On the limit of Frobenius in the Grothendieck group

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Dedicated to Professor Ngô Việt Trung for his 60th birthday.

Abstract

Considering the Grothendieck group modulo numerical equivalence, we obtain the finitely generated lattice $\overline{G_0(R)}$ for a Noetherian local ring R. Let $C_{CM}(R)$ be the cone in $\overline{G_0(R)}_{\mathbb{R}}$ spanned by cycles of maximal Cohen-Macaulay R-modules. We shall define the fundamental class $\overline{\mu_R}$ of R in $\overline{G_0(R)}_{\mathbb{R}}$, which is the limit of the Frobenius direct images (divided by their rank) $[{}^eR]/p^{de}$ in the case $ch(R) = p > 0$. The homological conjectures are deeply related to the problems whether $\overline{\mu_R}$ is in the Cohen-Macaulay cone $C_{CM}(R)$ or the strictly nef cone $SN(R)$ defined below. In this paper, we shall prove that $\overline{\mu_R}$ is in $C_{CM}(R)$ in the case where R is FFRT or F-rational.

1 Introduction

We shall define the Cohen-Macaulay cone $C_{CM}(R)$, the strictly nef cone $SN(R)$, and the fundamental class $\overline{\mu_R}$ for a Noetherian local domain R. They satisfy

$$\begin{array}{ccccc} \overline{G_0(R)}_{\mathbb{R}} & \supset & SN(R) & \supset & C_{CM}(R) - \{0\} \\ \cup & & & & \\ \overline{G_0(R)}_{\mathbb{Q}} & \ni & \overline{\mu_R} & & \end{array}$$

where $G_0(R)$ is the Grothendick group of finitely generated R-modules, $\overline{G_0(R)}$ is the Grothendick group modulo numerical equivalence, and $\overline{G_0(R)}_K = \overline{G_0(R)} \otimes_{\mathbb{Z}} K$. By [8], $\overline{G_0(R)}$ is a finitely generated free $\mathbb{Z}$-module. We define $C_{CM}(R)$ to be the cone in $\overline{G_0(R)}_{\mathbb{R}}$ spanned by cycles corresponding to maximal Cohen-Macaulay R-modules. If R is F-finite with residue class field algebraically closed, the fundamental class $\overline{\mu_R}$ is the limit of the Frobenius direct images (divided by their rank) $[{}^eR]/p^{de}$ as in Remark 8 (3). In the case where R contains a regular local ring S such that R is contained in a Galois extension B of S, then $\overline{\mu_R}$ is described in terms of B as in Remark 8 (2).

The fundamental class is deeply related to the homological conjectures as in Fact 10. The fundamental class $\overline{\mu_R}$ is in $C_{CM}(R)$ for any complete local domain R if and only if

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the small Mac conjecture is true. Roberts proved $\overline{\mu_R} \in SN(R)$ for any Noetherian local ring R of characteristic $p > 0$ in order to show the new intersection theorem in the mixed characteristic case [12]. In order to extend these results, we are mainly interested in the problem whether $\overline{\mu_R}$ is in such cones or not.

Problem 1 If R is an excellent Noetherian local domain, is $\overline{\mu_R}$ in $C_{CM}(R)$?

Problem 1 is affirmative if R is a complete intersection. However, even if R is a Gorenstein ring which contains a field, Problem 1 is an open question.

The following theorem is the main result in this paper. We define the terminologies later.

Theorem 2 *Assume that R is an F-finite Cohen-Macaulay local domain of characteristic $p > 0$ with residue class field algebraically closed.*

(1) If R is FFRT, then there exist a natural number n and a maximal Cohen-Macaulay R-module N such that $n\mu_R = [N]$ in $G_0(R)_{\mathbb{Q}}$. In particular, $\overline{\mu_R}$ is contained in $C_{CM}(R)$.

(2) If R is F-rational, then $\overline{\mu_R}$ is contained in $Int(C_{CM}(R))$.

In the case FFRT, we shall show that the cone generated by $[M_1]$, $\dots$, $[M_s]$ (in Definition 17) contains μ_R. In the case of F-rational, the key point in our proof is to use the dual F-signature defined by Sannai [14].

Finally we shall give a corollary (Corollary 22), which was first proved in [1].

2 Cohen-Macaulay cone

In this paper, let R be a d-dimensional Noetherian local domain such that one of the following conditions are satisfied:

(a) R is a homomorphic image of an excellent regular local ring containing $\mathbb{Q}$.

(b) R is essentially of finite type over a field, $\mathbb{Z}$ or a complete DVR.

If either (a) or (b) is satisfied, there exists a regular alteration of $\operatorname{Spec} R$ by de Jong's theorem [5].

We always assume that modules are finitely generated.

Let $G_0(R)$ be the Grothendieck group of finitely generated R-modules, that is,

$$G_0(R) = \frac{\bigoplus_{M\ :\ \text{f. g. } R\text{-module}} \mathbb{Z}[M]}{< [M] - [L] - [N] \mid 0 \to L \to M \to N \to 0 \text{ is exact} >},$$

where $[M]$ denotes the generator corresponding to an R-module M. Let $C(R)$ be the category of bounded complexes of finitely generated R-free modules such that every homology is of finite length. Let $C_d(R)$ be the subcategory of $C(R)$ consisting of complexes of length d with $H_0(\mathbb{F}.) \neq 0$. A complex $\mathbb{F}.$ in $C_d(R)$ is of the form

$$0 \to F_d \to F_{d-1} \to \cdots \to F_1 \to F_0 \to 0.$$

For example, the Koszul complex of a parameter ideal belongs to $C_d(R)$.

