Gorenstein isolated quotient singularities of odd prime dimension are cyclic

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Abstract

In this paper, we shall prove that Gorenstein isolated quotient singularities of odd prime dimension are cyclic. In the case where the dimension is bigger than 1 and is not an odd prime number, then there exist Gorenstein isolated non-cyclic quotient singularities.

Keywords. cyclic quotient singularity, isolated singularity, Gorenstein

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1 Introduction

Let G be a finite subgroup of $\mathrm{GL}(n, \mathbb{C})$, where $\mathbb{C}$ is the field of complex numbers and let $\mathrm{GL}(n, \mathbb{C})$ be the set of $n \times n$ invertible matrices with entries in $\mathbb{C}$. Then, G acts on a polynomial ring $R = \mathbb{C}[X_1, X_2, \ldots, X_n]$ linearly. Let R^G be the invariant subring, i.e.,

$$R^G = \{r \in R \mid g(r) = r \ \ \forall g \in G\}.$$

It is well-known that R^G is finitely generated over $\mathbb{C}$ (cf. Theorem 1.3.1 in [1]).

It is possible to classify finite subgroups in $\mathrm{SL}(2, \mathbb{C})$ (cf. Theorem 2.4.5 in [5]). Here, $\mathrm{SL}(n, \mathbb{C})$ is the subgroup of $\mathrm{GL}(n, \mathbb{C})$ consisting of all matrices of determinant 1. It is well-known that the invariant subring of $\mathbb{C}[X_1, X_2]$ under the linear action of a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$ is a hypersurface in $\mathbb{C}^3$ with isolated singularity.

It is also possible to classify finite subgroups in $\mathrm{SL}(3, \mathbb{C})$ (cf. Yau-Yu [7]). Using the classification, it was proven that Gorenstein isolated quotient singularities of dimension three are cyclic (Theorem A and Theorem 23 in Yau-Yu [7]).

The purpose of this paper is to prove the following theorem:

Theorem 1.1 *Let n be an odd prime number. Let G be a finite subgroup of* $\mathrm{GL}(n, \mathbb{C})$ *which contains no pseudo-reflection. Assume that the invariant subring R^G is Gorenstein with isolated singularity. Then, R^G has a cyclic quotient singularity.*

For a finite subgroup G of $\mathrm{GL}(n, \mathbb{C})$, we set

$$\Sigma_i = \{g \in G \mid 1 \text{ is an eigenvalue of } g \text{ with multiplicity at least } i\}$$

for $i = 0, 1, \ldots, n$. Each element in $\Sigma_{n-1} \backslash \{e\}$ is called a *pseudo-reflection.* Set

$$H_i = < \Sigma_i >,$$

which is the subgroup of G generated by Σ_i. By definition we have

$$\begin{array}{l} G = \Sigma_0 \supset \Sigma_1 \supset \cdots \supset \Sigma_{n-1} \supset \Sigma_n = \{e\} \text{ and} \\ G = H_0 \supset H_1 \supset \cdots \supset H_{n-1} \supset H_n = \{e\}. \end{array}$$

Here, remark that Σ_n is equal to $\{e\}$, since any matrix in G is diagonalizable. These are very important subgroups, because the ring homomorphism $R^G \to R^{H_l}$ is étale in codimension s if and only if $l \leq n - s$.

Suppose $n \geq 2$. Let l be an integer such that $0 \leq l \leq n - 2$. By purity of branch locus (cf. Theorem 41.1 in [2]) and the Shephard-Todd theorem (cf. Theorem 7.2.1 in [1]), we know that the following two conditions are equivalent:

(1) $H_l \supsetneqq H_{l+1} = \cdots = H_{n-1}$,

(2) $\mathrm{Sing} R^G \neq \emptyset$ and $\dim \mathrm{Sing} R^G = l$.

Here $\mathrm{Sing} R^G$ is the *singular locus* of R^G, i.e.,

$$\mathrm{Sing} R^G = \{P \in \mathrm{Spec} R^G \mid (R^G)_P \text{ is not a regular local ring}\}.$$

If $\mathrm{Sing} A$ is not empty and if the dimension of $\mathrm{Sing} A$ is 0, we say that A has *isolated singularities.* Thus, the following two conditions are equivalent:

(1) R^G has isolated singularities.

(2) $H_0 \supsetneqq H_1 = \cdots = H_{n-1}$.

