{b}'$ in the proof of (1) is contained in $\mathbb{Q}^1$. Put $\boldsymbol{b} = (\mathbf{0}, \boldsymbol{b}') \in \mathbb{Q}^n$. Then we have $(\phi(\boldsymbol{a}), \boldsymbol{b}') = (\boldsymbol{a}, \boldsymbol{b})$ for any $\boldsymbol{a} \in \mathbb{R}^n$. Hence we have

$$A = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^n,\ (\phi(\boldsymbol{a}),\boldsymbol{b}')\geq 0} A_{\boldsymbol{a}} = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^n,\ (\boldsymbol{a},\boldsymbol{b})\geq 0} A_{\boldsymbol{a}},$$

$$P = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^n,\ (\phi(\boldsymbol{a}),\boldsymbol{b}')> 0} A_{\boldsymbol{a}} = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^n,\ (\boldsymbol{a},\boldsymbol{b})> 0} A_{\boldsymbol{a}}.$$

We have completed the proof of Claim 8.10.

Since A is a Krull domain, there exists only finitely many height one prime ideals containing J_σ. Let $\{P_1, P_2, \ldots, P_m\}$ be the set of height one prime ideals of A containing J_σ. By Claim 8.10 (2), there exist $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_m$ in $\mathbb{Q}^n$ such that

$$A = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^n,\ (\boldsymbol{a},\boldsymbol{b}_i)\geq 0} A_{\boldsymbol{a}} \quad \text{and} \quad P_i = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^n,\ (\boldsymbol{a},\boldsymbol{b}_i)> 0} A_{\boldsymbol{a}} \tag{8.7}$$

for $i = 1, 2, \ldots, m$. We put

$$E_i = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^n,\ (\boldsymbol{a},\boldsymbol{b}_i)\geq 0} (A_{(0)})_{\boldsymbol{a}} \quad \text{and} \quad U_i = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^n,\ (\boldsymbol{a},\boldsymbol{b}_i)> 0} (A_{(0)})_{\boldsymbol{a}}.$$

Since $A \subset E_i$ and $P_i = A \cap U_i$, we have $A_{P_i} \subset (E_i)_{U_i}$. Since A_{P_i} is a discrete valuation ring, we have $A_{P_i} = (E_i)_{U_i}$. In particular, we have

$$A_{P_i} \supset E_i$$

for each i.

Choose $\boldsymbol{a} \in \mathrm{int}(\sigma) \cap \mathbb{Q}^n$. Take a homogeneous generating system $\{y_1, y_2, \ldots, y_t\}$ of the ideal $I_{\boldsymbol{a}}(A_\sigma)$ with degree in $\mathbb{R}_{>0}\boldsymbol{a}$. Consider the ideal transform $D_{\underline{y}}(A)$. By definition, it is a $\mathbb{Z}^n$-graded ring. Let $H_1(A)$ be the set of height one prime ideals of A . Since A is Krull, we have

$$A = \bigcap_{P\in H_1(A)} A_P$$

and

$$D_{\underline{y}}(A) = \bigcap_{P\in H_1(A),\ P\not\supset(\underline{y})A} A_P. \tag{8.8}$$

Then we obtain

$$A = D_{\underline{y}}(A) \cap \left(\bigcap_{i=1}^{m} A_{P_i} \right) = D_{\underline{y}}(A) \cap \left(\bigcap_{i=1}^{m} E_i \right).$$

Here we define τ to be

$$\tau = \bigcap_{i=1}^{m} \{ \boldsymbol{a} \in \mathbb{R}^n \mid (\boldsymbol{a}, \boldsymbol{b}_i) \geq 0 \}. \tag{8.9}$$