For $\mathbb{F}. \in C(R)$, we have a well-defined map

$$\chi_{\mathbb{F}.} : G_0(R) \longrightarrow \mathbb{Z}$$

by $\chi_{\mathbb{F}.}([M]) = \sum_i (-1)^i \ell_R(H_i(\mathbb{F}. \otimes_R M))$. We have the induced maps $\chi_{\mathbb{F}.} : G_0(R)_{\mathbb{Q}} \longrightarrow \mathbb{Q}$ and $\chi_{\mathbb{F}.} : G_0(R)_{\mathbb{R}} \longrightarrow \mathbb{R}$. We say that $\alpha \in G_0(R)$ ($\alpha \in G_0(R)_{\mathbb{Q}}$ or $\alpha \in G_0(R)_{\mathbb{R}}$) is numerically equivalent to 0 if $\chi_{\mathbb{F}.}(\alpha) = 0$ for any $\mathbb{F}. \in C(R)$. We define the Grothendieck group modulo numerical equivalence as follows:

$$\overline{G_0(R)} = G_0(R)/\{\alpha \in G_0(R) \mid \chi_{\mathbb{F}.}(\alpha) = 0 \text{ for any } \mathbb{F}. \in C(R)\}.$$

Then, by Theorem 3.1 and Remark 3.5 in [8], we know that $\overline{G_0(R)}$ is a non-zero finitely generated $\mathbb{Z}$-free module.[1]

Example 3 (1) If $d \leq 2$, then $\overline{G_0(R)} = \mathbb{Z}$ (Proposition 3.7 in [8]). If $d \geq 3$, there exists an example of d-dimensional Noetherian local domain R such that $\operatorname{rank} \overline{G_0(R)} = m$ for any positive integer m as in (2) (b) (i) below.

(2) Let X be a smooth projective variety with embedding $X \hookrightarrow \mathbb{P}^n$. Let R (resp. D) be the affine cone (resp. the very ample divisor) of this embedding. Let $A_*(R)$ be the Chow group of R. By [8], we can define numerical equivalence also on $A_*(R)$, that is compatible with the Riemann-Roch theory as below. Let $CH^{\cdot}(X)$ (resp. $CH^{\cdot}_{num}(X)$) be the Chow ring (resp. Chow ring modulo numerical equivalence) of X. It is well-known that $CH^{\cdot}_{num}(X)_{\mathbb{Q}}$ is a finite dimensional $\mathbb{Q}$-vector space. Then, we have the following commutative diagram:

$$\begin{array}{ccccc} G_0(R)_{\mathbb{Q}} & \underset{\sim}{\xrightarrow{\tau_R}} & A_*(R)_{\mathbb{Q}} & \xleftarrow{\sim} & CH^{\cdot}(X)_{\mathbb{Q}}/D \cdot CH^{\cdot}(X)_{\mathbb{Q}} \\ \downarrow & & \downarrow & & \downarrow \\ \overline{G_0(R)}_{\mathbb{Q}} & \underset{\sim}{\xrightarrow{\overline{\tau_R}}} & \overline{A_*(R)}_{\mathbb{Q}} & \xleftarrow{\phi} & CH^{\cdot}_{num}(X)_{\mathbb{Q}}/D \cdot CH^{\cdot}_{num}(X)_{\mathbb{Q}} \end{array}$$

(a) By the commutativity of this diagram, ϕ is a surjection. Therefore, we have

$$\operatorname{rank} \overline{G_0(R)} \leq \dim_{\mathbb{Q}} CH^{\cdot}_{num}(X)_{\mathbb{Q}}/D \cdot CH^{\cdot}_{num}(X)_{\mathbb{Q}}. \qquad (1)$$

(b) If $CH^{\cdot}(X)_{\mathbb{Q}} \simeq CH^{\cdot}_{num}(X)_{\mathbb{Q}}$, then we can prove that ϕ is an isomorphism ([8], [13]). In this case, the equality holds in (1). Using it, we can show the following:

(i) If X is a blow-up at n points of $\mathbb{P}^k$ ($k \geq 2$), then $\operatorname{rank} \overline{G_0(R)} = n + 1$.

(ii) If $X = \mathbb{P}^m \times \mathbb{P}^n$, then $\operatorname{rank} \overline{G_0(R)} = min\{m, n\}$.

(c) There exists an example such that ϕ is not an isomorphism [13].

Further, Roberts and Srinivas [13] proved the following: Assume that the standard conjecture and Bloch-Beilinson conjecture are true. Then ϕ is an isomorphism if the defining ideal of R is generated by polynomials with coefficients in the algebraic closure of the prime field.

[1] We need the existence of a regular alteration in the proof of this result.

Consider the groups $\overline{G_0(R)} \subset \overline{G_0(R)}_{\mathbb{Q}} \subset \overline{G_0(R)}_{\mathbb{R}}$. We shall define some cones in $\overline{G_0(R)}_{\mathbb{R}}$.