If $\Sigma_{n-1} = \{e\}$, then the above two conditions are equivalent to the following:

(3) $\Sigma_1 = \{e\}$, i.e., 1 is not an eigenvalue of any matrix in G except for e.

On the other hand, remember the following theorem due to Watanabe [4]:

Theorem 1.2 (Watanabe) *Let G be a finite subgroup of* $\mathrm{GL}(n, \mathbb{C})$ *and suppose that G acts on $R := \mathbb{C}[X_1, X_2, \ldots, X_n]$ linearly.*

- *If $G \subset \mathrm{SL}(n, K)$, then R^G is a Gorenstein ring.*
- *If R^G is a Gorenstein ring and if $\Sigma_{n-1} = \{e\}$, then $G \subset \mathrm{SL}(n, K)$.*

Since $R^{H_{n-1}}$ is isomorphic to a polynomial ring and G/H_{n-1} acts on $R^{H_{n-1}}$ linearly, the case where $\Sigma_{n-1} = \{e\}$ is very important.

By these arguments, if $\Sigma_{n-1} = \{e\}$, we have the following assertions:

- $G \subset \mathrm{SL}(n, K)$ if and only if R^G is Gorenstein.
- R^G has isolated singularities if and only if 1 is not an eigenvalue of any matrix in G except for e.

Thus Theorem 1.1 immediately follows from Lemma 1.3 below.

Lemma 1.3 *Let n be an odd prime number. Let G be a finite subgroup of* $\mathrm{SL}(n, K)$*, where K is a field such that the characteristic of K is* 0 *or does not divide the order of G. Assume that* 1 *is not an eigenvalue of any matrix in G except for the unit matrix. Then, G is a cyclic group.*

We remark that the pair (G, ρ) of a finite group G and its irreducible fixed point free complex representation ρ are classified, where fixed point free means that $\rho(s)$ does not have 1 as its eigenvalue for $s \neq e$. This classification is obtained in Theorem 7.2.18 in [6]. Therefore, Lemma 1.3 follows from the classification.

In this paper, we give a very simple and elementary proof to Lemma 1.3.

We shall prove Lemma 1.3 in Section 2. In Section 3, we shall give examples of non-cyclic subgroups in the case where n is bigger than 1 and is not an odd prime integer.

2 Proof of Lemma 1.3

We shall prove Lemma 1.3 in this section.

We may assume that K is an algebraically closed field.

Remark that each matrix in G is diagonalizable because the characteristic of K is 0 or does not divide the order of G.

First we shall prove Lemma 1.3 in the case where G is an abelian group. Next we shall do in the case where G is a solvable group. Finally we prove Lemma 1.3 without any additional assumptions.

2.1 The case where G is abelian

In this subsection, we prove Lemma 1.3 in the case where G is an abelian group.

Assume that G is a finite abelian subgroup of $\mathrm{SL}(n, K)$.

Since the characteristic of K is 0 or does not divide the order of G, there exists $c \in \mathrm{GL}(n, K)$ such that $c^{-1}gc$ is a diagonal matrix for any $g \in G$. Set $c^{-1}Gc := \{c^{-1}gc | g \in G\}$. Remember that g and $c^{-1}gc$ have the same characteristic polynomial. So, g and $c^{-1}gc$ have the same determinant and the same eigenvalues. Replacing G with $c^{-1}Gc$, we may assume that all matrices in G are diagonal.

We define

$$\psi : G \longrightarrow K^{\times}$$

by letting $\psi(g)$ be the $(1, 1)$th entry of each diagonal matrix g in G. Then, it is a group homomorphism. Since 1 is not an eigenvalue of any matrix in G except for the unit matrix, ψ is injective.

Since any finite subgroup of $K^{\times}$ is cyclic, so is G.

2.2 The case where G is solvable

In this subsection, we prove Lemma 1.3 in the case where G is a solvable group by induction on ${}^{\#}G$ (the order of G).

Let G be a finite solvable subgroup of $\mathrm{SL}(n, K)$ satisfying the assumption in Lemma 1.3. Assume ${}^{\#}G > 1$. By induction, any finite solvable subgroup G' of $\mathrm{SL}(n, K)$ satisfying the assumption in Lemma 1.3 is cyclic if ${}^{\#}G > {}^{\#}G'$. In particular, any proper subgroup of G is cyclic.

