THE CONE SPANNED BY MAXIMAL COHEN-MACAULAY MODULES AND AN APPLICATION

C-Y. JEAN CHAN AND KAZUHIKO KURANO

Abstract. The aim of this paper is to define the notion of the *Cohen-Macaulay cone* of a Noetherian local domain R and to present its applications to the theory of Hilbert-Kunz functions. It has been shown in [16] that, with a mild condition on R, the Grothendieck group $\overline{G_0(R)}$ of finitely generated R-modules modulo numerical equivalence is a finitely generated torsion-free abelian group. The Cohen-Macaulay cone of R is the cone in $\overline{G_0(R)}_{\mathbb{R}}$ spanned by cycles represented by maximal Cohen-Macaulay modules. We study basic properties on the Cohen-Macaulay cone in this paper. As an application, various examples of Hilbert-Kunz functions in the polynomial type will be produced. Precisely, for any given integers $\epsilon_i = 0, \pm 1$ $(d/2 < i < d)$, we shall construct a d-dimensional Cohen-Macaulay local ring R (of characteristic p) and a maximal primary ideal I of R such that the function $\ell_R(R/I^{[p^n]})$ is a polynomial in p^n of degree d whose coefficient of $(p^n)^i$ is the product of ϵ_i and a positive rational number for $d/2 < i < d$. The existence of such ring is proved by using Segre products to construct a Cohen-Macaulay ring such that the Chow group of the ring is of certain simplicity and that *test modules* exists for it.

1. Introduction

Let R be a Noetherian local domain. In this paper, we introduce the notion of the *Cohen-Macaulay cone* and *test modules* of R. As an application, for any given integers $\epsilon_i = 0, \pm 1$ $(d/2 < i < d)$, we shall construct a d-dimensional Cohen-Macaulay local ring R (of characteristic p) and a maximal primary ideal I of R such that the function $\ell_R(R/I^{[p^n]})$ is a polynomial in p^n of degree d whose coefficient of $(p^n)^i$ is the product of ϵ_i and a positive rational number for $d/2 < i < d$.

The main materials are divided into three parts. First part is an introduction to the theory of the Cohen-Macaulay cone and test modules. Then we prove

Date: July 14, 2015.

2010 Mathematics Subject Classifications: 13C14, 13D15, 13D40, 14C17, 14C40.

Key words and phrases. Numerical rational equivalence, Cohen-Macaulay cone, test module, Hilbert-Kunz function, Segre product.

The first author was partially supported by Early Career Investigator's Grant #C61368 of Central Michigan University. The second author was partially supported by KAKENHI (24540054).

the existence of a Cohen-Macaulay ring such that its Chow group of the ring is of certain simplicity and it has a test module. Using the ring just constructed, we shall further build a ring whose Hilbert-Kunz function satisfies the required conditions.

We now describe in more detail these new notions, and in the case of positive characteristic, their contribution to the theory of Hilbert-Kunz functions.

Let R be a Noetherian local domain of dimension d. The Grothendieck group $\overline{G_0(R)}$ of finitely generated R-modules modulo numerical equivalence is defined and studied in [16] where it is proven that under a mild condition $\overline{G_0(R)}$ is a finitely generated torsion-free abelian group (see also Theorem 2.1 in Section 2). Let $\rho(R)$ denote the rank of $\overline{G_0(R)}$. In this paper, we let R be a Cohen-Macaulay local domain and introduce a cone inside $\mathbb{R}^{\rho(R)} = \overline{G_0(R)} \otimes_{\mathbb{Z}} \mathbb{R}$ consisting of all nonnegative linear combinations of maximal Cohen-Macaulay modules. This is called the *Cohen-Macaulay cone* of R. A module M is a *test module* if M is a maximal Cohen-Macaulay module such that its Todd class consists of only the top term; *i.e.*, $\overline{\tau_R}([M]) \in \overline{\mathrm{A}_d(R)}_{\mathbb{Q}}$. In the case where R is an F-finite Cohen-Macaulay local ring (of positive characteristic p) with algebraically closed residue class field, M is a test module if and only if it is a maximal Cohen-Macaulay module such that

$$[{}_{F^e}M] = p^{de}[M] \ \text{ in } \overline{G_0(R)}_{\mathbb{Q}}$$

for some (any) $e > 0$, where $[{}_{F^e}M]$ denotes the R-module M whose R-module structure is given by the e-th power of the Frobenius map. If the small Cohen-Macaulay conjecture is affirmative, then any Noetherian local ring has a test module. However, the authors do not know whether test modules exist or not even if R is a Gorenstein ring. We refer the readers to [14] for test modules; however, the definition of test modules in this paper is slightly different from that in [14]. In this paper, we need test modules which contain the ring as a direct summand, that is, test modules which are in the interior of the Cohen-Macaulay cone of R. The ideas of these new notions are motivated by the studies of Hilbert-Kunz functions.

Assume additionally that R has a positive characteristic p with dimension d. Let I be a maximal primary ideal. The Hilbert-Kunz function with respect to I, named by Monsky [20], is defined as

$$\varphi_R(n) = \text{ length } (R/I^{[p^n]}R)$$

where $I^{[p^n]}$ is the Frobenius n-th power of I. Unlike the usual Hilbert function, the shape of the Hilbert-Kunz function varies from case to case. Monsky proved that $\varphi_M(n) = e_{HK}(I, M)\, p^{nd} + \mathcal{O}(p^{n(d-1)})$ for some positive constant $e_{HK}(I, M)$.

Its stability has been studied in Huneke-McDermott-Monsky [12], Fakhruddin-Trivedi [8], Brenner [1], Hochster-Yao [11], Chan-Kurano [3], etc.

Classically from Macaulay's theorem one knows what numerical functions are Hilbert functions (*c.f.* [2, Section 4.2]). Similarly for the Hilbert-Kunz function, it is natural to ask what functions are Hilbert-Kunz functions. But the latter is a much more subtle question since the shape of a Hilbert-Kunz function is unpredictable in general. In [17, Example 3.1(3)], the second author proved that if I is a maximal primary ideal of a local ring R that satisfies the following two conditions

- R is an F-finite Cohen-Macaulay local ring whose residue class field is algebraically closed, and
- I has finite projective dimension,

then the Hilbert-Kunz function of R with respect to I is a polynomial of p^n (see also [16] and [3, Lemma 3.4]). We refer the reader to MacDonnell [19] in the case where I is a homogeneous ideal. One aim of this paper is to prove the following theorem:

Theorem 1.1. *Let d be a positive integer and p a prime number. Let ϵ_0, ϵ_1, ..., ϵ_d be integers such that*

$$\epsilon_i = \begin{cases} 1 & i = d, \\ -1,\ 0 \text{ or } 1 & d/2 < i < d, \\ 0 & i \le d/2. \end{cases}$$

Then, there exists a d-dimensional Cohen-Macaulay local ring R of characteristic p, a maximal primary ideal I of R of finite projective dimension, and positive rational numbers α, β_{d-1}, β_{d-2}, ..., β_0 such that

$$\ell_R(R/I^{[p^n]}) = \epsilon_d \alpha p^{dn} + \sum_{i=0}^{d-1} \epsilon_i \beta_i p^{in}$$

for any $n > 0$.

The proof of Theorem 1.1 is established by constructing a test module M over a Cohen-Macaulay ring whose rational Chow group $\mathrm{A}_*(R)_{\mathbb{Q}}$ is of certain simplicity. To determine how the Todd class of a module looks is very difficult in general since $G_0(R)_{\mathbb{Q}}$ is too big. As mentioned earlier $\overline{G_0(R)_{\mathbb{R}}}$ is a finite dimensional $\mathbb{R}$-vector space of dimension $\rho(R)$. We denote the Cohen-Macaulay cone $\sum_{M:\mathrm{MCM}} \mathbb{R}_{\ge 0}[M]$ in $\mathbb{R}^{\rho(R)} = \overline{G_0(R)_{\mathbb{R}}}$ by $C_{CM}(R)$. Then we prove that the existence of a test module is equivalent to certain properties on the projections of the Cohen-Macaulay cone $C_{CM}(R)$ in the Chow group $\mathrm{A}_*(R)_{\mathbb{R}}$ via the Riemann-Roch map $\overline{\tau_R}$ (Theorem 2.12).

Assume that R is a Cohen-Macaulay local domain. If $\dim R \leq 2$, then $\rho(R) = 1$. If $\dim R \geq 3$, then there is no upper bound for $\rho(R)$. In either case, the Cohen-Macaulay cone has the maximal possible dimension, *i.e.*, $\dim_{\mathbb{R}} C_{CM}(R) = \rho(R)$. Indeed if $\rho(R) = 1$, then $C_{CM}(R)$ is obviously a half line. For arbitrary $\rho(R)$, we prove that there is an open neighborhood U of $[R]$ in $\mathbb{R}^{\rho(R)}$ such that U is contained in the interior of $C_{CM}(R)$. This is proved in Lemma 2.5 along with other properties of the Cohen-Macaulay cone.

If the ring R is of finite Cohen-Macaulay type—namely, there exist only finitely many indecomposable isomorphism classes of maximal Cohen-Macaulay modules—then the cone is finitely generated and so it is closed in $\mathbb{R}^{\rho(R)}$ under the usual topology for the Euclidean space. The Cohen-Macaulay cone in general may not be finitely generated, but the authors do not know a Cohen-Macaulay ring R whose $C_{CM}(R)$ is not closed.

In order to know how the notion of Cohen-Macaulay cones and test modules are applied in the study of Hilbert-Kunz functions, we provide a conceptual sketch of a key step in the proof of Theorem 1.1. The idea presented below is loosely about Step 2.

By the singular Riemann-Roch theorem, the coefficients of the Hilbert-Kunz function with respect to I of finite projective dimension over a Cohen-Macaulay ring can be expressed in terms of the localized Chern characters. Precisely it says that the coefficient of $(p^n)^i$ is $\mathrm{ch}_i(\mathbb{G}_\bullet)(\tau_R([R])_i)$ where $\mathbb{G}_\bullet$ is the resolution of R/I and $\tau_R([R])_i$ in $\mathrm{A}_i(R)_{\mathbb{R}}$ is the i-th Todd class of R. Thus to obtain desired coefficients for the Hilbert-Kunz function is equivalent to obtaining the values of the corresponding localized Chern characters when applied to the Todd classes of R.

We prove Theorem 1.1 by constructing a module over a Cohen-Macaulay ring A such that the i-th Todd class of the module in $\mathrm{A}_i(A)_{\mathbb{Q}}$ for each i has the desired value when the localized Chern character is applied to it. Then we take the idealization of the module to obtain a ring whose Hilbert-Kunz function has the reqired form. The module just described is constructed by induction on i, so the initial step that shows the existence of a ring A that possesses certain properties and a test module is crucial. This is done in Lemma 3.1.

Last, we would like to make a remark without the intension of getting into any technical detail in the present paper. In Theorem 1.1, the coefficients of the polynomial are assumed to be zero in the terms of degree $d/2$ or lower. This assumption is made due to the fact that $\overline{\mathrm{A}_i(R)_{\mathbb{Q}}} = 0$ for $i \leq d/2$ (for a homogeneous coordinate ring R of a smooth projective variety) if the Grothendieck's standard conjecture holds (*c.f.* [16, Remark 7.12]). There is no known example where $\overline{\mathrm{A}_i(R)_{\mathbb{Q}}}$ does not vanish for some $i \leq d/2$.

The paper is organized as follows. Section 2 introduces the Cohen-Macaulay cone of an arbitrary local domain and the definition of test modules. Basic properties of these new notions are proved. In Example 2.10, assuming R is complete or essentially of finite type over a field, we construct some examples of test modules. We also prove equivalence conditions of the existence of test modules; for a general local domain, it can be found in Remark 2.9. Main results about such equivalence for Cohen-Macaulay rings are stated and proven in Theorem 2.12.