Definition 4 Let $C_{CM}(R)$ be the cone (in $\overline{G_0(R)}_{\mathbb{R}}$) spanned by all maximal Cohen-Macaulay R-modules.

$$C_{CM}(R) = \sum_{M:MCM} \mathbb{R}_{\geq 0}[M] \subset \overline{G_0(R)}_{\mathbb{R}}.$$

We call it the *Cohen-Macaulay cone* of R. Thinking a free basis of $\overline{G_0(R)}$ as an orthonormal basis of $\overline{G_0(R)}_{\mathbb{R}}$, we think $\overline{G_0(R)}_{\mathbb{R}}$ as a metric space. Let $C_{CM}(R)^-$ be the closure of $C_{CM}(R)$ with respect to this topology on $\overline{G_0(R)}_{\mathbb{R}}$.

We define the *strictly nef cone* by

$$SN(R) = \{\alpha \mid \ \chi_{\mathbb{F}.}(\alpha) > 0 \text{ for any } \mathbb{F}. \in C_d(R)\}.$$

By the depth sensitivity, $\chi_{\mathbb{F}.}([M]) = \ell_R(H_0(\mathbb{F}. \otimes M)) > 0$ for any maximal Cohen-Macaulay module M $(\neq 0)$ and $\mathbb{F}. \in C_d(R)$. Therefore,

$$SN(R) \supset C_{CM}(R) - \{0\}.$$

Remark 5 Assume that R is a Cohen-Macaulay local domain. Let M be a torsion R-module. Taking sufficiently high syzygies of M, we know

$$\pm[M] + n[R] \in C_{CM}(R) \quad \text{for } n \gg 0.$$

Therefore, we have $\dim C_{CM}(R) = \operatorname{rank} \overline{G_0(R)}$ and

$$C_{CM}(R)^- \supset C_{CM}(R) \supset Int(C_{CM}(R)^-) = Int(C_{CM}(R)) \ni [R],$$

where $Int(\ \ )$ denotes the interior.

Example 6 The following examples are given in [2]. Assume that k is an algebraically closed field of characteristic zero.

(1) Put $R = k[x,y,z,w]_{(x,y,z,w)}/(xy - f_1 f_2 \cdots f_t)$. Here, we assume that f_1, f_2, ..., f_t are pairwise coprime linear forms in $k[z,w]$ with $t \geq 2$. In this case, we have $\operatorname{rank} \overline{G_0(R)} = t$. We know (see [2]) that the Cohen-Macaulay cone is minimally spanned by the following $2^t - 2$ maximal Cohen-Macaulay modules of rank one:

$$\{(x, f_{i_1} f_{i_2} \cdots f_{i_s}) \mid 1 \leq s < t, \quad 1 \leq i_1 < i_2 < \cdots < i_s \leq t\}$$

Here, remark that this ring is of finite representation type if and only if $t \leq 3$.

(2) The Cohen-Macaulay cone of $k[x_1, x_2, \ldots, x_6]_{(x_1,x_2,\ldots,x_6)}/(x_1x_2 + x_3x_4 + x_5x_6)$ is not spanned by maximal Cohen-Macaulay modules of rank one. It is of finite representation type since it has a simple singularity.

3 Fundamental class

Definition 7 Let R be a d-dimensional Noetherian local domain. We put

$$\mu_R = {\tau_R}^{-1}([\operatorname{Spec} R]) \in G_0(R)_{\mathbb{Q}},$$

where $\tau_R : G_0(R)_{\mathbb{Q}} \xrightarrow{\sim} A_*(R)_{\mathbb{Q}}$ is the singular Riemann-Roch map, and $[\operatorname{Spec} R]$ denotes the cycle in $A_*(R)$ corresponding to the scheme $\operatorname{Spec} R$ itself.

$$\begin{array}{ccc} G_0(R)_{\mathbb{Q}} & \longrightarrow & \overline{G_0(R)}_{\mathbb{Q}} \\ \mu_R & \mapsto & \overline{\mu_R} \end{array}$$

We call the image of μ_R in $\overline{G_0(R)}_{\mathbb{Q}}$ the *fundamental class* of R, and denote it by $\overline{\mu_R}$.

Remark that $\overline{\mu_R} \neq 0$ since $\operatorname{rank}_R \mu_R = 1$.

Put $R = T/I$, where T is a regular local ring. The map τ_R is defined using not only R but also T. Therefore, μ_R may depend on the choice of T.[2] However, we can prove that $\overline{\mu_R}$ is independent of T (Theorem 5.1 in [8]).

We shall explain the reason why we call $\overline{\mu_R}$ the fundamental class of R.

Remark 8 (1) If X $(= \operatorname{Spec} R)$ is a d-dimensional affine variety over $\mathbb{C}$, we have the cycle map cl such that $cl([\operatorname{Spec} R])$ coincides with the fundamental class μ_X in $H_{2d}(X, \mathbb{Q})$ in the usual sense, where $H_*(X, \mathbb{Q})$ is the Borel-Moore homology. Here μ_X is the generator of $H_{2d}(X, \mathbb{Q}) \simeq \mathbb{Z}$.

$$\begin{array}{ccccc} G_0(R)_{\mathbb{Q}} & \xrightarrow{\tau_R} & A_*(R)_{\mathbb{Q}} & \xrightarrow{cl} & H_*(X, \mathbb{Q}) \\ \mu_R & \mapsto & [\operatorname{Spec} R] & \mapsto & \mu_X \end{array}$$

The map cl induces the map $\overline{A_d(R)}_{\mathbb{Q}} \longrightarrow H_{2d}(X, \mathbb{Q})$ such that the fundamental class μ_X is the image of $\overline{\tau_R}(\overline{\mu_R})$. Hence, we call $\overline{\mu_R}$ the fundamental class of R.