Let H be a maximal subgroup of G that contains the commutator subgroup of G. We remark that such a subgroup exists since G is solvable. Then H is a normal subgroup of G. Since H is a proper subgroup of G, H is a cyclic group. Let a be a generator of H, and take $b \in G \setminus H$. Then,

$$H = <a> \text{ and } G = <a, b>,$$

where $<a_1, \ldots, a_t>$ means the subgroup generated by $a_1, \ldots, a_t$.

Let s be the order of a. Since H is a normal subgroup of G, $b^{-1}ab$ is in H. There exists $u \in (\mathbb{Z}/s\mathbb{Z})^\times$ such that $b^{-1}ab = a^u$.

Let $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be the set of the eigenvalues of a, where each λ_i is a primitive sth root of 1. We regard it as a multi-set.

Then, by a famous theorem due to Frobenius, $\{\lambda_1^u, \lambda_2^u, \ldots, \lambda_n^u\}$ is the set of eigenvalues of a^u.

Since $b^{-1}ab = a^u$,

$$\{\lambda_1, \lambda_2, \ldots, \lambda_n\} = \{\lambda_1^u, \lambda_2^u, \ldots, \lambda_n^u\}$$

is satisfied as a multi-set. Repeating it, we have

$$\{\lambda_1, \lambda_2, \ldots, \lambda_n\} = \{\lambda_1^{(u^m)}, \lambda_2^{(u^m)}, \ldots, \lambda_n^{(u^m)}\} \tag{1}$$

as a multi-set for any positive integer m. Let $\mathrm{ord}(u)$ be the order of u in the multiplicative group $(\mathbb{Z}/s\mathbb{Z})^\times$. Then, for any i,

$$\left\{\lambda_i, \lambda_i^u, \lambda_i^{(u^2)}, \ldots, \lambda_i^{(u^{\mathrm{ord}(u)-1})}\right\} \tag{2}$$

is a subset of mutually distinct eigenvalues of the matrix a. By (1), we know that eigenvalues in (2) have the same multiplicity. Therefore, it is easy to see that $\mathrm{ord}(u)$ divides n. Since n is a prime number, $\mathrm{ord}(u)$ is equal to 1 or n.

(i) If $u = 1$, then G is abelian since $ab = ba$. In this case, G is cyclic as we have already seen in Subsection 2.1.

(ii) Suppose $\mathrm{ord}(u) = n$. Then, we may assume that

$$\{\lambda, \lambda^u, \lambda^{(u^2)}, \ldots, \lambda^{(u^{n-1})}\}$$

is the set of eigenvalues of a, where λ is a primitive sth root of 1. Here, remark that the multiplicity of each eigenvalue is one. Then there exists $c \in \mathrm{GL}(n, K)$ such that

$$c^{-1}ac = \begin{pmatrix} \lambda & & & & O \\ & \lambda^u & & & \\ & & \lambda^{(u^2)} & & \\ & & & \ddots & \\ O & & & & \lambda^{(u^{n-1})} \end{pmatrix}. \tag{3}$$

Replacing G with $c^{-1}Gc$, we may assume that a is equal to the right-hand-side of (3). Then,

$$b^{-1}ab = a^u = \begin{pmatrix} \lambda^u & & & & O \\ & \lambda^{(u^2)} & & & \\ & & \ddots & & \\ & & & \lambda^{(u^{n-1})} & \\ O & & & & \lambda \end{pmatrix}.$$

By the above equality, the matrix b coincides with

$$(\mathbf{b}_1\ \mathbf{b}_2\ \cdots\ \mathbf{b}_{n-1}\ \mathbf{b}_0),$$

where $\mathbf{b}_i$ is an eigenvector of a of eigenvalue $\lambda^{(u^i)}$ for $i = 0, 1, \ldots, n-1$. Therefore, we may assume that the matrix b is of the following form:

$$\begin{pmatrix} 0 & \cdots & \cdots & 0 & b_0 \\ b_1 & 0 & \cdots & \cdots & 0 \\ 0 & b_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{n-1} & 0 \end{pmatrix}$$