Section 3 discusses the Hilbert-Kunz functions and proves Theorem 1.1, which constructs rings whose Hilbert-Kunz function is of the required form. For Theorem 1.1, we need Lemma 3.1, which assures the existence of a Cohen-Macaulay ring such that it has test modules and its numerical Chow group satisfies certain properties. The proof of Lemma 3.1 deserves independent attention and so Section 4 is devoted to proving this lemma. The Chow group of $X = \mathbb{P}^m \times \mathbb{P}^n$ and the Riemann-Roch map $\tau_X : G_0(X)_{\mathbb{Q}} \to \mathrm{A}_*(X)_{\mathbb{Q}}$ have been carefully studied by the second author [13]. Lemma 3.1 is proven by taking appropriate Segre products of graded rings and utilizing the special properties on τ_X and $\mathrm{A}_*(X)_{\mathbb{Q}}$.

2. Cohen-Macaulay cone

Let (R, m) be a d-dimensional Noetherian local domain. We always assume that local domains are homomorphic images of regular local rings and assume that one of two conditions in Theorem 2.1 is satisfied.

Further, in this paper, we assume that all modules are finitely generated. Let $G_0(R)$ be the Grothendieck group of finitely generated R-modules.

We put

$$C(R) = \left\{ \mathbb{F}. \;\middle|\; \begin{array}{l} \text{bounded complex of finite } R\text{-free modules,} \\ H_i(\mathbb{F}.) \text{ has finite length for any } i \end{array} \right\}.$$

For $\mathbb{F}. \in C(R)$, we define an additive map

$$\chi_{\mathbb{F}.} : G_0(R) \longrightarrow \mathbb{Z}$$

by

$$\chi_{\mathbb{F}.}([M]) = \sum_i (-1)^i \ell_R(H_i(\mathbb{F}. \otimes M)).$$

We set

$$\overline{G_0(R)} = G_0(R)/\{c \in G_0(R) \mid \chi_{\mathbb{F}.}(c) = 0 \text{ for any } \mathbb{F}. \in C(R)\}.$$

Theorem 2.1 (Kurano, [16] Theorem 3.1 and Remark 3.5)**.** *Assume that a Noetherian local domain R satisfies one of the following two conditions:*

- *R is an excellent ring such that R contains $\mathbb{Q}$.*
- *R is essentially of finite type over a field, $\mathbb{Z}$ or a complete discrete valuation ring.*

Then, $\overline{G_0(R)}$ is a finitely generated free $\mathbb{Z}$-module.

If $\mathrm{Spec}(R)$ has a resolution of singularities or a regular alteration, then the above theorem is still true for such R without assuming one of two conditions above.

Let $\rho(R)$ be the rank of $\overline{G_0(R)}$. Note that $\rho(R)$ is always positive, that is $\overline{G_0(R)} \neq 0$. Indeed consider the Koszul complex $\mathbb{K}.$ of some system of parameters $\underline{a}$ of R. Then $\chi_{\mathbb{K}.}([R])$ is equal to the Hilbert-Samuel multiplicity of R with respect to the ideal generated by $\underline{a}$ (*c.f.* [25] Chapter IV Theorem 1). So $\chi_{\mathbb{K}.}([R]) \neq 0$. This shows that $[R]$ is not zero in $\overline{G_0(R)}$ by definition (*c.f.* [16] page 582). If $d \leq 2$, then $\rho(R) = 1$ (see [16] Proposition 3.7). For any given $d \geq 3$, there is no upper bound for $\rho(R)$ (see [16] Example 4.1).

Proposition 2.2. The following conditions are equivalent:

(1) $\rho(R) = 1$.
(2) $\overline{G_0(R)} = \mathbb{Z}[R]$.
(3) For any $\mathbb{F}. \in C(R)$ and any R-module M with $\dim M < d$, $\chi_{\mathbb{F}.}([M]) = 0$.
(4) For any $\mathbb{F}. \in C(R)$ and any R-module M,

$$\chi_{\mathbb{F}.}([M]) = \mathrm{rank}_R M \cdot \chi_{\mathbb{F}.}([R]).$$

Proof. It is easy to see (4) $\Longrightarrow$ (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1).

We shall prove (1) $\Longrightarrow$ (4). Let $\mathbb{K}.$ be the Koszul complex of some system of parameters $\underline{a}$. Then by Serre's theorem (*c.f.* [25] Chapter IV Theorem 1), $\chi_{\mathbb{K}.}([M]) = e_I(M)$ where $e_I(-)$ denotes the Hilbert-Samuel multiplicity with respect to the ideal I generated by $\underline{a}$. Since $e_I(M) = \mathrm{rank}_R M \cdot e_I(R)$, we have

$$\chi_{\mathbb{K}.}([M]) = \mathrm{rank}_R M \cdot \chi_{\mathbb{K}.}([R]).$$

On the other hand, note that $\chi_{\mathbb{K}.}([R]) = e_I(R) \neq 0$. Therefore, $[R] \neq 0$ in $\overline{G_0(R)}_{\mathbb{Q}}$. Thus $\overline{G_0(R)}_{\mathbb{Q}} = \mathbb{Q}[R]$ by the condition (1). We write $[M] = r[R]$ in $\overline{G_0(R)}_{\mathbb{Q}}$ for some rational number r. Thus for every $\mathbb{F}. \in C(R)$, $\chi_{\mathbb{F}.}([M]) = r \cdot \chi_{\mathbb{F}.}([R])$. In particular,

$$r \cdot \chi_{\mathbb{K}.}([R]) = \chi_{\mathbb{K}.}([M]) = \mathrm{rank}_R M \cdot \chi_{\mathbb{K}.}([R]).$$

This shows $r = \mathrm{rank}_R M$ and $[M] = \mathrm{rank}_R M \cdot [R]$ in $\overline{G_0(R)}_{\mathbb{Q}}$. Therefore, the condition (4) is satisfied. $\square$

Remark 2.3. Suppose $\mathbb{F}. \in C(R)$. Assume that $\mathbb{F}.$ is not exact and the length of $\mathbb{F}.$ is d, that is,

$$\mathbb{F}. \ : \ 0 \to F_d \to F_{d_1} \to \cdots \to F_1 \to F_0 \to 0.$$

Let M be a finitely generated R-module. By induction on the depth, we have $\mathrm{depth}(M) = d - \max\{i | H_i(\mathbb{F}. \otimes_R M) \neq 0\}$. If M is a maximal Cohen-Macaulay module, then $H_i(\mathbb{F}. \otimes_R M) = 0$ for $i > 0$. Thus,

$$\chi_{\mathbb{F}.}([M]) = \ell(H_0(\mathbb{F}. \otimes_R M)) > 0.$$

We think that cycles in $\overline{G_0(R)}_{\mathbb{R}}$ represented by maximal Cohen-Macaulay modules are *positive* elements in a sense.

In this paper, $\mathbb{Q}_{\geq 0}$ (resp. $\mathbb{R}_{\geq 0}$) denotes the set of non-negative rational (resp. real) numbers. Further, $\mathbb{Q}_+$ (resp. $\mathbb{R}_+$) denotes the set of positive rational (resp. real) numbers.

Definition 2.4. Set

$$C_{CM}(R) = \sum_{M:MCM} \mathbb{R}_{\geq 0}[M] \subset \overline{G_0(R)}_{\mathbb{R}},$$

where M runs over all maximal Cohen-Macaulay R-modules in the above summation. We call it the *CM (Cohen-Macaulay) cone* of R. Let $C_{CM}(R)^-$ be the closure of $C_{CM}(R)$ in $\overline{G_0(R)}_{\mathbb{R}}$ with respect to the usual topology on the Euclidean space $\mathbb{R}^{\rho(R)}$.

The CM cone $C_{CM}(R)$ and its closure $C_{CM}(R)^-$ are convex cones by definition. The authors do not have an example where $C_{CM}(R)$ is a proper subset of its closure. (We know that $C_{CM}(R)^-$ is a strongly convex cone by a recent result due to Dao and Kurano [7]. We do not need this result in this paper.)

Set

$$\mathrm{Nef}(R) = \{c \in \overline{G_0(R)}_{\mathbb{R}} \mid \chi_{\mathbb{F}.}(c) \geq 0 \text{ for any } \mathbb{F}. \in C(R) \text{ of length } d\}.$$

We call it the *nef (numerically effective) cone* of R.

Lemma 2.5. *Let R be a Cohen-Macaulay local domain.*

(1) Let c be in $\sum_{M:MCM} \mathbb{Q}_{\geq 0}[M]$. Then, there exists a positive integer n and a maximal Cohen-Macaulay module M such that

$$nc = [M] \quad \textit{in } \overline{G_0(R)}_{\mathbb{R}}.$$

(2) *Let c be an element in $C_{CM}(R)$. For any open subset U of $\overline{G_0(R)}_{\mathbb{R}}$ containing c,*

$$U \cap \sum_{M:MCM} \mathbb{Q}_{\geq 0}[M] \neq \emptyset.$$

(3)

$$C_{CM}(R) \cap \overline{G_0(R)}_{\mathbb{Q}} \subset \sum_{M:MCM} \mathbb{Q}_{\geq 0}[M] \subset \overline{G_0(R)}_{\mathbb{R}}$$

is satisfied.

(4)

$$C_{CM}(R) \subset C_{CM}(R)^{-} \subset \mathrm{Nef}(R) \subset \overline{G_0(R)}_{\mathbb{R}}.$$

(5)

$$\mathrm{Nef}(R) \cap -\mathrm{Nef}(R) = \{0\}.$$

(6) *If R is of finite Cohen-Macaulay representation type, then*

$$C_{CM}(R) = C_{CM}(R)^{-}.$$

(7) *There exists an open set U of $\overline{G_0(R)}_{\mathbb{R}}$ such that $[R] \in U \subset C_{CM}(R)$.*

(8)

$$Int(C_{CM}(R)^{-}) \subset C_{CM}(R).$$

Proof. It is easy to see (1).

Here, we shall prove (2). Put $c = \sum_i r_i[M_i]$, where $r_i \in \mathbb{R}_+$ and M_i is a maximal Cohen-Macaulay module for each i. We choose $r_i' \in \mathbb{Q}_+$ sufficiently near r_i for each i. Then $\sum_i r_i'[M_i]$ is in U.

We shall prove (3). It is sufficient to show that, for a finite number of maximal Cohen-Macaulay modules $M_1, \ldots, M_s$,

$$\sum_{i=1}^{s} \mathbb{R}_{\geq 0}[M_i] \cap \overline{G_0(R)}_{\mathbb{Q}} \subset \sum_{i=1}^{s} \mathbb{Q}_{\geq 0}[M_i]$$

in $\overline{G_0(R)}_{\mathbb{R}}$.

Let $c = \sum_i h_i[M_i]$ ($h_i \in \mathbb{R}_+$) be in the left-hand side in the above. First, remark that

$$\sum_{i=1}^{s} \mathbb{R}[M_i] \cap \overline{G_0(R)}_{\mathbb{Q}} = \sum_{i=1}^{s} \mathbb{Q}[M_i].$$

Therefore,

$$c = \sum_{i=1}^{s} q_i[M_i]$$

for $q_i \in \mathbb{Q}$ ($i = 1, \ldots, s$). Here, we put

$$W = \{(\alpha_1, \ldots, \alpha_s) \in \mathbb{Q}^s \mid \sum_i \alpha_i[M_i] = 0\} \subset \mathbb{Q}^s.$$

Then,

$$(h_1 - q_1, \dots, h_s - q_s) \in W \otimes_{\mathbb{Q}} \mathbb{R}$$

Therefore, there exists $(\beta_1, \dots, \beta_s) \in W$ sufficiently near $(h_1 - q_1, \dots, h_s - q_s)$. Then,

$$c = \sum_{i=1}^{s} (q_i + \beta_i)[M_i]$$

where $q_i + \beta_i \in \mathbb{Q}_+$ for $i = 1, \dots, s$.