(2) Let R have a subring S such that S is a regular local ring and R is a localization of a finite extension of S. Let L be a finite-dimensional normal extension of $Q(S)$ containing $Q(R)$. Let B be the integral closure of R in L. Then, we have

$$\mu_R = \tfrac{1}{\operatorname{rank}_R B}[B] \text{ in } G_0(R)_{\mathbb{Q}}.$$

In particular, $\overline{\mu_R} = \frac{[B]}{\operatorname{rank}_R B}$ in $\overline{G_0(R)}_{\mathbb{Q}}$ (see the proof of Theorem 1.1 in [6]).

(3) Assume that R is of characteristic $p > 0$ and F-finite. Assume that the residue class field is algebraically closed. By the singular Riemann-Roch theorem, we have

$$\overline{\mu_R} = \lim_{e \to \infty} \frac{[{}^e R]}{p^{de}} \text{ in } \overline{G_0(R)}_{\mathbb{R}},$$

where ${}^e R$ is the e-th Frobenius direct image (see Definition 13, 14 below). It immediately follows from the equations (7) and (9) below.

[2]There is no example that the map τ_R actually depend on the choice of T. For some excellent rings, it had been proved that τ_R is independent of the choice of T (Proposition 1.2 in [7]).

Example 9 (1) If R is a complete intersection, then μ_R is equal to $[R]$ in $G_0(R)_{\mathbb{Q}}$, therefore $\overline{\mu_R} = [R]$ in $\overline{G_0(R)}_{\mathbb{Q}}$. There exists a Gorenstein ring such that $\overline{\mu_R} \neq [R]$. However there exist many examples of rings satisfying $\overline{\mu_R} = [R]$ ([7]). Roberts ([10], [11]) proved the vanishing property of intersection multiplicities for rings satisfying $\overline{\mu_R} = [R]$.

(2) Let R be a normal domain. Then, we have

$$\begin{array}{ccl} G_0(R)_{\mathbb{Q}} & \xrightarrow{\tau_R} & A_*(R)_{\mathbb{Q}} = A_d(R)_{\mathbb{Q}} \oplus A_{d-1}(R)_{\mathbb{Q}} \oplus \cdots \\ [R] & \mapsto & [\mathrm{Spec}\, R] - \frac{K_R}{2} + \cdots \\ [\omega_R] & \mapsto & [\mathrm{Spec}\, R] + \frac{K_R}{2} + \cdots , \end{array}$$

where K_R is the Weil divisor corresponding to the canonical module ω_R. If $\tau_R^{-1}(K_R) \neq 0$ in $\overline{G_0(R)}_{\mathbb{Q}}$, then $[R] \neq \overline{\mu_R}$. Although the equality

$$\overline{\mu_R} = \frac{1}{2}([R] + [\omega_R])$$

is sometimes satisfied, it is not true in general.

(3) Let $R = k[x_{ij}]/I_2(x_{ij})$, where (x_{ij}) is the generic $(m+1) \times (n+1)$-matrix, and k is a field. Suppose $0 < m \leq n$. Then, we have

$$\begin{array}{rcl} G_0(R)_{\mathbb{Q}} \simeq \overline{G_0(R)}_{\mathbb{Q}} & \simeq & A_*(R)_{\mathbb{Q}} \simeq \mathbb{Q}[a]/(a^{m+1}) \\ [R] & \mapsto & \left(\frac{a}{1-e^{-a}}\right)^m \left(\frac{-a}{1-e^{a}}\right)^n \\ & & = 1 + \frac{1}{2}(m-n)a + \frac{1}{24}(\cdots)a^2 + \cdots \\ [\omega_R] & \mapsto & \left(\frac{-a}{1-e^{a}}\right)^m \left(\frac{a}{1-e^{-a}}\right)^n \\ \overline{\mu_R} & \mapsto & 1 \\ \tau_R^{-1}(K_R) & \mapsto & (n-m)a \end{array}$$

(4) By Remark 2.9 in [1], if $\overline{\mu_R} \in C_{CM}(R)$, then there exists a maximal Cohen-Macaulay R-module N such that $[N] = \mathrm{rank}_R N \cdot \overline{\mu_R}$ in $\overline{G_0(R)}_{\mathbb{Q}}$.

Here, we shall explain the connection between the fundamental class $\overline{\mu_R}$ and the homological conjectures.

Fact 10 (1) The small Mac conjecture is true if and only if $\overline{\mu_R} \in C_{CM}(R)$ for any complete local domain R (Theorem 1.3 in [6]). We give an outline of the proof here.