Then,

$$\det(b) = (-1)^{n-1} b_0 b_1 \cdots b_{n-1} = 1.$$

On the other hand,

$$\begin{aligned} & \det(te - b) \\ = \ & \det \begin{pmatrix} t & 0 & \cdots & 0 & -b_0 \\ -b_1 & t & \ddots & \ddots & 0 \\ 0 & -b_2 & t & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -b_{n-1} & t \end{pmatrix} \\ = \ & \det \begin{pmatrix} t & 0 & \cdots & \cdots & 0 \\ -b_1 & t & 0 & \cdots & \vdots \\ 0 & -b_2 & t & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -b_{n-1} & t \end{pmatrix} + \det \begin{pmatrix} 0 & 0 & \cdots & \cdots & -b_0 \\ -b_1 & t & 0 & \cdots & \vdots \\ 0 & -b_2 & t & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -b_{n-1} & t \end{pmatrix} \\ = \ & t^n + (-1)^{n+(n-1)} b_0 b_1 \cdots b_{n-1} \\ = \ & t^n + (-1)^n. \end{aligned}$$

Since n is an odd number, we know that 1 is an eigenvalue of the matrix b. It is a contradiction. Therefore, $\mathrm{ord}(u)$ is not n.

We have completed a proof in the case where G is solvable.

2.3 Final step in our proof of Lemma 1.3

In this subsection, we prove Lemma 1.3 without any additional assumptions.

Let G be a group satisfying the assumption of Lemma 1.3. We prove Lemma 1.3 by induction on ${}^{\#}G$. By induction, any proper subgroup of G is cyclic. Let S_p be a p-Sylow subgroup of G for each prime number p.

First, assume that S_p is a normal subgroup of G for any prime number p. Then it is well known that G is isomorphic to the direct product of all Sylow subgroups. Therefore, in this case, G is solvable. Thus, G is cyclic as we have already seen in Subsection 2.2.

Next, we assume that there exists a prime number p such that S_p is not a normal subgroup of G. The following subgroup is called the *normalizer* of S_p.

$$N_G(S_p) = \{c \in G \mid cS_pc^{-1} = S_p\}$$

Since S_p is not a normal subgroup of G, $G \neq N_G(S_p)$.

Remember the following famous theorem due to Burnside (cf. Theorem 7.50 in [3]):

Theorem 2.1 (Burnside) *Let F be a finite group. Assume that there exists a prime number q such that a q-Sylow subgroup S_q of F is contained in the center of its normalizer $N_F(S_q)$.*

Then there exists a normal subgroup H of F such that

$$F = HS_q \quad \text{and} \quad H \cap S_q = \{e\}.$$

In our case, S_p is contained in the center of $N_G(S_p)$ because $N_G(S_p)$ is cyclic. By the above theorem due to Burnside, there exists a normal subgroup H of G such that

$$G = HS_p \quad \text{and} \quad H \cap S_p = \{e\}.$$

Since $S_p \neq \{e\}$, H is a proper subgroup of G. Therefore, H is cyclic. Since S_p is a proper subgroup of G, S_p is also cyclic. Then, G is solvable because of

$$G/H \simeq S_p.$$

Since G is solvable, it is a cyclic group as we have already seen in Subsection 2.2.

We have completed a proof of Lemma 1.3.

3 The case where n is not an odd prime number

Suppose that n is an integer bigger than 1.

In this section, we give examples of non-abelian finite subgroups of $\mathrm{SL}(n,\mathbb{C})$ that satisfy the assumption in Lemma 1.3 except for that n is an odd prime number.

These examples are of type I of Theorem 6.1.11 and the representations are given in Theorem 5.5.6 in [6].

3.1 The case where n is an even number

In this subsection, we assume that n is an even number.

Let H be a non-abelian finite subgroup of $\mathrm{SL}(2,\mathbb{C})$. For example, $H = < A, B >$, where

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

It is easy to see that 1 is not an eigenvalue of any matrix in H except for e.

Here we define as

$$G = \left\{ \begin{pmatrix} M & 0 & \cdots & \cdots & 0 \\ 0 & M & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & M \end{pmatrix} \in \mathrm{SL}(n,\mathbb{C}) \;\middle|\; M \in H \right\}.$$

Then 1 is not an eigenvalue of any matrix in G except for e. Since G is isomorphic to H as a group, G is not abelian.

3.2 The case where n is an odd composite number

In this subsection, assume that n is an odd composite number.

Set

$$n = qn', \tag{4}$$

where q is an odd prime number and n' is an odd number such that $q \leq n'$.