(4) immediately follows from Remark 2.3.

We shall prove (5). Let $c \in \mathrm{Nef}(R) \cap -\mathrm{Nef}(R)$. For $\mathbb{F}. \in C(R)$ of length d, we have $\chi_{\mathbb{F}.}(c) \geq 0$ and $\chi_{\mathbb{F}.}(-c) \geq 0$. Thus, we have $\chi_{\mathbb{F}.}(c) = 0$. By Proposition 2 in [23], the set of complexes of length d generates the Grothendieck group of $C(R)$. Therefore, c is numerically equivalent to 0.

(6) is easy.

Next, we shall prove (7). Let $T_1, \dots, T_\rho$ be torsion R-modules such that

- $\{[T_1], \dots, [T_{\rho-1}], [R]\}$ is a basis of the $\mathbb{Q}$-vector space $\overline{G_0(R)_{\mathbb{Q}}}$, and
- $[T_1] + \cdots + [T_{\rho-1}] + [T_\rho] = 0$ in $\overline{G_0(R)_{\mathbb{Q}}}$.

Let M_i be the k-th sygyzy of T_i, where k is an even integer bigger than d. Then, M_i is a maximal Cohen-Macaulay module such that

$$[M_i] = (\mathrm{rank}_R M_i)[R] + [T_i]$$

in $\overline{G_0(R)_{\mathbb{Q}}}$. Then, we have

$$[M_1] + \cdots + [M_{\rho-1}] + [M_\rho] = \left(\sum_i \mathrm{rank}_R M_i \right) [R]$$

in $\overline{G_0(R)_{\mathbb{Q}}}$. By the above equations,

$$\{[M_1], \dots, [M_{\rho-1}], [M_\rho]\}$$

is a basis of the $\mathbb{R}$-vector space $\overline{G_0(R)_{\mathbb{R}}}$. We know that $[R]$ is in interior of the cone spanned by $[M_1], \dots, [M_{\rho-1}], [M_\rho]$.

Lastly, we shall prove (8). Suppose that $0 \neq c \in Int(C_{CM}(R)^-)$. There exists an open neighborhood U of c such that U is a subset of $C_{CM}(R)^-$. Choose $e_1, \dots, e_{\rho-1} \in \overline{G_0(R)_{\mathbb{R}}}$ such that $\{e_1, \dots, e_{\rho-1}, c\}$ is an $\mathbb{R}$-basis of $\overline{G_0(R)_{\mathbb{R}}}$. Taking e_i's small enough, we may assume $c + e_i \in U$ for $i = 1, \dots, \rho - 1$, and $c - e_1 - \cdots - e_{\rho-1} \in U$. We put $s_i = c + e_i$ for $i = 1, \dots, \rho - 1$ and $s_\rho = c - e_1 - \cdots - e_{\rho-1}$. Then $s_1, \dots, s_\rho$ are in U such that

$$s_1 + \cdots + s_\rho = \rho \cdot c$$

in $\overline{G_0(R)}_{\mathbb{R}}$, and $\{s_1, \cdots, s_{\rho-1}, s_\rho\}$ is a basis of the $\mathbb{R}$-vector space $\overline{G_0(R)}_{\mathbb{R}}$. Each point in U is the limit of a sequence in $C_{CM}(R)$. Therefore, there exist $s'_1, \ldots, s'_\rho$ such that

- $s'_1, \ldots, s'_\rho \in C_{CM}(R)$,
- $\{s'_1, \ldots, s'_\rho\}$ is a basis of the $\mathbb{R}$-vector space $\overline{G_0(R)}_{\mathbb{R}}$, and
- c is in the cone spanned by $s'_1, \ldots, s'_\rho$.

Hence c is in $C_{CM}(R)$. □

Example 2.6. Let

$$R = k[x, y, z, w]_{(x,y,z,w)}/(xy - zw)$$

Then,

$$\{R, P, Q\}$$

is the set of isomorphism classes of indecomposable maximal Cohen-Macaulay modules (see Yoshino [27]), where $P = (x, z)$, $Q = (x, w)$. In this case, $\rho(R) = 2$, and

$$C_{CM}(R) = C_{CM}(R)^- = \mathbb{R}_{\geq 0}[P] + \mathbb{R}_{\geq 0}[Q] \subset \mathrm{Nef}(R).$$

Remark 2.7 (Riemann-Roch theory)**.** We have an isomorphism of $\mathbb{Q}$-vector spaces

$$G_0(R)_{\mathbb{Q}} \xrightarrow{\tau_R} \mathrm{A}_*(R)_{\mathbb{Q}} = \oplus_{i=0}^{d} \mathrm{A}_i(R)_{\mathbb{Q}}$$

as in [9] and [22]. Then, we have $\mathrm{A}_d(R)_{\mathbb{Q}} = \mathbb{Q}[\mathrm{Spec}(R)]$ and $p_d\tau_R([M]) = \mathrm{rank}_R M \cdot [\mathrm{Spec}(R)]$ where p_d is the projection $\mathrm{A}_*(R) \to \mathrm{A}_d(R)$.

Put $\tau_R([R]) = c_d + c_{d-1} + \cdots + c_0$, where $c_i \in \mathrm{A}_i(R)_{\mathbb{Q}}$. Then,

(1) $c_d = [\mathrm{Spec}(R)]$.
(2) If R is a complete intersection, $\tau_R([R]) = c_d$.
(3) If R is Cohen-Macaulay,
$$\tau_R([\omega_R]) = c_d - c_{d-1} + c_{d-2} - c_{d-3} + \cdots ,$$
where ω_R is the canonical module of R.
(4) If R is Gorenstein,
$$c_{d-1} = c_{d-3} = c_{d-5} = \ldots = 0.$$
(5) If R is normal, we have an isomorphism $\mathrm{A}_{d-1}(R) \simeq \mathrm{Cl}(R)$ by $[\mathrm{Spec}(R/I)] \mapsto -\mathrm{cl}(I)$, where $\mathrm{cl}(I)$ denotes the isomorphism class of a divisorial ideal I. Then, we have
$$c_{d-1} = -\frac{\mathrm{cl}(\omega_R)}{2}.$$
(6) Localization of Galois extension of a regular local ring satisfies $\tau_R([R]) = c_d$.

As in [16], we can define $\overline{\mathrm{A}_i(R)}$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
G_0(R)_{\mathbb{Q}} & \xrightarrow{\tau_R} & \mathrm{A}_*(R)_{\mathbb{Q}} \\
\downarrow & & \downarrow \\
\overline{G_0(R)}_{\mathbb{Q}} & \xrightarrow{\overline{\tau_R}} & \overline{\mathrm{A}_*(R)}_{\mathbb{Q}} \\
\downarrow & & \downarrow \\
\overline{G_0(R)}_{\mathbb{R}} & \xrightarrow{\overline{\tau_R}} & \overline{\mathrm{A}_*(R)}_{\mathbb{R}}
\end{array}
\tag{2.1}
$$

Definition 2.8. We say that R-module M is an R-test module if the following two conditions are satisfied:

(1) M is a non-zero maximal Cohen-Macaulay module.
(2) $\overline{\tau_R}([M]) = \mathrm{rank}_R\, M \cdot [\mathrm{Spec}(R)]$ in $\overline{\mathrm{A}_*(R)}_{\mathbb{Q}}$.

The above condition (2) is equivalent to $\overline{\tau_R}([M]) \in \overline{\mathrm{A}_d(R)}_{\mathbb{Q}}$.

The definition of test modules here is a little different from that in [14].

For $\mathbb{F}. \in C(R)$, the Dutta multiplicity (limit multiplicity) is defined to be

$$\chi_\infty(\mathbb{F}.) = \chi_{\mathbb{F}.}(\tau_R^{-1}([\mathrm{Spec}(R)])).$$

If M is an R-test module and $\mathbb{F}.$ is a complex in $C(R)$ of length d, then

$$\chi_\infty(\mathbb{F}.) = \frac{1}{\mathrm{rank}_R M}\chi_{\mathbb{F}.}([M]) = \frac{1}{\mathrm{rank}_R M}\ell_R(H_0(\mathbb{F}. \otimes_R M)) > 0. \tag{2.2}$$

For a non-exact complex in $C(R)$ of length d, $\chi_\infty(\mathbb{F}.)$ is positive if R contains a field ([21], [18], [14]). Positivity of $\chi_\infty(\mathbb{F}.)$ for a non-exact complex in $C(R)$ of length d is an open question for the mixed characteristic case. By (2.2), this question is true for R which possesses a test module.

Remark 2.9. A local ring R has a test module if and only if

$$\overline{\tau_R}^{-1}([\mathrm{Spec}(R)]) \in C_{CM}(R) \tag{2.3}$$

as follows. The key point is that $\overline{\tau_R}^{-1}([\mathrm{Spec}(R)])$ is in $\overline{G_0(R)}_{\mathbb{Q}}$ by the commutativity of the diagram (2.1). If (2.3) is satisfied, then we have

$$\overline{\tau_R}^{-1}([\mathrm{Spec}(R)]) \in C_{CM}(R) \cap \overline{G_0(R)}_{\mathbb{Q}} \subset \sum_{M:MCM} \mathbb{Q}_{\geq 0}[M]$$

by Lemma 2.5 (3). Then, by Lemma 2.5 (1), we know the existence of R-test modules.

Example 2.10. Let R be a Noetherian local domain of dimension d. Suppose that R contains an excellent regular local ring S, and let A be the integral closure of S in R. We assume that A is a finitely generated S-module, and R coincides with A_P for some prime ideal P of A. (We remark that such S exists if R is complete or essentially of finite type over a field.)

When the characteristic of S is positive, we further assume that S is F-finite. Let L be a finite dimensional normal extension of $Q(S)$ containing $Q(A)$ where $Q(S)$ and $Q(A)$ denote the field of fractions of S and A respectively. Let B be the integral closure of S in L. Since S is excellent, B is a finitely generated A-module. Thus $B \otimes_A R$ is a finitely generated R-module.

(1) Applying the method in [15], we obtain

$$\tau_R([B \otimes_A R]) \in \mathrm{A}_d(R)_{\mathbb{Q}},$$

that is,

$$[B \otimes_A R] = \mathrm{rank}_R(B \otimes_A R) \cdot \tau_R^{-1}([\mathrm{Spec}(R)])$$

in $G_0(R)_{\mathbb{Q}}$. Therefore, if $B \otimes_A R$ is a Cohen-Macaulay ring, then $B \otimes_A R$ is an R-test module.

(2) Put $G = \mathrm{Aut}_{Q(S)}(L)$. Assume that N is a maximal Cohen-Macaulay B-module. For each $g \in G$, we give another B-module structure to N by

$$\begin{array}{ccc} B \times N & \longrightarrow & N \\ (b, n) & \mapsto & g(b)n. \end{array}$$

We denote this B-module by ${}_gN$. We put

$$M = \bigoplus_{g \in G} {}_gN.$$

Then, we have

$$[M] = \mathrm{rank}_A(M) \cdot \tau_A^{-1}([\mathrm{Spec}(A)])$$

in $G_0(A)_{\mathbb{Q}}$. Changing the base regular scheme using Lemma 4.1 (c) in [15], we obtain

$$[M \otimes_A R] = \mathrm{rank}_R(M \otimes_A R) \cdot \tau_R^{-1}([\mathrm{Spec}(R)])$$

in $G_0(R)_{\mathbb{Q}}$; therefore $M \otimes_A R$ is an R-test module.