"If" part is trivial. We shall show "only if" part. Suppose that S is a regular local ring such that R is a finite extension over S. Let L be a finite-dimensional normal extension of $Q(S)$ containing $Q(R)$. Let B be the integral closure of R in L. Then, B is finite over R, and B is a complete local domain. Here, *assume that there exists an maximal Cohen-Macaulay B-module M.* Put $\mathrm{Aut}_{Q(S)}(L) = \{g_1, \ldots, g_t\}$ and $N = \oplus_i ({}_{g_i}M)$, where ${}_{g_i}M$ denotes M with R-module structure given by $a \times m = g_i(a)m$. Then N is a maximal Cohen-Macaulay R-module such that $[N] = \mathrm{rank}_R N \cdot \mu_R$ in $G_0(R)_{\mathbb{Q}}$. Therefore, $\overline{\mu_R} = \frac{[N]}{\mathrm{rank}_R N} \in C_{CM}(R)$.

Even if R is an equi-characteristic Gorenstein ring, it is not known whether $\overline{\mu_R}$ is in $C_{CM}(R)$ or not. If R is a complete intersection, then $\overline{\mu_R} = [R] \in C_{CM}(R)$ as in (1) in Example 9.

(2) If $\overline{\mu_R} = [R]$ in $\overline{G_0(R)}_{\mathbb{Q}}$, then the vanishing property of intersection multiplicities holds (Roberts [10], [11]).

(3) Roberts [12] proved $\overline{\mu_R} \in SN(R)$ if $ch(R) = p > 0$. Using it, he proved the new intersection theorem in the mixed characteristic case.

(4) If R contains a field, then $\overline{\mu_R} \in SN(R)$ (Kurano-Roberts [9]). *Even if R is a Gorenstein ring (of mixed characteristic), we do not know whether* $\overline{\mu_R} \in SN(R)$ *or not.*

(5) If $\overline{\mu_R} \in SN(R)$ for any R, then Serre's positivity conjecture is true in the case where one of two modules is (not necessary maximal) Cohen-Macaulay.

It is well-known that Serre's positivity conjecture follows from the small Mac conjecture.

Remark 11 (1) If R is Cohen-Macaulay of characteristic $p > 0$, then ${}^eR$ is a maximal Cohen-Macaulay module. Since $\overline{\mu_R}$ is the limit of $[{}^eR]/p^{de}$ in $\overline{G_0(R)}_{\mathbb{R}}$ as in Remark 8 (3), $\overline{\mu_R}$ is contained in $C_{CM}(R)^-$. If we know that $C_{CM}(R)$ is a closed set of $\overline{G_0(R)}_{\mathbb{R}}$, we have $\overline{\mu_R} \in C_{CM}(R)^- = C_{CM}(R)$. If the cone $C_{CM}(R)$ is finitely generated, then it is a closed subset. We do not know any example that the cone $C_{CM}(R)$ is not finitely generated.

In the case where R is not of characteristic $p > 0$, *we do not know whether* $\overline{\mu_R}$ *is contained in* $C_{CM}(R)^-$ *even if R is a Gorenstein ring.*

(2) As we have already seen in Remark 5, if R is Cohen-Macaulay, then $[R] \in Int(C_{CM}(R)) \subset C_{CM}(R)$.

There is an example of non-Cohen-Macaulay ring R containing a field such that $[R] \notin SN(R)$.[3] On the other hand, it is expected that $\overline{\mu_R} \in SN(R)$ for any R (Fact 10 (4)). Therefore, for a non-Cohen-Macaulay local ring R, $\overline{\mu_R}$ behaves better than $[R]$ in a sense.

4 Main theorem

In Fact 10, we saw that the fundamental class $\overline{\mu_R}$ is deeply related to the homological conjectures. We propose the following question.

Question 12 Assume that R is a "good" Cohen-Macaulay local domain (for example, equi-characteristic, Gorenstein, etc). Is $\overline{\mu_R}$ in $C_{CM}(R)$?

If R is a Cohen-Macaulay local domain such that the rank of $\overline{G_0(R)}$ is one, then $[R] = \overline{\mu_R} \in C_{CM}(R)$, therefore Question 12 is true in this case. There are a lot of such examples (for instance, invariant subrings with respect to finite group actions, etc.).

[3] It was conjectured above 50 years ago that $[R]$ was in $SN(R)$ for any local ring R. Essentially, the famous counter example due to Dutta-Hochster-MacLaughlin [3] gives an example $[R] \notin SN(R)$.

Definition 13 Let p be a prime number and R be a Noetherian ring of characteristic p. Let $e > 0$ be an integer and

$$F^e : R \longrightarrow R$$

be the e-th Frobenius map. We denote by ${}^eR$ the R-module R with R-module structure given by $r \times x = F^e(r)x$. It is called the e-th *Frobenius direct image*.

Definition 14 Let p be a prime number and R be a Noetherian ring of characteristic p. We say that R is *F-finite* if the Frobenius map $F : R \longrightarrow R$ is finite.