By a famous theorem due to Dirichlet, there exists an odd prime number l such that

$$l \equiv 1 \pmod{2q}.$$

Then, there exists $\alpha \in (\mathbb{Z}/l\mathbb{Z})^\times$ such that the order of α is q, i.e., it satisfies

$$\alpha^q \equiv 1 \pmod{l} \quad \text{and} \quad \alpha \not\equiv 1 \pmod{l}. \tag{5}$$

Let z (resp. x) be a primitive lth root (resp. qth root) of 1.

Here, set

$$A = \left(\begin{array}{ccc|c} & O & & x \\ \hline 1 & & O & \\ & \ddots & & O \\ O & & 1 & \end{array}\right), \quad B = \begin{pmatrix} z & & & O \\ & z^{\alpha} & & \\ & & \ddots & \\ O & & & z^{(\alpha^{q-1})} \end{pmatrix} \in \mathrm{GL}(q, \mathbb{C}).$$

Lemma 3.1 *Set* $G = <A, B> \subset \mathrm{GL}(q, \mathbb{C})$. *Then we have the following:*

- **(i)** $\det A = x$, $\det B = 1$.
- **(ii)** $AB \neq BA$. *In particular,* G *is not abelian.*
- **(iii)** G *is a finite group.*
- **(iv)** 1 *is not an eigenvalue of any matrix in* G *except for the unit matrix.*

Proof. We have

$$\begin{aligned} \det A &= (-1)^{q-1} x = x \\ \det B &= \prod_{i=0}^{q-1} z^{(\alpha^i)} = z^{\frac{\alpha^q - 1}{\alpha - 1}}. \end{aligned}$$

Since l divides $\frac{\alpha^q-1}{\alpha-1}$ by (5),

$$z^{\frac{\alpha^q-1}{\alpha-1}} = 1.$$

The assertion (i) has been proven.

$$
\begin{aligned}
A^{-1}BA &= \left(\begin{array}{c|ccc} & 1 & & O \\ O & & \ddots & \\ & O & & 1 \\ \hline x^{-1} & & O & \end{array}\right)
\begin{pmatrix} z & & & O \\ & z^{\alpha} & & \\ & & \ddots & \\ O & & & z^{(\alpha^{q-1})} \end{pmatrix}
\left(\begin{array}{ccc|c} & O & & x \\ \hline 1 & & O & \\ & \ddots & & O \\ O & & 1 & \end{array}\right) \\
&= \begin{pmatrix} z^{\alpha} & & & & O \\ & z^{(\alpha^2)} & & & \\ & & \ddots & & \\ & & & z^{(\alpha^{q-1})} & \\ O & & & & z \end{pmatrix} = B^{\alpha}
\end{aligned}
$$

Since $z \neq z^{\alpha}$, we have $AB \neq BA$. The assertion (ii) has been proven.

It is easy to see that the order of B is l. Since

$$
A^q = \begin{pmatrix} x & & O \\ & \ddots & \\ O & & x \end{pmatrix},
$$

the order of A is q^2. Since $BA = AB^{\alpha}$, we have

$$
G = \{A^r B^s | r = 0, 1, \ldots, q^2 - 1;\ s = 0, 1, \ldots, l - 1\}.
$$

In particular, the order of G is finite. The assertion (iii) has been proven.

Now, we shall show that 1 is not an eigenvalue of $A^r B^s$ for $r = 0, 1, \ldots, q^2 - 1$, $s = 0, 1, \ldots, l - 1$ except for the case $r = s = 0$.

Set

$$
r = uq + v,
$$

where u and v are integers such that $0 \leq u, v < q$.

First, assume $v = 0$. Since

$$
A^r B^s = x^u \begin{pmatrix} z^s & & & O \\ & z^{s\alpha} & & \\ & & \ddots & \\ O & & & z^{s\alpha^{q-1}} \end{pmatrix} = \begin{pmatrix} x^u z^s & & & O \\ & x^u z^{s\alpha} & & \\ & & \ddots & \\ O & & & x^u z^{s\alpha^{q-1}} \end{pmatrix},
$$

$\{x^u z^s,\ x^u z^{s\alpha},\ \dots\ , x^u z^{s\alpha^{q-1}}\}$ is the set of the eigenvalues of $A^r B^s$. Here assume that $x^u z^{s\alpha^t} = 1$ for some $0 \le t \le q-1$. Since q and l are relatively prime, we have

$$\begin{aligned} u &\equiv 0 \pmod{q} \\ s\alpha^t &\equiv 0 \pmod{l}. \end{aligned}$$

Therefore, we have $r = s = 0$.