Remark 2.11. (1) If any local ring has a test module, then a conjecture (a positivity conjecture of Dutta multiplicity) is true (see (2.2), [14] Conjecture 3.3 and Proposition 4.3).

(2) Let R be a complete local domain. If the small Cohen-Macaulay conjecture is true, then R has a test module (see Example 2.10 and [14] Theorem 1.3).

(3) Even if R is a Gorenstein ring, we do not know whether R has a test module or not. If R is a complete intersection, then R itself is an R-test module.

The following is the main theorem of this section. We denote by p_i the projection

$$\overline{\mathrm{A}_*(R)_{\mathbb{R}}} = \oplus_{i=0}^{d} \overline{\mathrm{A}_i(R)_{\mathbb{R}}} \to \overline{\mathrm{A}_i(R)_{\mathbb{R}}}.$$

Theorem 2.12. *Let (R, m) be a Cohen-Macaulay local domain. Consider the following conditions:*

(i) $\overline{\tau_R}^{-1}([\mathrm{Spec}(R)]) \in Int(C_{CM}(R)^-)$

(ii) R has a test module which contains R as a direct summand.

(iii) For $i = 0, 1, \ldots, d-1$, $p_i\overline{\tau_R}(C_{CM}(R)) = \overline{\mathrm{A}_i(R)_{\mathbb{R}}}$.

(iv) For $i = 0, 1, \ldots, d-1$, $p_i\overline{\tau_R}(C_{CM}(R)^-) = \overline{\mathrm{A}_i(R)_{\mathbb{R}}}$.

Then, we have

$$(i) \iff (ii) \implies (iii) \iff (iv).$$

If R is F-finite of characteristic $p > 0$ and R/m is algebraically closed, then above four conditions are equivalent to each other.

Proof. $(i) \implies (ii)$. There exists a positive integer n such that

$$n\overline{\tau_R}^{-1}([\mathrm{Spec}(R)]) - [R] \in Int(C_{CM}(R)^-) \cap \overline{G_0(R)_{\mathbb{Q}}} \subset C_{CM}(R) \cap \overline{G_0(R)_{\mathbb{Q}}}$$

by Lemma 2.5 (8). By Lemma 2.5 (1), (3), there exists a maximal Cohen-Macaulay module M such that

$$[R] + [M] = n'\overline{\tau_R}^{-1}([\mathrm{Spec}(R)]) \tag{2.4}$$

for some $n' > 0$.

$(ii) \implies (i)$. Let N_0 be a module over R such that $N = R \oplus N_0$ is a test module. Let $M = N_0 \oplus N$. Then M is a maximal Cohen-Macaulay module, and we have an equality as (2.4) in which $n' = 2\,\mathrm{rank}_R N$. Since $[R] \in Int(C_{CM}(R)^-)$ by Lemma 2.5 (7) and $[M] \in C_{CM}(R)$, $[R \oplus M]$ is also in $Int(C_{CM}(R)^-)$.

$(iii) \implies (iv)$ is trivial.

$(iv) \implies (iii)$. Since

$$p_i\overline{\tau_R}(C_{CM}(R)^-) \subset (p_i\overline{\tau_R}(C_{CM}(R)))^-,$$

we have $(p_i\overline{\tau_R}(C_{CM}(R)))^- = \overline{\mathrm{A}_i(R)_{\mathbb{R}}}$. Note that $p_i\overline{\tau_R}(C_{CM}(R))$ is a cone in $\overline{\mathrm{A}_i(R)_{\mathbb{R}}}$. Since the convexity is preserved under $\overline{\tau_R}$ and the projection p_i, if $p_i\overline{\tau_R}(C_{CM}(R)) \neq \overline{\mathrm{A}_i(R)_{\mathbb{R}}}$, then it must be contained in a closed half-space, and so must its closure. This contradicts the above fact $(p_i\overline{\tau_R}(C_{CM}(R)))^- = \overline{\mathrm{A}_i(R)_{\mathbb{R}}}$ resulted from the condition (iv). Hence $p_i\overline{\tau_R}(C_{CM}(R)) = \overline{\mathrm{A}_i(R)_{\mathbb{R}}}$.

$(i) \implies (iii)$. Remark that

$$p_i\overline{\tau_R}(C_{CM}(R)) \supset p_i\overline{\tau_R}(Int(C_{CM}(R)^-)) \ni p_i([\mathrm{Spec}(R)]) = \mathbf{0}$$

if $i < d$. Since $p_i\overline{\tau_R}$ is an open map, $p_i\overline{\tau_R}(Int(C_{CM}(R)^-))$ contains an open neighbourhood of $\mathbf{0}$. Then, we have $p_i\overline{\tau_R}(C_{CM}(R)) = \overline{\mathrm{A}_i(R)}_{\mathbb{R}}$ since $p_i\overline{\tau_R}(C_{CM}(R))$ is a cone in $\overline{\mathrm{A}_i(R)}_{\mathbb{R}}$.

Now, we shall prove $(iii) \Longrightarrow (i)$ in the case where R is F-finite of characteristic $p > 0$ and R/m is algebraically closed.

Step 1 First we want to show $\overline{\tau_R}^{-1}([\mathrm{Spec}(R)]) \in C_{CM}(R)^-$. We put $\overline{\tau_R}([R]) = c_d + c_{d-1} + \cdots + c_0$, where $c_i \in \overline{\mathrm{A}_i(R)}_{\mathbb{Q}}$.

$$\begin{array}{ccc} \overline{G_0(R)}_{\mathbb{R}} & \xrightarrow{\overline{\tau_R}} & \overline{\mathrm{A}_*(R)}_{\mathbb{R}} = \oplus_{i=0}^{d}\overline{\mathrm{A}_i(R)}_{\mathbb{R}} \\ [R] & \longleftrightarrow & c_d + c_{d-1} + \cdots + c_0 \\ [R^{\frac{1}{p^e}}] & \longleftrightarrow & p^{de}c_d + p^{(d-1)e}c_{d-1} + \cdots + p^{0e}c_0. \end{array}$$

Therefore

$$\frac{1}{p^{de}}[R^{\frac{1}{p^e}}] = \overline{\tau_R}^{-1}\left(c_d + \frac{1}{p^e}c_{d-1} + \cdots + \frac{1}{p^{de}}c_0\right) \in C_{CM}(R).$$

Take the limit. Then, we have

$$\overline{\tau_R}^{-1}([\mathrm{Spec}(R)]) = \overline{\tau_R}^{-1}(c_d) = \lim_{e\to\infty}\frac{1}{p^{de}}[R^{\frac{1}{p^e}}] \in C_{CM}(R)^-.$$

Step 2 We shall show $\overline{\tau_R}^{-1}([\mathrm{Spec}(R)]) \in Int(C_{CM}(R)^-)$.

Assume that $\overline{\tau_R}^{-1}([\mathrm{Spec}(R)])$ is in the boundary of the cone $C_{CM}(R)^-$. Then $[\mathrm{Spec}\, R]$ is in the boundary of the image of the cone under $\overline{\tau_R}$. If $\rho(R) = 1$, then it can never happen. Therefore we may assume that $\rho(R) > 1$. For any R-module M,

$$\overline{\tau_R}([M]) = \mathrm{rank}_R\, M[\mathrm{Spec}(R)] + (\text{lower dimensional terms}).$$

Therefore, $\overline{\tau_R}(C_{CR}(R)^-) \neq \overline{\mathrm{A}_*(R)}_{\mathbb{R}}$. Note that $\overline{\tau_R}(C_{CM}(R)^-)$ is a convex cone since $\overline{\tau_R}$ is an $\mathbb{R}$-linear map. Thus there exists a hyperplane through the origin that contains the boundary possessing $[\mathrm{Spec}(R)]$. Indeed such a hyperplane is a supporting hyperplane of the cone $\overline{\tau_R}(C_{CM}(R)^-)$; namely, there exists a vector $\mathbf{v}$ normal to the hyperplane such that the inner product $< \mathbf{v}, \mathbf{u} >$ is nonnegative for every $\mathbf{u}$ in $\overline{\tau_R}(C_{CR}(R)^-)$. Let ξ be the projection of $\overline{\mathrm{A}_*(R)}_{\mathbb{R}}$ onto the line generated by $\mathbf{v}$. Then

$$\xi : \overline{\mathrm{A}_*(R)}_{\mathbb{R}} \longrightarrow \mathbb{R}$$

is a non-zero $\mathbb{R}$-linear map with the properties

$$\begin{cases} \xi([\mathrm{Spec}(R)]) = 0, \\ \xi\overline{\tau_R}(C_{CM}(R)^-) \subset \mathbb{R}_{\geq 0}. \end{cases} \tag{2.5}$$

Since $\overline{\mathrm{A}_d(R)_{\mathbb{R}}} = \mathbb{R}[\mathrm{Spec}(R)]$, we have $\xi(\overline{\mathrm{A}_d(R)_{\mathbb{R}}}) = 0$. Since $\xi \neq 0$, we can choose $0 \leq j < d$ such that

$$\begin{cases} \xi(\overline{\mathrm{A}_i(R)_{\mathbb{R}}}) = 0 \text{ for } i = j+1, j+2, \ldots, d, \\ \xi(\overline{\mathrm{A}_j(R)_{\mathbb{R}}}) \neq 0. \end{cases} \tag{2.6}$$

Therefore, $\xi(\overline{\mathrm{A}_j(R)_{\mathbb{R}}}) = \mathbb{R}$. Since $p_j\overline{\tau_R}(C_{CM}(R)) = \overline{\mathrm{A}_j(R)_{\mathbb{R}}}$ by the condition (iii), we have

$$\mathbb{R} = \xi p_j \overline{\tau_R}(C_{CM}(R)) = \sum_{M:MCM} \mathbb{R}_{\geq 0} \xi p_j \overline{\tau_R}([M]).$$

Therefore, there exists a maximal Cohen-Macaulay module N such that

$$\xi p_j \overline{\tau_R}([N]) < 0. \tag{2.7}$$

Set

$$\overline{\tau_R}([N]) = s_d + s_{d-1} + \cdots + s_0,$$

where $s_i \in \overline{\mathrm{A}_i(R)_{\mathbb{Q}}}$. By (2.7), we have

$$\xi(s_j) < 0. \tag{2.8}$$

Then,

$$\overline{\tau_R}([F^e(N)]) = p^{de}s_d + p^{(d-1)e}s_{d-1} + \cdots + p^{0e}s_0.$$

By the assumption (2.6), we have

$$\xi\overline{\tau_R}([F^e(N)]) = p^{je}\xi(s_j) + p^{(j-1)e}\xi(s_{j-1}) + \cdots + p^{0e}\xi(s_0).$$

Since (2.8), we know

$$\xi\overline{\tau_R}([F^e(N)]) < 0$$

for a sufficiently large e. Since $F^e(N)$ is Cohen-Macaulay,

$$\xi\overline{\tau_R}([F^e(N)]) \in \xi\overline{\tau_R}(C_{CM}(R)) \subset \mathbb{R}_{\geq 0}$$

by (2.5). It is a contradiction. $\square$

For a positive integer ℓ, we define

$$\psi^\ell : \overline{\mathrm{A}_*(R)_{\mathbb{R}}} \longrightarrow \overline{\mathrm{A}_*(R)_{\mathbb{R}}}$$

to be

$$\psi^\ell(s_d + s_{d-1} + \cdots + s_0) = \ell^d s_d + \ell^{d-1}s_{d-1} + \cdots + \ell^0 s_0,$$

where $s_i \in \overline{\mathrm{A}_i(R)_{\mathbb{R}}}$ for $i = 0, 1, \ldots, d$.