Remark 15 Let R be a d-dimensional F-finite Noetherian local ring. We have the following commutative diagram (2) where the horizontal map τ_R is the singular Riemann-Roch map and the vertical maps are induced by F^e:

$$\begin{array}{ccc} G_0(R)_{\mathbb{Q}} & \xrightarrow{\tau_R} & A_*(R)_{\mathbb{Q}} \\ {\scriptstyle F^e_*}\downarrow & & \downarrow{\scriptstyle F^e_*} \\ G_0(R)_{\mathbb{Q}} & \xrightarrow{\tau_R} & A_*(R)_{\mathbb{Q}} \end{array} \tag{2}$$

By diagram (2), we have

$$\tau_R([{}^eR]) = F^e_*\big(\tau_R([R])\big). \tag{3}$$

We set

$$\tau_R([R]) = \tau_R([R])_d + \tau_R([R])_{d-1} + \cdots + \tau_R([R])_0$$

where $\tau_R([R])_i \in A_i(R)_{\mathbb{Q}}$ for $i = 0, \ldots, d$. Then, by the top term property [4], we know

$$\tau_R([R])_d = [\operatorname{Spec} R] \in A_*(R)_{\mathbb{Q}}. \tag{4}$$

Assume that $(R, \mathfrak{m})$ is a d-dimensional F-finite Noetherian local domain with residue class field $R/\mathfrak{m}$ algebraically closed. For $\alpha \in A_i(R)_{\mathbb{Q}}$ we have

$$F_*(\alpha) = p^i \alpha \tag{5}$$

by Lemma 16 below and the definition of F_* [4]. Therefore

$$F^e_*\big(\tau_R([R])\big) = p^{de}[\operatorname{Spec} R] + \sum_{0 \le i \le d-1} p^{ie} \tau_R([R])_i. \tag{6}$$

Hence, by the equations (3), (6), we have

$$\tau_R([{}^eR])_i = p^{ie} \tau_R([R])_i.$$

Therefore,

$$[{}^eR] = p^{de} \tau_R^{-1}\big([\operatorname{Spec} R]\big) + \sum_{0 \le i \le d-1} p^{ie} \tau_R^{-1}(\tau_R([R])_i) \tag{7}$$

in $G_0(R)_{\mathbb{Q}}$.

The following lemma is well-known. We omit a proof.

Lemma 16 *Assume that R is an F-finite Noetherian local domain of characteristic p with residue class field algebraically closed. Then, for any $e > 0$, we have*

$$\mathrm{rank}_R\, {}^eR = p^{(\dim R)e}.$$

Definition 17 Let R be a Cohen-Macaulay ring of characteristic $p > 0$. We say that R is *FFRT* (*of finite F-representation type*) if there exist finitely many indecomposable maximal Cohen-Macaulay R-modules $M_1, \ldots, M_s$ such that there exist nonnegative integers $a_{e1}, \ldots, a_{es}$ with

$${}^eR \simeq M_1^{a_{e1}} \oplus \cdots \oplus M_s^{a_{es}}$$

for each $e > 0$.

Definition 18 Let p be a prime number and R be a Noetherian ring of characteristic p. Let R° be the set of elements of R that are not contained in any minimal prime ideals of R. Let I be an ideal of R. Given a natural number e, set $q = p^e$. The ideal generated by the q-th powers of elements of I is called the q-th Frobenius power of I, denoted by $I^{[q]}$. We define the *tight closure* I^* of I as follows:

$$I^* = \{x \in R \mid \text{there exists } c \in R^\circ \text{ such that } cx^q \in I^{[q]} \text{ for } q \gg 0\}.$$

We say that I is *tightly closed* if $I = I^*$.

Definition 19 Let R be a Noetherian local ring of characteristic $p > 0$. We say that R is *F-rational* if every parameter ideal is tightly closed.

Now, we start to prove Theorem 2 (1). Since R is FFRT, there exist finitely many indecomposable maximal Cohen-Macaulay R-modules $M_1, \ldots, M_s$ such that there exist nonnegative integers $a_{e1}, \ldots, a_{es}$ with

$${}^eR \simeq M_1^{a_{e1}} \oplus \cdots \oplus M_s^{a_{es}} \tag{8}$$

for each $e > 0$. Let U be the $\mathbb{Q}$-vector subspace of $G_0(R)_\mathbb{Q}$ spanned by

$$\{[M_1], \ldots, [M_s]\} \cup \{\tau_R^{-1}\big(\tau_R([R])_j\big) \mid 0 \le j \le d\}.$$

Here, recall that $\mu_R = \tau_R^{-1}(\tau_R([R])_d) \in U$ by the top term property (4). Although we can show that U is spanned by $\{[M_1], \ldots, [M_s]\}$, we do not need it in this proof. Thinking a basis of U as an orthonormal basis of $U_\mathbb{R}$, we think $U_\mathbb{R}$ as a metric space. Set $C = \sum_{i=1}^{s} \mathbb{R}_{\ge 0}[M_i] \subset U_\mathbb{R}$. Then C is a closed subset of $U_\mathbb{R}$. We shall show $\mu_R \in C$.

Since the residue field is algebraically closed, $\mathrm{rank}_R\, {}^eR = p^{de}$ for any $e > 0$ by Lemma 16. Since

$$[{}^eR] = a_{e1}[M_1] + \cdots + a_{es}[M_s]$$

by (8), we have

$$\frac{1}{p^{de}}[{}^eR] \in C$$

for any $e > 0$. By the equation (7),

$$\frac{1}{p^{de}}[{}^eR] = \sum_{0\leq i\leq d} \frac{1}{p^{ie}} \tau_R^{-1}\big(\tau_R([R])_{d-i}\big). \tag{9}$$

By the definition of U, every term of the right-hand side is in $U_{\mathbb{R}}$. Hence we have

$$\lim_{e\to\infty} \frac{1}{p^{de}}[{}^eR] = \tau_R^{-1}\big(\tau_R([R])_d\big) = \tau_R^{-1}\big([\operatorname{Spec} R]\big) = \mu_R \ \text{ in } U_{\mathbb{R}}.$$

Since C is a closed set of $U_{\mathbb{R}}$, we have $\mu_R \in C$. By the same argument as in Example 9 (4), there exist a natural number n and a maximal Cohen-Macaulay R-module N such that $n\mu_R = [N]$ in $G_0(R)_{\mathbb{Q}}$.