Then

$$A = D_{\underline{y}}(A) \cap (A_{(0)})_\tau = D_{\underline{y}}(A)_\tau.$$

Since

$$\sigma \subset C(A) \subset \tau,$$

we know $D_{\underline{y}}(A)_\sigma = A_\sigma$, and it is Noetherian by (I) (b). By Proposition 2.4 (3), σ is a chamber of $D_{\underline{y}}(A)$. By (8.8), there is no height one prime ideal of $D_{\underline{y}}(A)$ containing $J_{\boldsymbol{a}}(D_{\underline{y}}(A))$ (cf. Theorem12.3 in [14]). Therefore $D_{\underline{y}}(A)$ satisfies the condition (I) in Theorem 8.6. Then, by Theorem 8.6, there exist a normal projective variety X over $A_{\boldsymbol{0}}$ and $\mathbb{Q}$-divisors D_1, D_2, $\ldots$, D_n satisfying conditions (a), (b) in (III) in Theorem 8.6, and

$$D_{\underline{y}}(A) = R(X; D_1, D_2, \ldots, D_n).$$

Therefore we have

$$A = R(X; D_1, D_2, \ldots, D_n)_\tau.$$

By Remark 8.7, the vector $(c_1, c_2, \ldots, c_n)$ is in $\operatorname{int}(\sigma)$. Therefore $(c_1, c_2, \ldots, c_n)$ is in $C(A)$.

We have completed the proof of Theorem 8.9. □

Example 8.11. Let $A = k[x, y, z]$ be a $\mathbb{Z}^2$-graded polynomial ring with $\deg(x) = (1, 0)$, $\deg(y) = (1, 1)$ and $\deg(z) = (0, 1)$. Then A does not satisfy the condition (I) in Theorem 8.9.

Remark 8.12. Let A be a $\mathbb{Z}^n$-graded Noetherian normal domain such that $A_{\boldsymbol{0}}$ is a field and $(A_{(0)})_{\boldsymbol{e}_i} \neq 0$ for $i = 1, 2, \ldots, n$. Even if all the ray ideals of chambers are of height less than 2, A is isomorphic to a section ring $R(X; D_1, D_2, \ldots, D_n)$ as follows.

By the map

$$\mathbb{Z}^n \simeq \mathbb{Z}^n \times \{0\} \hookrightarrow \mathbb{Z}^{n+1},$$

we think that A is a $\mathbb{Z}^{n+1}$-graded ring. Let $\boldsymbol{e}_i$ be the ith unit vector in $\mathbb{Z}^n$. Consider the polynomial ring

$$B = A[x_1, x_2, \ldots, x_{n+1}, y_1, y_2, \ldots, y_{n+1}]$$

with $\deg(x_i) = \deg(y_i) = (\boldsymbol{e}_i, 1)$ for $i = 1, 2, \ldots, n$ and $\deg(x_{n+1}) = \deg(y_{n+1}) = (-\boldsymbol{e}_1 - \boldsymbol{e}_2 - \cdots - \boldsymbol{e}_n, 1)$. Let $\boldsymbol{a}$ be a point in $\mathbb{Q}^{n+1}$ sufficiently near $(0, \ldots, 0, 1)$. Then $J_{\boldsymbol{a}}(B)$ contains $x_1 x_2 \cdots x_{n+1}$ and $y_1 y_2 \cdots y_{n+1}$, and therefore the height of $J_{\boldsymbol{a}}(B)$ is at least 2. Then there exists a chamber σ of B such that B and σ satisfy the condition (I) in Theorem 8.6. Then there exist a normal projective variety X of $\dim X > 0$

and $\mathbb{Q}$-divisors D_1, D_2, ..., D_{n+1} such that B is equal to $R(X; D_1, D_2, \ldots, D_{n+1})$. Therefore A coincides with $R(X; D_1, D_2, \ldots, D_n)$. In this case, X does not inherit properties of the ring A as in the following example.

Let $A = k[x]$ be a graded polynomial ring over an algebraically closed field k with $\deg(x) = 1$. Let X be a blow-up of a smooth projective variety over k at a closed point. Let E be the exceptional divisor of X. Then we have $A = R(X; E)$.

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