If there exists $\ell \geq 2$ such that

$$\psi^\ell\left(\overline{\tau_R}(C_{CM}(R))\right) \subset \overline{\tau_R}(C_{CM}(R)),$$

the conditions (i), (ii), (iii), (iv) in Theorem 2.12 are equivalent to each other without assuming that R is of positive characteristic.

If R is of characteristic prime p, then

$$\psi^p\left(\overline{\tau_R}(C_{CM}(R))\right) \subset \overline{\tau_R}(C_{CM}(R)).$$

Therefore, it is natural to ask the following for an arbitrary Cohen-Macaulay local domain:

Question 2.13. Is there an integer $\ell \geq 2$ such that

$$\psi^\ell\left(\overline{\tau_R}(C_{CM}(R))\right) \subset \overline{\tau_R}(C_{CM}(R))?$$

3. Examples of various Hilbert-Kunz functions

In the rest of this paper, we shall prove Theorem 1.1 in the introduction.

We need the following lemma.

Lemma 3.1. *Let d be a positive integer, and p be a prime number.*

Then, there exists a ring A of characteristic p which satisfies the following conditions:

- *A is a d-dimensional F-finite Cohen-Macaulay normal local domain and the residue class field of A is algebraically closed.*
- $\mathrm{A}_i(A)_{\mathbb{Q}} = \overline{\mathrm{A}_i(A)_{\mathbb{Q}}} = \begin{cases} \mathbb{Q} & (\frac{d}{2} < i \leq d) \\ 0 & (otherwise) \end{cases}$.
- *There exists a maximal Cohen-Macaulay A-module M such that $\tau_A([A \oplus M]) \in \mathrm{A}_d(A)_{\mathbb{Q}}$; that is, $A \oplus M$ is an A-test module containing A as a direct summand.*

The above lemma will be proven in the next section. In this section, using Lemma 3.1, we shall prove Theorem 1.1.

Let A be a ring satisfying three conditions in Lemma 3.1.

Step 1. We set

$$\{i_1, i_2, \ldots, i_t\} = \{i \mid \epsilon_i \neq 0\}.$$

In Step 1, we shall show that there exists $a_{i_k} \in \mathrm{A}_{i_k}(A)_{\mathbb{Q}}$ for $k = 1, 2, \ldots, t$, and a finite free A-complex $\mathbb{F}.$ of length d with support at the maximal ideal m such that

$$\mathrm{ch}(\mathbb{F}.)(a_{i_k}) \neq 0$$

for all $k = 1, 2, \ldots, t$.

Recall that $\epsilon_j = 0$ if $j \leq \frac{d}{2}$. Then, by the assumption on the ring A, $\overline{\mathrm{A}_{i_k}(A)_{\mathbb{Q}}} = \mathbb{Q}$ for $k = 1, 2, \ldots, t$. By the definition of $\overline{\mathrm{A}_i(A)_{\mathbb{Q}}}$ (see [16]), there exists $a_{i_k} \in \mathrm{A}_{i_k}(A)_{\mathbb{Q}}$

and a finite free A-complex $\mathbb{F}^{(k)}.$ of length d with support at the maximal ideal m such that

$$\mathrm{ch}(\mathbb{F}^{(k)}.)(a_{i_k}) \neq 0$$

for $k = 1, 2, \dots, t$. Here, we recall that, since A is a Cohen-Macaulay local ring, the Grothendieck group of bounded finite A-free complexes with support in $\{m\}$ is generated by free resolutions of modules of finite length and of finite projective dimesion (cf. Proposition 2 in [23]).

By induction, it is easy to show that there exist positive integers n_1, n_2, $\dots$, n_t such that

$$\mathbb{F}. = \mathbb{F}^{(1)}.^{\oplus n_1} \oplus \mathbb{F}^{(2)}.^{\oplus n_2} \oplus \cdots \oplus \mathbb{F}^{(t)}.^{\oplus n_t}$$

satisfies the required condition.

Step 2. Take the complex $\mathbb{F}.$ that we have constructed in Step 1.

In Step 2, we shall show that there exists a maximal Cohen-Macaulay A-module N and positive rational numbers β_0, β_1, $\dots$, β_d such that

$$\mathrm{ch}(\mathbb{F}.)(\tau_A([A \oplus N])_i) = \epsilon_i \beta_i$$

for $i = 0, 1, \dots, d$. Here, $\tau_A([A \oplus N])_i$ is the element in $\mathrm{A}_i(A)_{\mathbb{Q}}$ such that $\tau_A([A \oplus N]) = \sum_{i=0}^{d} \tau_A([A \oplus N])_i$.

By the induction on j, we shall prove the following: There exists a maximal Cohen-Macaulay A-module N and positive rational numbers β_{d-j}, β_{d-j+1}, $\dots$, β_d such that

$$\mathrm{ch}(\mathbb{F}.)(\tau_A([A \oplus N])_i) = \epsilon_i \beta_i$$

for $i = d-j, d-j+1, \dots, d$.

Consider the case $j = 0$. Here, recall that $\mathbb{F}.$ is a bounded finite A-free complex of length d with support in $\{m\}$. Set $N = A$. Then, $\tau_A([A \oplus A])_d = 2[\mathrm{Spec}(A)]$ and

$$\mathrm{ch}(\mathbb{F}.)([\mathrm{Spec}(A)]) > 0$$

by a theorem of Roberts [21]. Here, recall that $\epsilon_d = 1$. Therefore, $N = A$ and $\beta_d = \mathrm{ch}(\mathbb{F}.)(2[\mathrm{Spec}(A)])$ satisfy the required condition.

Next suppose $0 \leq j < d$. We assume that there exists a maximal Cohen-Macaulay A-module N' and positive rational numbers β'_{d-j}, β'_{d-j+1}, $\dots$, β'_d such that

$$\mathrm{ch}(\mathbb{F}.)(\tau_A([A \oplus N'])_i) = \epsilon_i \beta'_i$$

for $i = d-j, d-j+1, \dots, d$.

Compare the rational number $\mathrm{ch}(\mathbb{F}.)(\tau_A([A \oplus N'])_{d-j-1})$ with ϵ_{d-j-1}.

If there exists a positive rational number β'_{d-j-1} such that

$$\mathrm{ch}(\mathbb{F}.)(\tau_A([A \oplus N'])_{d-j-1}) = \epsilon_{d-j-1}\beta'_{d-j-1},$$

we have nothing to prove. (Here, if $d-j-1 \leq \frac{d}{2}$, both $\mathrm{ch}(\mathbb{F}.)(\tau_A([A \oplus N'])_{d-j-1})$ and ϵ_{d-j-1} are 0. Therefore, we have only to set $\beta'_{d-j-1} = 1$ in this case.)

We assume that there does not exist a positive rational number β'_{d-j-1} satisfying the above condition.

(*) If $\epsilon_{d-j-1} = 0$, we set $b = -\tau_A([A \oplus N'])_{d-j-1}$. If $\epsilon_{d-j-1} \neq 0$, we choose $b \in \mathrm{A}_{d-j-1}(A)_{\mathbb{Q}}$ such that the sign of $\mathrm{ch}(\mathbb{F}.)(b)$ is the same as that of ϵ_{d-j-1}.

Here, remark that, by the construction of $\mathbb{F}.$ in Step 1, we can choose an element b satisfying the above condition. We shall show the following claim:

Claim 3.2. *There exists a maximal Cohen-Macaulay A-module L and a positive integer n such that*

$$\tau_A([L]) = \mathrm{rank}_A(L)[\mathrm{Spec}(A)] + nb + (\textit{lower dimensional terms}).$$

We prove this claim.

Since $b \in \mathrm{A}_{d-j-1}(A)_{\mathbb{Q}}$ and $d-j-1 < d$, there exist (not necessary distinct) prime ideals $P_1, \ldots, P_s$ of height $j+1$ such that

$$nb = [\mathrm{Spec}(A/P_1)] + \cdots + [\mathrm{Spec}(A/P_s)]$$

in $\mathrm{A}_{d-j-1}(A)_{\mathbb{Q}}$ for some positive integer n.

Consider the following exact sequence

$$0 \to N_1 \to F_{2u-1} \to \cdots \to F_1 \to F_0 \to A/P_1 \oplus \cdots \oplus A/P_s \to 0,$$

where $F_0, F_1, \ldots, F_{2u-1}$ are finitely generated A-free modules and u is a large enough number such that N_1 is a maximal Cohen-Macaulay A-module.

Then, we have

$$\begin{aligned} [N_1] &= [A/P_1 \oplus \cdots \oplus A/P_s] - [F_0] + [F_1] - \cdots + [F_{2u-1}] \\ &= [A/P_1 \oplus \cdots \oplus A/P_s] + \mathrm{rank}_A(N_1)[A] \end{aligned}$$

in $\mathrm{G}_0(A)_{\mathbb{Q}}$.

By the assumption, there exists a maximal Cohen-Macaulay A-module M such that $\tau_A([A \oplus M]) \in \mathrm{A}_d(A)_{\mathbb{Q}}$. Adding $\mathrm{rank}_A(N_1)[M]$ to the both sides, we obtain

$$[N_1] + \mathrm{rank}_A(N_1)[M] = [A/P_1 \oplus \cdots \oplus A/P_s] + \mathrm{rank}_A(N_1)[A \oplus M]$$

in $\mathrm{G}_0(A)_{\mathbb{Q}}$.

Then, we have

$$\begin{aligned} & \tau_A([N_1 \oplus M^{\oplus \mathrm{rank}_A(N_1)}]) \\ = \quad & \mathrm{rank}_A(N_1 \oplus M^{\oplus \mathrm{rank}_A(N_1)})[\mathrm{Spec}(A)] + nb + \text{(lower dimensional terms)} \end{aligned}$$

since

$$\tau_A([A/P_1 \oplus \cdots \oplus A/P_s]) = nb + \text{(lower dimensional terms)}$$

by the top term property (Theorem 18.3 (5) in [9]). Thus, $N_1 \oplus M^{\oplus \mathrm{rank}_A(N_1)}$ satisfies the condition on L in Claim 3.2. We have completed the proof of Claim 3.2.

Here, we set

$$N = A^{\oplus e-1} \oplus N'^{\oplus e} \oplus L^{\oplus f}.$$

for some positive integers e and f, and a maximal Cohen-Macaulay module L in Claim 3.2. Then,

$$\tau_A([A \oplus N])_d = \mathrm{rank}_A(A \oplus N)[\mathrm{Spec}(A)] = \frac{\mathrm{rank}_A(A \oplus N)}{\mathrm{rank}_A(A \oplus N')} \tau_A([A \oplus N'])_d$$

and

$$\tau_A([A \oplus N])_i = e\tau_A([A \oplus N'])_i$$

for $i = d-j, d-j+1, \ldots, d-1$. Therefore, we have

$$\mathrm{ch}(\mathbb{F}.)(\tau_A([A \oplus N])_d) = \epsilon_d \beta_d$$

where

$$\beta_d = \frac{\mathrm{rank}_A(A \oplus N)}{\mathrm{rank}_A(A \oplus N')} \beta'_d > 0,$$

and

$$\mathrm{ch}(\mathbb{F}.)(\tau_A([A \oplus N])_i) = \epsilon_i \beta_i$$

where

$$\beta_i = e\beta'_i > 0$$

for $i = d-j, d-j+1, \ldots, d-1$.