Next, we start to prove Theorem 2 (2).

First, we shall prove that $[\omega_R] \in Int(C_{CM}(R))$ if R is Cohen-Macaulay. We have a homomorphism $\xi : G_0(R)_{\mathbb{R}} \to G_0(R)_{\mathbb{R}}$ given by $\xi([M]) = \sum_i (-1)^i [\operatorname{Ext}^i_R(M, \omega_R)]$. For a maximal Cohen-Macaulay module M, $\operatorname{Ext}^i_R(M, \omega_R) = 0$ for $i > 0$ and $\operatorname{Hom}_R(\operatorname{Hom}_R(M, \omega_R), \omega_R) \simeq M$. Therefore, ξ^2 is equal to the identity, and ξ is an isomorphism. By the definition of τ_R, we have a commutative diagram[4]

$$\begin{array}{ccc} G_0(R)_{\mathbb{R}} & \stackrel{\tau_R\otimes 1}{\longrightarrow} & A_*(R)_{\mathbb{R}} \\ \xi \downarrow & & \phi \downarrow \\ G_0(R)_{\mathbb{R}} & \stackrel{\tau_R\otimes 1}{\longrightarrow} & A_*(R)_{\mathbb{R}} \end{array}$$

where $\phi : A_*(R)_{\mathbb{R}} \to A_*(R)_{\mathbb{R}}$ is the map given by

$$\phi(q_d + q_{d-1} + \cdots + q_i + \cdots + q_0) = q_d - q_{d-1} + \cdots + (-1)^{d-i} q_i + \cdots + (-1)^d q_0 \tag{10}$$

for $q_i \in A_i(R)_{\mathbb{R}}$. Since the numerical equivalence is graded in $A_*(R)_{\mathbb{Q}}$ as in Proposition 2.4 in [8], ϕ preserves the numerical equivalence. Therefore we have the induced map

$$\overline{\xi} : \overline{G_0(R)}_{\mathbb{R}} \to \overline{G_0(R)}_{\mathbb{R}}.$$

Remark that $\overline{\xi}$ is an isomorphism of $\mathbb{R}$-vector spaces since $\overline{\xi}^2$ is the identity. The map $\overline{\xi}$ satisfies $\overline{\xi}([R]) = [\omega_R]$ and $\overline{\xi}(C_{CM}(R)) = C_{CM}(R)$. Since $[R] \in Int(C_{CM}(R))$ by Remark 5, we obtain $[\omega_R] \in Int(C_{CM}(R))$.

Assume that M is a maximal Cohen-Macaulay module. For $e > 0$, consider the following exact sequence

$$0 \longrightarrow L_e \longrightarrow F^e_*(M) \longrightarrow M^{\oplus b_e} \longrightarrow 0$$

where $F^e_*(M)$ is the e-th Frobenius direct image of M. Take b_e as large as possible. Recall that L_e is a maximal Cohen-Macaulay module. Put $r = \operatorname{rank}_R M$.

[4]Put $R = T/I$, where T is a regular local ring. Then, $\xi([M]) = (-1)^{\operatorname{ht}(I)} \sum_i (-1)^i [\operatorname{Ext}^i_T(M, T)]$. Let $\mathbb{F}.$ be a T-free resolution of M. Then, by the definition of τ_R, we have $\tau_R([M]) = \operatorname{ch}(\mathbb{F}.) \cap [\operatorname{Spec} T]$, where $\operatorname{ch}(\mathbb{F}.)$ is the localized Chern character of $\mathbb{F}$. (§18 in [4]). By the local Riemann-Roch formula (Example 18.3.12 in [4]), $\tau_R(\xi([M])) = \operatorname{ch}(\mathbb{F}.^*[\operatorname{ht}(I)]) \cap [\operatorname{Spec} T]$. By Example 18.1.2, we obtain the equality (10).

Here we define the dual F-signature following Sannai [14] as follows:

$$s(M) = \limsup_{e\to\infty} \frac{b_e}{rp^{de}}$$

Then, taking a subsequence of $\{\frac{b_e}{rp^{de}}\}_e$, we may assume that $s(M) = \lim_{e\to\infty} \frac{b_e}{rp^{de}}$.