Next assume $v \neq 0$.

$$\begin{aligned} A^r B^s &= (A^q)^u A^v B^s \\ &= \begin{array}{c} \begin{array}{cc} \overbrace{\hphantom{x^u \quad \ddots \quad x^u}}^{q-v} & \overbrace{\hphantom{x^{u+1} \quad \ddots \quad x^{u+1}}}^{v} \end{array} \\ \left(\begin{array}{c|c} O & \begin{matrix} x^{u+1} & & 0 \\ & \ddots & \\ 0 & & x^{u+1} \end{matrix} \\ \hline \begin{matrix} x^u & & 0 \\ & \ddots & \\ 0 & & x^u \end{matrix} & O \end{array}\right) \end{array} \begin{pmatrix} z^s & & & O \\ & z^{s\alpha} & & \\ & & \ddots & \\ O & & & z^{s\alpha^{q-1}} \end{pmatrix} \\ &= \left(\begin{array}{c|c} O & \begin{matrix} x^{u+1} z^{s\alpha^{q-v}} & & O \\ & \ddots & \\ O & & x^{u+1} z^{s\alpha^{q-1}} \end{matrix} \\ \hline \begin{matrix} x^u z^s & & O \\ & \ddots & \\ O & & x^u z^{s\alpha^{q-v-1}} \end{matrix} & O \end{array}\right). \end{aligned}$$

Therefore, we know that

$$\text{the } (i,j)\text{th entry of } tE - A^r B^s = \begin{cases} t & (i = j) \\ -x^u z^{s\alpha^{j-1}} & (i = j + v) \\ -x^{u+1} z^{s\alpha^{j-1}} & (i = j + v - q) \\ 0 & (\text{otherwise}). \end{cases}$$

For each j, the (i,j)th entry of $tE - A^r B^s$ is not 0 if and only if $i = j$ or $i \equiv j + v \pmod{q}$. Since q and v are relatively prime, we have

$$\begin{aligned} \det(tE - A^r B^s) &= t^q + (-1)^{q+v(q-v)} x^{uq+v} z^{s(1+\alpha+\cdots+\alpha^{q-1})} \\ &= t^q - x^v. \end{aligned}$$

Since $x^v \neq 1$, 1 is not an eigenvalue of $A^r B^s$. **Q.E.D.**

We define a group homomorphism

$$f : G \longrightarrow \mathrm{GL}(qn', \mathbb{C})$$

by

$$f(C) \;\; = \;\; \begin{array}{c} \begin{array}{cc} \overbrace{\hphantom{C \quad \ddots \quad O}}^{\frac{q+n'}{2}} & \hphantom{\overline{C} \quad \ddots \quad O} \end{array} \\ \left(\begin{array}{ccc|ccc} C & & O & & & \\ & \ddots & & & O & \\ O & & C & & & \\ \hline & & & \overline{C} & & O \\ & O & & & \ddots & \\ & & & O & & \overline{C} \end{array}\right) \\ \begin{array}{cc} \hphantom{C \quad \ddots \quad O} & \underbrace{\hphantom{\overline{C} \quad \ddots \quad O}}_{\frac{n'-q}{2}} \end{array} \end{array}$$

for each $C \in G$, where $\overline{C}$ is the complex conjugate matrix of C. Here, remember that n' is an odd number satisfying (4). If C is not the unit matrix, 1 is an eigenvalue of neither C nor $\overline{C}$. Therefore, if C is not the unit matrix, 1 is not an eigenvalue of $f(C)$.

On the other hand,

$$\det f(A) = (\det A)^{\frac{q+n'}{2}} (\det \bar{A})^{\frac{n'-q}{2}} = x^{\frac{q+n'}{2}} (x^{-1})^{\frac{n'-q}{2}} = x^q = 1$$

and, obviously $\det f(B) = 1$. Therefore, $f(G) \subset \mathrm{SL}(n, \mathbb{C})$. Since $AB \neq BA$,

$$f(A)f(B) \neq f(B)f(A).$$

Therefore, $f(G)$ is not abelian.

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