On the other hand, we have

$$\tau_A([A \oplus N])_{d-j-1} = e\tau_A([A \oplus N'])_{d-j-1} + fnb.$$

If $\epsilon_{d-j-1} = 0$, then we suppose $e = fn$. Then, $\tau_A([A \oplus N])_{d-j-1} = 0$ by the definition of b (see (*) above Claim 3.2). Thus, putting $\beta_{d-j-1} = 1$,

$$\mathrm{ch}(\mathbb{F}.)(\tau_A([A \oplus N])_{d-j-1}) = 0 = \epsilon_{d-j-1}\beta_{d-j-1}.$$

Next, assume that $\epsilon_{d-j-1} \neq 0$. Consider the equality

$$\mathrm{ch}(\mathbb{F}.)(\tau_A([A \oplus N])_{d-j-1}) = e\ \mathrm{ch}(\mathbb{F}.)(\tau_A([A \oplus N'])_{d-j-1}) + fn\ \mathrm{ch}(\mathbb{F}.)(b).$$

Assume that f/e is big enough. Then the sign of the right-hand-side is the same as that of ϵ_{d-j-1} by the definition of b (see (*) above Claim 3.2). Therefore, there exists a positive rational number β_{d-j-1} such that

$$\mathrm{ch}(\mathbb{F}.)(\tau_A([A \oplus N])_{d-j-1}) = \epsilon_{d-j-1}\beta_{d-j-1}.$$

Step 3. Let $\mathbb{F}.$ and N be a complex and a maximal Cohen-Macaulay module as in Step 2, respectively. Let R be the idealization $A \ltimes N$. Then, R is a d-dimensional Cohen-Macaulay local ring.

Since $\mathrm{ch}(\mathbb{F}.)$ is a bivariant class (Chapter 17 in [9]), we have the commutative diagram

$$\begin{array}{ccc} \mathrm{A}_*(R/(m \ltimes mN))_{\mathbb{Q}} & \overset{\mathrm{ch}(\mathbb{F}.\otimes_A R)}{\longleftarrow} & \mathrm{A}_*(R)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ \mathrm{A}_*(A/m)_{\mathbb{Q}} & \overset{\mathrm{ch}(\mathbb{F}.)}{\longleftarrow} & \mathrm{A}_*(A)_{\mathbb{Q}} \end{array}$$

where the vertical maps are isomorphisms induced by finite morphisms $\mathrm{Spec}(R) \to \mathrm{Spec}(A)$ and $\mathrm{Spec}(R/(m \ltimes mN)) \to \mathrm{Spec}(A/m)$. Then, we obtain

$$\mathrm{ch}(\mathbb{F}. \otimes_A R)(\tau_R([R])_i) = \mathrm{ch}(\mathbb{F}.)(\tau_A([A \oplus N])_i) = \epsilon_i\beta_i$$

for $i = 0, 1, \ldots, d$. Since R is a Cohen-Macaulay local ring of dimension d, $\mathbb{F}. \otimes_A R$ is a finite free resolution of an R-module Q of finite length by [23]. Let $\mathcal{C}$ be the category of R-modules of finite length and finite projective dimension.

Then, by Kumar's method (cf. Lemma 9.10 in [26]), there exist maximal primary ideals $I_1, \ldots, I_\ell, I$ of R of finite projective dimension such that

- I_i is an ideal generated by a maximal regular sequence of R for $i = 1, \ldots, \ell$.
- $[Q] + \sum_{i=1}^{\ell}[R/I_i] = [R/I]$ in $\mathrm{K}_0(\mathcal{C})$.

Let $F : R \to R$ be the Frobenius map. It is a finite morphism since A is F-finite. We denote by ${}_{F^n}R$ the R-module R whose R-module structure is given by

$$r \times a := r^{p^n} a$$

for $r \in R$ and $a \in {}_{F^n}R$.

By the Riemann-Roch formula,

$$\tau_R([{}_{F^n}R]) = \sum_{i=0}^{d} p^{in}\tau_R([R])_i.$$

By the local Riemann-Roch formula, we have

$$\begin{aligned}\chi_{\mathbb{F}.\otimes_A R}({}_{F^n}R) &= \mathrm{ch}(\mathbb{F}.\otimes_A R)(\sum_{i=0}^{d}\tau_R([{}_{F^n}R])_i) \\ &= \mathrm{ch}(\mathbb{F}.\otimes_A R)(\sum_{i=0}^{d}p^{in}\tau_R([R])_i) \\ &= \sum_{i=0}^{d}\mathrm{ch}(\mathbb{F}.\otimes_A R)(\tau_R([R])_i)p^{in} \\ &= \sum_{i=0}^{d}\epsilon_i\beta_i p^{in}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\chi_{\mathbb{F}.\otimes_A R}({}_{F^n}R) &= \chi(Q, {}_{F^n}R) \\ &= \chi(R/I, {}_{F^n}R) - \sum_{i=1}^{\ell}\chi(R/I_i, {}_{F^n}R) \\ &= \ell_R(R/I^{[p^n]}) - \sum_{i=1}^{\ell}\ell_R(R/I_i^{[p^n]})\end{aligned}$$

since ${}_{F^n}R$ is a Cohen-Macaulay R-module and the residue class field of R is algebraically closed.

We set $I_i = (a_{i1}, \dots, a_{id})$, where $a_{i1}, \dots, a_{id}$ forms a maximal R-regular sequence. Then,

$$\ell_R(R/I_i^{[p^n]}) = \ell_R(R/(a_{i1}^{p^n}, \dots, a_{id}^{p^n})) = p^{dn}\ell_R(R/(a_{i1}, \dots, a_{id})).$$

Thus, we have

$$\ell_R(R/I^{[p^n]}) = \left(\epsilon_d\beta_d + \sum_{i=1}^{\ell}\ell_R(R/(a_{i1}, \dots, a_{id}))\right)p^{dn} + \sum_{i=0}^{d-1}\epsilon_i\beta_i p^{in}.$$

Remark that

$$\epsilon_d\beta_d + \sum_{i=1}^{\ell}\ell_R(R/(a_{i1}, \dots, a_{id})) = e_{HK}(I) > 0.$$

Putting $\alpha = e_{HK}(I)$, we know that I satisfies the required condition. We have completed the proof of Theorem 1.1.

In Theorem 1.1, the coefficients of the polynomial are assumed to be zero in the terms of degree $d/2$ or lower. This assumption is made due to the fact that $\overline{\mathrm{A}_i(R)_{\mathbb{Q}}} = 0$ for $i \leq d/2$ (for a homogeneous coordinate ring R of a smooth projective variety) if the Grothendieck's standard conjecture holds (*c.f.* [16, Remark 7.12]). There is no known example where $\overline{\mathrm{A}_i(R)_{\mathbb{Q}}}$ does not vanish for some $i \le d/2$.

Therefore, it is natural to ask the following:

Conjecture 3.3. *Let R be a d-dimensional Cohen-Macaulay local ring of characteristic p with perfect residue class field. Let I be an maximal primary ideal of R of finite projective dimension. We set*

$$\ell_R(R/I^{[p^n]}) = \sum_{i=0}^{d} \beta_i p^{in}$$

for $n > 0$. Then, if $i \le d/2$, $\beta_i = 0$.

4. Proof of Lemma 3.1

This section is devoted to proving Lemma 3.1.

We use the following basic properties on singular Riemann-Roch maps.

Fact 4.1. *Let X be a d-dimensional projective variety over k. Then, we have an isomorphism*

$$G_0(X)_{\mathbb{Q}} \xrightarrow{\tau_X} \mathrm{A}_*(X)_{\mathbb{Q}}.$$

Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module. Put

$$\tau_X([\mathcal{M}]) = s_d + s_{d-1} + \cdots + s_0,$$

where $s_i \in \mathrm{A}_i(X)_{\mathbb{Q}}$.

Let D be a very ample divisor on X. Put $S = k[x_0, x_1, \ldots, x_n]$, where S is a graded polynomial ring with $\deg(x_i) = 1$ for $i = 0, 1, \ldots, n$. Let

$$X = \mathrm{Proj}(B) \overset{i}{\hookrightarrow} \mathbb{P}^n = \mathrm{Proj}(S)$$

be the embedding corresponding to D, where we have

$$S \twoheadrightarrow B = k[B_1] \subset \bigoplus_m H^0(X, \mathcal{O}_X(mD)).$$

Here, B_1 denotes the homogeneous component of the graded ring B of degree 1. We note that B is standard graded; that is, B is a graded ring generated by elements of degree 1 over $B_0 = k$.

(1) We have a commutative diagram:

$$\begin{array}{ccc} G_0(X)_{\mathbb{Q}} & \xrightarrow{\tau_X} & \mathrm{A}_*(X)_{\mathbb{Q}} \\ i_* \downarrow & & \downarrow i_* \\ G_0(\mathbb{P}^n)_{\mathbb{Q}} & \xrightarrow{\tau_{\mathbb{P}^n}} & \mathrm{A}_*(\mathbb{P}^n)_{\mathbb{Q}} \end{array}$$

Put

$$M = \bigoplus_m H^0(X, \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(mD)).$$

Then, M is a graded $\bigoplus_m H^0(X, \mathcal{O}_X(mD))$-module. We have

$$\tau_{\mathbb{P}^n}([\tilde{M}]) = \tau_{\mathbb{P}^n} i_*([\mathcal{M}]) = i_*\tau_X([\mathcal{M}]) = i_*(s_d) + i_*(s_{d-1}) + \cdots + i_*(s_0).$$

Put

$$H_i = \mathrm{Proj}(S/(x_{i+1}, \ldots, x_n)).$$

Then, we have

$$\mathrm{A}_i(\mathbb{P}^n)_{\mathbb{Q}} = \mathbb{Q}[H_i]$$

for $i = 0, 1, \ldots, n$. Let ℓ_i be a rational number such that

$$i_*(s_i) = \ell_i[H_i]$$

for $i = 0, 1, \ldots, d$. Then, we have

$$P_M(t) = \dim_k M_t = \frac{\ell_d}{d!}t^d + \frac{\ell_{d-1}}{(d-1)!}t^{d-1} + \cdots + \frac{\ell_0}{0!}t^0$$

for $t \gg 0$. (See Proposition in p3005 in Chan-Miller [4]*, Proposition 4.1 in Roberts-Singh* [24]*)*

(2) Let m be the homogeneous maximal ideal of B. We have the following commutative diagram:

$$\begin{array}{ccc} G_0(X)_{\mathbb{Q}} & \xrightarrow{\tau_X} & \mathrm{A}_*(X)_{\mathbb{Q}} \\ \alpha \downarrow & & \downarrow \beta \\ G_0(B)_{\mathbb{Q}} & \xrightarrow{\tau_B} & \mathrm{A}_*(B)_{\mathbb{Q}} \\ \gamma \downarrow & & \downarrow \delta \\ G_0(B_m)_{\mathbb{Q}} & \xrightarrow{\tau_{B_m}} & \mathrm{A}_*(B_m)_{\mathbb{Q}} \end{array}$$

The horizontal maps are isomorphisms. Here, γ and δ are isomorphisms induced by localization $B \to B_m$. For a graded B-module M, we have

$$\alpha([\tilde{M}]) = [M].$$

Here, we remark that α is well-defined since $[T] = 0$ in $G_0(B)_{\mathbb{Q}}$ for a graded B-module T whose homogeneous graded pieces are zero except for finitely many degrees. The map β is the sum of the maps

$$\mathrm{A}_i(X)_{\mathbb{Q}} \overset{\beta}{\twoheadrightarrow} \frac{\mathrm{A}_i(X)_{\mathbb{Q}}}{c_1(D)\mathrm{A}_{i+1}(X)_{\mathbb{Q}}} = \mathrm{A}_{i+1}(B)_{\mathbb{Q}},$$

where this map is given by

$$[\mathrm{Proj}(B/P)] \mapsto [\mathrm{Spec}(B/P)]$$

for each homogeneous prime ideal P with $\dim B/P > 0$.

Thus, we obtain

$$\tau_{B_m}([M \otimes_B B_m]) = \delta\beta(s_d) + \delta\beta(s_{d-1}) + \cdots + \delta\beta(s_0),$$

where $\delta\beta(s_i) \in \mathrm{A}_{i+1}(B_m)_{\mathbb{Q}}$. *We refer the reader to section 4 in* [13] *for maps* α, β, γ, δ.