On the other hand, consider

$$\tau_R([M]) = \tau_R([M])_d + \tau_R([M])_{d-1} + \cdots + \tau_R([M])_0.$$

Here, we have $\tau_R([M])_d = r[\mathrm{Spec}\, R]$ since $[M] - r[R]$ is a sum of cycles of torsion modules. By (2) and (5),

$$\begin{aligned}\tau_R([F_*^e(M)]) &= F_*^e(\tau_R([M])_d + \tau_R([M])_{d-1} + \cdots + \tau_R([M])_0) \\ &= p^{de}\tau_R([M])_d + p^{(d-1)e}\tau_R([M])_{d-1} + \cdots + \tau_R([M])_0.\end{aligned}$$

Then, we have

$$\overline{\tau_R}(\lim_{e\to\infty} \frac{[F_*^e(M)]}{rp^{de}}) = \frac{\tau_R([M])_d}{r} = [\mathrm{Spec}\, R] \ \text{ in } \overline{A_*(R)_{\mathbb{R}}}.$$

Thus,

$$\lim_{e\to\infty} \frac{[F_*^e(M)]}{rp^{de}} = \overline{\mu_R} \ \text{ in } \overline{G_0(R)_{\mathbb{R}}}.$$

Then, $\frac{[L_e]}{rp^{de}}$ converges to some element in $\overline{G_0(R)_{\mathbb{R}}}$, say $\alpha(M)$.

$$\begin{array}{ccccccc} \frac{[F_*^e(M)]}{rp^{de}} & = & \frac{b_e[M]}{rp^{de}} & + & \frac{[L_e]}{rp^{de}} & \in \overline{G_0(R)_{\mathbb{R}}} \\ \downarrow & & \downarrow & & \downarrow & {\scriptstyle (e\to\infty)} \\ \overline{\mu_R} & = & s(M)[M] & + & \alpha(M) & \end{array}$$

Since L_e is a maximal Cohen-Macaulay module, we know $\alpha(M) \in C_{CM}(R)^-$.

Here set $M = \omega_R$. Then

$$\overline{\mu_R} = s(\omega_R)[\omega_R] + \alpha(\omega_R) \in \overline{G_0(R)_{\mathbb{R}}}, \tag{11}$$

where

$$\alpha(\omega_R) \in C_{CM}(R)^- \tag{12}$$

and

$$[\omega_R] \in Int(C_{CM}(R)) = Int(C_{CM}(R)^-). \tag{13}$$

The most important point in this proof is the fact that

$$R \text{ is F-rational if and only if } s(\omega_R) > 0$$

due to Sannai [14].

Therefore, if R is F-rational, then $\overline{\mu_R} \in Int(C_{CM}(R))$ by (11), (12), (13) and Remark 5.

q.e.d.

Remark 20 If R is a toric ring (a normal semi-group ring over a field k), then we can prove $\overline{\mu_R} \in C_{CM}(R)$ as in the case FFRT without assuming that $ch(k)$ is positive.

Problem 21 (1) As in the above proof, if there exists a maximal Cohen-Macaulay module in $Int(C_{CM}(R))$ such that its generalized F-signature or its dual F-signature is positive, then $\overline{\mu_R}$ is in $Int(C_{CM}(R))$.

Without assuming that R is F-rational, do there exist such a maximal Cohen-Macaulay module?

(2) How do we make mod p reduction? (for example, the case of rational singularity)

(3) If R is Cohen-Macaulay, is $\overline{\mu_R}$ in $C_{CM}(R)^-$? If R is a Cohen-Macaulay ring containing a field of positive characteristic, then $\overline{\mu_R}$ in $C_{CM}(R)^-$ as in (1) in Remark 11.

(4) If R is of finite representation type, is $\overline{\mu_R}$ in $C_{CM}(R)$?

(5) Find more examples of $C_{CM}(R)$ and $SN(R)$.

In order to prove the following corollary, it is enough to construct a d-dimensional Cohen-Macaulay local domain A satisfying the following two conditions (Lemma 3.1 in [1]):

(1) $\overline{A_i(A)} \neq 0$ for $d/2 < i \leq d$, and

(2) $\overline{\mu_A}$ is contained in $Int(C_{CM}(A))$.

The ring R in Corollary 22 is the idealization of A and certain maximal Cohen-Macaulay A-module M. We can simplify the proof of Corollary 22 using Theorem 2. We know that $k[x_{ij}]_{(x_{ij})}/I_2(x_{ij})$ satisfies the conditions (1) and (2) above, where (x_{ij}) is the generic $n \times n$ or $n \times (n+1)$ matrix, and $I_2(x_{ij})$ stands for the ideal generated by 2-minors of (x_{ij}). In fact, by Example 3 (2) (b) and Example 9 (3), the condition (1) is satisfied. Since $k[x_{ij}]_{(x_{ij})}/I_2(x_{ij})$ is F-rational, the condition (2) is satisfied by Theorem 2 (2).

Corollary 22 ([1]) *Let d be a positive integer and p a prime number. Let $\epsilon_0, \epsilon_1, \ldots, \epsilon_d$ be integers such that*

$$\epsilon_i = \begin{cases} 1 & i = d, \\ -1,\ 0\ \text{or}\ 1 & d/2 < i < d, \\ 0 & i \leq d/2. \end{cases}$$

Then, there exists a d-dimensional Cohen-Macaulay local ring R of characteristic p, a maximal primary ideal I of R of finite projective dimension, and positive rational numbers α, β_{d-1}, $\beta_{d-2}, \ldots, \beta_0$ such that

$$\ell_R(R/I^{[p^n]}) = \epsilon_d \alpha p^{dn} + \sum_{i=0}^{d-1} \epsilon_i \beta_i p^{in}$$

for any $n > 0$.

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