Example 4.2. *Set* $X = \mathbb{P}^m \times \mathbb{P}^n$. *Let* p_1 *and* p_2 *be the first and second projections, respectively. Assume* $m \geq n \geq 2$.

Then, we have

$$G_0(X)_{\mathbb{Q}} \xrightarrow{\tau_X} \mathrm{A}_*(X)_{\mathbb{Q}} = \mathbb{Q}[a,b]/(a^{m+1}, b^{n+1}),$$

where

$$\mathrm{A}_{m+n-c}(X)_{\mathbb{Q}} = \bigoplus_{i+j=c} \mathbb{Q}a^i b^j,$$

and $a = c_1(p_1^*\mathcal{O}_{\mathbb{P}^m}(1)) \in \mathrm{A}_{m+n-1}(X)$ *and* $b = c_1(p_2^*\mathcal{O}_{\mathbb{P}^n}(1)) \in \mathrm{A}_{m+n-1}(X)$. *We put*

$$\mathcal{O}_X(s,t) = p_1^*\mathcal{O}_{\mathbb{P}^m}(s) \otimes_{\mathcal{O}_X} p_2^*\mathcal{O}_{\mathbb{P}^n}(t).$$

Put

$$f(x) = \frac{x}{1-e^{-x}}.$$

Then, we have

$$\begin{aligned} \tau_X([\mathcal{O}_X(s,t)]) &= \mathrm{ch}(p_1^*\mathcal{O}_{\mathbb{P}^m}(s))\mathrm{ch}(p_2^*\mathcal{O}_{\mathbb{P}^n}(t))\mathrm{td}(\Omega_X^\vee) \\ &= \mathrm{ch}(p_1^*\mathcal{O}_{\mathbb{P}^m}(s))\mathrm{ch}(p_2^*\mathcal{O}_{\mathbb{P}^n}(t))\mathrm{td}(p_1^*\Omega_{\mathbb{P}^m}^\vee)\mathrm{td}(p_2^*\Omega_{\mathbb{P}^n}^\vee) \\ &= e^{sa}f(a)^{m+1}e^{tb}f(b)^{n+1}. \end{aligned}$$

Here, take a very ample divisor $a + b \in \mathrm{A}_{m+n-1}(X)$. *Then, the homogeneous coordinate ring* B *is defined by all the* 2*-minors of the generic* $(m+1) \times (n+1)$*-matrix. In this case, the cycle in* X *corresponding to* $a^i b^j$ *is the closed subscheme defined by the ideal generated by the entries in the top* i *rows and the left* j *columns. Then, we have*

$$G_0(B)_{\mathbb{Q}} \xrightarrow{\tau_B} \mathrm{A}_*(B)_{\mathbb{Q}} = \mathbb{Q}[a,b]/(a^{m+1}, b^{n+1}, a+b) = \mathbb{Q}[b]/(b^{n+1}),$$

where we identify a *with* $-b$. *Let* P *(resp.* Q*) be the ideals generated by the elements in the first row (resp. the first colummn). Then, for* $s > 0$ *and* $t > 0$,

$$\begin{aligned} [P^{(s)}] &= [P^s] = \alpha([\mathcal{O}_X(-s,0)]) \\ [Q^{(t)}] &= [Q^t] = \alpha([\mathcal{O}_X(0,-t)]). \end{aligned}$$

Here, for an ideal I, $I^{(s)}$ denotes the s-th symbolic power of I. Then,

$$\begin{aligned}\tau_B([P^{(s)}]) &= e^{sb}f(-b)^{m+1}f(b)^{n+1} = f(-b)^{m+1-s}f(b)^{n+1+s} \in \mathbb{Q}[b]/(b^{n+1})\\ \tau_B([Q^{(t)}]) &= f(-b)^{m+1}e^{-tb}f(b)^{n+1} = f(-b)^{m+1+t}f(b)^{n+1-t} \in \mathbb{Q}[b]/(b^{n+1}),\end{aligned}$$

since $e^b = f(b)/f(-b)$. Here,

$$\{P^{(m)}, P^{(m-1)}, \dots, P^{(1)}, B, Q^{(1)}, \dots, Q^{(n-1)}, Q^{(n)}\}$$

is the set of rank one maximal Cohen-Macaulay modules. It is easily verified calculating local cohomology modules of Segre products [10]*. If there exists non-negative integers q_0, q_1, $\dots$, q_{m+n} satisfying*

$$\sum_{k=0}^{m+n} q_k > 0 \quad \text{and}$$

$$\sum_{k=0}^{m+n} q_k f(-b)^{1+k}f(b)^{m+n+1-k} = (\sum_{k=0}^{m+n} q_k) + b^{n+1}(\cdots),$$

then

$$(P^{(m)})^{\oplus q_0} \oplus \cdots \oplus (P^{(1)})^{\oplus q_{m-1}} \oplus B^{\oplus q_m} \oplus (Q^{(1)})^{\oplus q_{m+1}} \oplus \cdots \oplus (Q^{(n)})^{\oplus q_{m+n}}$$

is a B-test module. If $q_m > 0$, then it contains B as a direct summand.

The authors do not know whether a test module (having B as a direct summand) exists or not in this case.

In order to prove Lemma 3.1, it is enough to show the following Claim.

Claim 4.3. *Let d be a positive integer and p a prime number. Let k be an algebraically closed field of characteristic p. If d is even, then we put*

$$S = k[x_0, x_1, \dots, x_{d/2}] \quad \text{and} \quad T = k[y_0, y_1, \dots, y_{(d/2)-1}].$$

If d is odd, then we put

$$S = k[x_0, x_1, \dots, x_{(d-1)/2}] \quad \text{and} \quad T = k[y_0, y_1, \dots, y_{(d-1)/2}].$$

We think that S and T are graded rings with $\deg(x_i) = \deg(y_j) = 1$ for each i and j. Let ℓ be a sufficiently large integer. We denote by $S\#T^{(\ell)}$ be the Segre product of S and $T^{(\ell)}$, that is, $S\#T^{(\ell)} = \oplus_{m\geq 0}(S\#T^{(\ell)})_m$ with $(S\#T^{(\ell)})_m = S_m \otimes_k T_{m\ell}$ (see [10]*).*[1] *Let A be the localization of $S\#T^{(\ell)}$ at the homogeneous maximal ideal.*

Then, the ring A satisfies the following conditions:

[1]For a graded ring T, $T^{(\ell)}$ denotes the ℓ-th Veronese subring of T. Please do not confuse it with the symbolic power of ideals as in Example 4.2.

(1) *The ring A is a d-dimensional F-finite Cohen-Macaulay normal local domain and the residue class field of A is algebraically closed.*
(2) $\mathrm{A}_i(A)_{\mathbb{Q}} = \overline{\mathrm{A}_i(A)}_{\mathbb{Q}} = \begin{cases} \mathbb{Q} & (\frac{d}{2} < i \le d), \\ 0 & (\textit{otherwise}). \end{cases}$
(3) *There exists a maximal Cohen-Macaulay A-module M such that $\tau_A([A \oplus M]) \in \mathrm{A}_d(A)_{\mathbb{Q}}$.*

If d is even, then we set $m = d/2$ and $n = d/2 - 1$. If d is odd, then we set $m = n = (d-1)/2$. Let ℓ be a positive integer. Then, $a + \ell b$ is a very ample divisor on $X = \mathbb{P}^m \times \mathbb{P}^n$. Put $B = S\#T^{(\ell)}$.

Calculating local cohomologies of Segre products (see [10]), (1) will be easily proved.

(2) will be proved by the method due to Roberts-Srinivas [23]. In fact, the rational equivalence on cycles on $X = \mathbb{P}^m \times \mathbb{P}^n$ coincides with the numerical equivalence. Put $A = B_m$. Then, by Theorem 7.7 in [16], we have isomorphisms

$$\mathrm{A}_*(B)_{\mathbb{Q}} \simeq \mathrm{A}_*(A)_{\mathbb{Q}} \simeq \overline{\mathrm{A}_*(A)}_{\mathbb{Q}}.$$

We know

$$\mathrm{A}_i(B)_{\mathbb{Q}} = \begin{cases} \mathbb{Q} & (d/2 < i \le d), \\ 0 & (\text{otherwise}) \end{cases}$$

by [13].

In the rest, using Theorem 2.12, we shall prove (3). We shall prove that

$$p_i \overline{\tau_A}(C_{CM}(A)) = \overline{\mathrm{A}_i(A)}_{\mathbb{R}} \text{ for } d/2 < i < d. \tag{4.1}$$

Here, we define

$$N_q = \bigoplus_{s \in \mathbb{Z}} H^0(X, \mathcal{O}_X(q+s, \ell s)).$$

and prove the following lemma:

Lemma 4.4. *For any $\ell > 0$, N_q is a maximal Cohen-Macaulay B-module if $-m \le q \le 0$.*

Proof. We have

$$N_q = S(q)\#T^{(\ell)}.$$

Let m_1 (resp. m_2) be the homogeneous maximal ideal of S (resp. $T^{(\ell)}$).

Then, $H^i_{m_1}(S(q))_s \ne 0$ if and only if

$$i = 0 \text{ and } s \ge -q$$

or

$$i = m+1 \text{ and } s \le -q - m - 1.$$

Further, $H^i_{m_2}(T^{(\ell)})_s \neq 0$ if and only if

$$i = 0 \text{ and } s \geq 0$$

or

$$i = n+1 \text{ and } s \leq -\lceil \tfrac{n+1}{\ell} \rceil.$$

Here $\lceil \frac{n+1}{\ell} \rceil$ denotes the minimal integer which is bigger than or equal to $\frac{n+1}{\ell}$. We refer the reader to [10] for local cohomologies of Segre products. Therefore, N_q is a maximal Cohen-Macaulay module if and only if

$$\begin{cases} -\lceil \frac{n+1}{\ell} \rceil < -q \\ -q - m - 1 < 0. \end{cases}$$

It is equivalent to

$$-m - 1 < q < \lceil \frac{n+1}{\ell} \rceil.$$

Therefore, if $-m \leq q \leq 0$, then N_q is a maximal Cohen-Macaulay module. $\square$

We set

$$h_{m,q}(x) = (x+q+m)(x+q+m-1)\cdots(x+q+1).$$

Consider the polynomials

$$\begin{aligned}
h_{m,0}(x) &= (x+m)(x+m-1)\cdots(x+2)(x+1), \\
h_{m,-1}(x) &= (x+m-1)(x+m-2)\cdots(x+1)x, \\
h_{m,-2}(x) &= (x+m-2)(x+m-3)\cdots x(x-1), \\
&\vdots \\
h_{m,q}(x) &= (x+m+q)(x+m+q-1)\cdots(x+1+q), \\
&\vdots \\
h_{m,-m}(x) &= x(x-1)(x-2)\cdots(x-(m-1)).
\end{aligned}$$

The following lemma will be used later.

Lemma 4.5. *Suppose $m \geq 2$ and $m > u > 0$. The set of the coefficients of x^u in*

$$h_{m,-1}(x),\ h_{m,-2}(x),\ \ldots,\ h_{m,-m}(x)$$

contains a negative value.

Proof. We shall prove it by induction on m.

Suppose $m = 2$. Then,

$$\begin{aligned}
h_{2,-1}(x) &= (x+1)x = x^2 + x, \\
h_{2,-2}(x) &= x(x-1) = x^2 - x.
\end{aligned}$$

Assume that $m \geq 2$ and the assertion is true for m.

Suppose $1 < u < m+1$. By the induction hypothesis, there exists $-m \leq q < 0$ such that the coefficient of x^{u-1} in $h_{m,q}(x)$ is negative. If the coefficient of x^u in $h_{m,q}(x)$ is negative, the coefficient of x^u in

$$h_{m+1,q}(x) = (x + q + m + 1)h_{m,q}(x)$$

is negative. If the coefficient of x^u in $h_{m,q}(x)$ is non-negative, the coefficient of x^u in

$$h_{m+1,q-1}(x) = h_{m,q}(x)(x + q)$$

is negative.

Suppose $u = 1$. By the induction hypothesis, there exists $-m \leq q < 0$ such that the coefficient of x in $h_{m,q}(x)$ is negative. Remark that $h_{m,q}(0) = 0$. Then, the coefficient of x in

$$h_{m+1,q}(x) = (x + q + m + 1)h_{m,q}(x)$$

is negative. □

Consider

$$\tau_X(\mathcal{O}_X(q, 0)) = e^{qa} f(a)^{m+1} f(b)^{n+1} \in \mathrm{A}_*(X)_{\mathbb{Q}} = \mathbb{Q}[a, b]/(a^{m+1}, b^{n+1}).$$

Lemma 4.6. *Suppose that v is an integer such that $1 \leq v \leq n$.*

(1) Assume $v < m$. Then, the set of the coefficients of a^v in

$$\tau_X(\mathcal{O}_X(-m, 0)),\ \tau_X(\mathcal{O}_X(-m+1, 0)),\ \ldots,\ \tau_X(\mathcal{O}_X(0, 0))$$

contains a positive value and a negative value.

(2) Assume $v = m = n$. Then, the coefficient of a^m in $\tau_X(\mathcal{O}_X(0, 0))$ is positive. The coefficient of a^m in $\tau_X(\mathcal{O}_X(-1, 0))$ is zero. The coefficient of $a^{m-1}b$ in $\tau_X(\mathcal{O}_X(-1, 0))$ is positive.

Proof. The coefficient of a^v in

$$e^{qa} f(a)^{m+1} f(b)^{n+1}$$

is equal to the coefficient of a^v in

$$e^{qa} f(a)^{m+1}.$$

Since

$$\tau_{\mathbb{P}^m}([\mathcal{O}_{\mathbb{P}^m}(q)]) = e^{qa} f(a)^{m+1} \in \mathbb{Q}[a]/(a^{m+1}),$$

the coefficient of a^v in

$$e^{qa} f(a)^{m+1}$$

is equal to

$$(m-v)!\left\{\text{the coefficient of } x^{m-v} \text{ in the polynomial } \tbinom{x+q+m}{m}\right\} \tag{4.2}$$

by Fact 4.1 (1). Furthermore, (4.2) is equal to

$$\frac{(m-v)!}{m!}\left\{\text{the coefficient of } x^{m-v} \text{ in the polynomial } h_{m,q}(x)\right\}.$$

It is easy to see that, for $0 < u < m$, the coefficient of x^u in $h_{m,0}(x)$ is positive. Therefore, Lemma 4.6 (1) immediately follows from Lemma 4.5.

Assume that $v = m = n$. Since the constant term of $h_{m,0}(x)$ is positive, the coefficient of a^m in

$$f(a)^{m+1}f(b)^{m+1}$$

is positive. Since the constant term of $h_{m,-1}(x)$ is zero, the coefficient of a^m in

$$e^{-a}f(a)^{m+1}f(b)^{m+1}$$

is zero. The coefficient of $a^{m-1}b$ in

$$e^{-a}f(a)^{m+1}f(b)^{m+1}$$

is equal to

$$\left\{\text{the coefficient of } a^{m-1} \text{ in } e^{-a}f(a)^{m+1}\right\} \times \left(\frac{m+1}{2}\right),$$

where $\frac{m+1}{2}$ is the coefficient of b in $f(b)^{m+1}$. The sign of the coefficient of a^{m-1} in $e^{-a}f(a)^{m+1}$ is the same as the sign of the coefficient of x in $h_{m,-1}(x)$, that is obviously positive. □

We return to the proof of Claim 4.3 (3) and take an ample divisor $a + \ell b$ on $X = \mathbb{P}^m \times \mathbb{P}^n$ for $\ell > 0$. Remark that $S\#T^{(\ell)}$ is a homogeneous coordinate ring of X under the embedding corresponding to $a+\ell b$. We denote this ring simply by B. Then the commutative diagram from Fact 4.1(2) with the current X is precisely

$$\begin{array}{ccll} G_0(X)_{\mathbb{Q}} & \xrightarrow{\tau_X} & \mathrm{A}_*(X)_{\mathbb{Q}} & = \mathbb{Q}[a,b]/(a^{m+1}, b^{n+1}) \\ \alpha \downarrow & & \downarrow \beta & \\ G_0(B)_{\mathbb{Q}} & \xrightarrow{\tau_B} & \mathrm{A}_*(B)_{\mathbb{Q}} & = \mathbb{Q}[a,b]/(a^{m+1}, b^{n+1}, a+\ell b) = \mathbb{Q}[b]/(b^{n+1}) \end{array}$$

where $\beta(a) = -\ell b$.

Recall that

$$N_0,\ N_{-1}, \ldots, N_{-m}$$

are graded Cohen-Macaulay B-modules. Since $\overline{\mathrm{A}_i(A)_{\mathbb{R}}} = \mathbb{R}$, in order to show (4.1), it is enough to prove that, for $v = 1, 2, \ldots, n$, the set of the coefficients of b^v in

$$\tau_B([N_0]),\ \tau_B([N_{-1}]), \ldots, \tau_B([N_{-m}])$$

contains a positive value and a negative value. Note that

$$\tau_B([N_q]) = \tau_B \alpha(\mathcal{O}_X(q,0)) = \beta \tau_X(\mathcal{O}_X(q,0)) = \beta\left(e^{qa} f(a)^{m+1} f(b)^{n+1}\right).$$

Here recall that the map

$$\beta : \mathbb{Q}[a,b]/(a^{m+1}, b^{n+1}) \longrightarrow \mathbb{Q}[b]/(b^{n+1})$$

is given by $\beta(a^s b^t) = (-1)^s \ell^s b^{s+t}$. Thus, we have

$$\beta(\sum_{s,t \geq 0} q_{st} a^s b^t) = \sum_{s,t \geq 0} (-1)^s q_{st} \ell^s b^{s+t} = \sum_{v=0}^{n} (\sum_{s=0}^{v} (-1)^s q_{s,v-s} \ell^s) b^v.$$

If $(-1)^v q_{v0} > 0$ (resp. $(-1)^v q_{v0} < 0$), the coefficient of b^v in the above is positive (resp. negative) for $\ell \gg 0$.

First suppose $1 \leq v \leq n$ and $v < m$. By Lemma 4.6 (1), the set of coefficients of b^v in

$$\beta\left(f(a)^{m+1} f(b)^{n+1}\right),\ \beta\left(e^{-a} f(a)^{m+1} f(b)^{n+1}\right),\ \ldots,\ \beta\left(e^{-ma} f(a)^{m+1} f(b)^{n+1}\right)$$

contains a positive value and a negative value for $\ell \gg 0$.

Next suppose $v = m = n$. By Lemma 4.6 (2), the sign of the coefficients of b^m in

$$\beta\left(f(a)^{m+1} f(b)^{m+1}\right) \text{ and } \beta\left(e^{-a} f(a)^{m+1} f(b)^{m+1}\right)$$

are different for $\ell \gg 0$. We have complete the proof of Claim 4.3.

References

[1] H. Brenner, *The Hilbert-Kunz function in graded dimension two*, Comm. Algebra **35** (2007), 3199-3213.

[2] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press (1998).

[3] C-Y. J. Chan and K. Kurano, *Hilbert-Kunz functions over rings regular in codimension one*, to appear in Comm. Algebra.

[4] C-Y. J. Chan and C. Miller, *A Reimann-Roch formula for blow-up of a non-singular affine scheme*, J. Algebra, **322** (2009), 3003-3025.

[5] H. L. Dao, *Decent intersection and Tor-rigidity for modules over local hypersurfaces*, Trans. Amer. Math. Soc. **365** (2003) 2803-2821.

[6] H. L. Dao and K. Kurano, *Hochster's theta pairing and numerical equivalence*, to appear in J. K-theory.

[7] H. L. Dao and K. Kurano, *Boundary and shape of Cohen-Macaulay cone*, to appear in Math. Ann..

[8] N. Fakhruddin and V. Trivedi, *Hilbert-Kunz functions and multiplicities for full flag varieties and elliptic curves*, J. Pure Appl. Algebra **181** (2003) 23-52.

[9] W. Fulton, *Intersection Theory, 2nd Edition*, Springer-Verlag, Berlin, New York, 1997.

[10] S. Goto and K.-i. Watanabe, *On graded rings. I*, J. Math. Soc. Japan **30** (1978), 179–213.

[11] M. Hochster and Y. Yao, *Second coefficients of Hilbert-Kunz functions for domains*, preliminary preprint: *http://www.math.lsa.umich.edu/∼hochster/hk.pdf.*

[12] C. Huneke, M. McDermott and P. Monsky, *Hilbert-Kunz functions for normal rings*, Math. Res. Letters **11** (2004) 539-546.

[13] K. Kurano, *A remark on the Riemann-Roch formula for affine schemes associated with Noetherian local rings*, Tohoku Math. J., **48** (1996), 121–138.

[14] K. Kurano, *Test modules to calculate Dutta Multiplicities*, J. Algebra, **236** (2001), 216-235.

[15] K. Kurano, *On Roberts rings*, J. Math. Soc. Japan, **53** (2001), 333-355.

[16] K. Kurano, *Numerical equivalence defined on Chow groups of Noetherian local rings*, Invent. Math. **157** (2004), 575–619.

[17] K. Kurano, *The singular Riemann-Roch theorem and Hilbert-Kunz functions*, J. Algebra **304** (2006), 487-499.

[18] K. Kurano and P. Roberts, *Adams operations, localized Chern characters, and the positivity of Dutta multiplicity in characteristic* 0, Trans. Amer. Math. Soc. **352** (2000), 3103-3116.

[19] L. MacDonnell, *A note on a conjecture of Watanabe and Yoshida*, arXiv 10044224.

[20] P. Monsky *The Hilbert-Kunz function*, Math. Ann. **263** (1983) 43-49.

[21] P. Roberts, *Intersection theorems*, Commutative algebra, 417–436, Math. Sci. Res. Inst. Publ., **15**, Springer, New York, Berlin, 1989.

[22] P. Roberts, *Multiplicities and Chern Classes in Local Algebra,* Cambridge Tracts in Mathematics 133, Cambridge University Press, 1998.

[23] P. C. Roberts and V. Srinivas, *Modules of finite length and finite projective dimension*, Invent. Math., **151** (2003), 1–27.

[24] P. Roberts and A. Singh, *Reconciling Riemann-Roch results*, Contemporary Math, **555** (2011), 165–172.

[25] J.-P. Serre, *Local Algebra*, Springer Monographs in Mathematics, Springer-Verlag Berlin Heidelberg New York, 2000.

[26] V. Srinivas, *Algebraic K-theory, Second edition*, Progress in Mathematics, 90. Birkhauser Boston, Inc., Boston, MA, 1996.

[27] Y. Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings*, Lon. Math. Soc. Lect. Note 146, Cambridge University Press, 1992.

C-Y. Jean Chan
Department of Mathematics
Central Michigan University
Mt. Pleasant, MI 48858
U. S. A.
chan1cj@cmich.edu
http://people.cst.cmich.edu/chan1cj

Kazuhiko Kurano
Department of Mathematics
School of Science and Technology
Meiji University
Higashimita 1-1-1, Tama-ku
Kawasaki 214-8571, Japan
kurano@isc.meiji.ac.jp
http://www.isc.meiji.ac.jp/~kurano