A local ring such that the map between Grothendieck groups with rational coefficients induced by completion is not injective

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Abstract

In this paper, we construct a local ring A such that the kernel of the map $G_0(A)_{\mathbb{Q}} \to G_0(\hat{A})_{\mathbb{Q}}$ is not zero, where $\hat{A}$ is the completion of A with respect to the maximal ideal, and $G_0(\ )_{\mathbb{Q}}$ is the Grothendieck group of finitely generated modules with rational coefficients. In our example, A is a two-dimensional local ring which is essentially of finite type over $\mathbb{C}$, but it is not normal.

1 Introduction

For a Noetherian ring R, we set

$$G_0(R) = \frac{\bigoplus_{M:\text{ f. g. } R\text{-mod.}} \mathbb{Z}[M]}{\langle [L] + [N] - [M] \,|\, 0 \to L \to M \to N \to 0 \text{ is exact} \rangle},$$

that is called the *Grothendieck group* of finitely generated R-modules. Here, $[M]$ denotes the free basis (corresponding to a finitely generated R-module M) of the free module $\bigoplus \mathbb{Z}[M]$, where $\mathbb{Z}$ is the ring of integers.

For a flat ring homomorphism $R \to A$, we have the induced map $G_0(R) \to G_0(A)$ defined by $[M] \mapsto [M \otimes_R A]$.

We are interested in the following problem (Question 1.4 in [7]):

Problem 1.1 Let R be a Noetherian local ring. Is the map $G_0(R)_{\mathbb{Q}} \to G_0(\widehat{R})_{\mathbb{Q}}$ injective?

Here, $\widehat{R}$ denotes the $\mathfrak{m}$-adic completion of R, where $\mathfrak{m}$ is the unique maximal ideal of R. For an abelian group N, $N_{\mathbb{Q}}$ denotes the tensor product with the field of rational numbers $\mathbb{Q}$.

We shall explain motivation and applications.

Assume that S is a regular scheme and X is a scheme of finite type over S. Then, by the singular Riemann-Roch theorem [3], we obtain an isomorphism

$$\tau_{X/S} : G_0(X)_{\mathbb{Q}} \xrightarrow{\sim} A_*(X)_{\mathbb{Q}},$$

where $G_0(X)$ (resp. $A_*(X)$) is the *Grothendieck group* of coherent sheaves on X (resp. *Chow group* of X). We refer the reader to Chapters 1, 18, 20 in [3] for definition of $G_0(X)$, $A_*(X)$ and $\tau_{X/S}$. Note that $G_0(X)$ (resp. $\tau_{X/S}$) is denoted by $K_0(X)$ (resp. τ_X) in [3]. The map $\tau_{X/S}$ usually depends on the choice of S. In fact, we have

$$\begin{array}{rcl} \tau_{\mathbb{P}^1_k/\mathbb{P}^1_k}([\mathcal{O}_{\mathbb{P}^1_k}]) & = & [\mathbb{P}^1_k] \in A_*(\mathbb{P}^1_k)_{\mathbb{Q}} = \mathbb{Q}[\mathbb{P}^1_k] \oplus \mathbb{Q}[t] \\ \tau_{\mathbb{P}^1_k/\operatorname{Spec} k}([\mathcal{O}_{\mathbb{P}^1_k}]) & = & [\mathbb{P}^1_k] + \chi(\mathcal{O}_{\mathbb{P}^1_k})[t] = [\mathbb{P}^1_k] + [t] \in A_*(\mathbb{P}^1_k)_{\mathbb{Q}}, \end{array}$$

where t is a k-rational closed point of $\mathbb{P}^1_k$ over a field k. Here, for a closed subvariety Y, $[Y]$ denotes the algebraic cycle corresponding to Y. Hence,

$$\tau_{\mathbb{P}^1_k/\mathbb{P}^1_k}([\mathcal{O}_{\mathbb{P}^1_k}]) \neq \tau_{\mathbb{P}^1_k/\operatorname{Spec} k}([\mathcal{O}_{\mathbb{P}^1_k}])$$

in this case. However, for a local ring R which is a homomorphic image of a regular local ring T, the map $\tau_{\operatorname{Spec} R/\operatorname{Spec} T}$ is independent of the choice of T in many cases. In fact, if R is a complete local ring or R is essentially of finite type over either a field or the ring of integers, it is proved in Propopsition 1.2 of [9] that the map $\tau_{\operatorname{Spec} R/\operatorname{Spec} T}$ is actually independent of T.

From now on, for simplicity, we denote $\tau_{\operatorname{Spec} R/\operatorname{Spec} T}$ by $\tau_{R/T}$. It is natural to ask the following:

Problem 1.2 Let R be a homomorphic image of a regular local ring T. Is the map $\tau_{R/T}$ independent of T?

Remark that, by the singular Riemann-Roch theorem, the diagram

$$\begin{array}{ccc} G_0(R)_{\mathbb{Q}} & \xrightarrow{\tau_{R/T}} & A_*(R)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ G_0(\widehat{R})_{\mathbb{Q}} & \xrightarrow{\tau_{\widehat{R}/\widehat{T}}} & A_*(\widehat{R})_{\mathbb{Q}} \end{array}$$

is commutative, where the vertical maps are induced by the completion $R \to \widehat{R}$. We want to emphasize that the bottom map, as well as the vertical maps is independent of the choice of T since $\widehat{R}$ is complete (Propopsition 1.2 of [9]). Therefore, if the vertical maps are injective, then the top map is also independent of T.

Therefore, if Problem 1.1 is affirmative, then so is Problem 1.2.

We shall explain another motivation.

Roberts [11] and Gillet-Soulé [4] proved the vanishing theorem of intersection multiplicities for complete intersections. If a local ring R is a complete intersection, then $\tau_{R/T}([R]) = [\mathrm{Spec}\, R]$ holds, where

$$[\mathrm{Spec}\, R] = \sum_{\substack{\mathfrak{p} \in \mathrm{Spec}\, R \\ \dim R/\mathfrak{p} = \dim R}} \ell_{R_\mathfrak{p}}(R_\mathfrak{p})[\mathrm{Spec}\, R/\mathfrak{p}] \in A_{\dim R}(R)_\mathbb{Q}.$$

In [11], Roberts proved the vanishing theorem of intersection multiplicities not only for complete intersections but also for local rings satisfying $\tau_{R/T}([R]) = [\mathrm{Spec}\, R]$. Inspired by his work, Kurano [9] started to study local rings which satisfy the condition $\tau_{R/T}([R]) = [\mathrm{Spec}\, R]$, and call them *Roberts* rings. If R is a Roberts ring, then the completion, the henselization and localizations of it are also Roberts rings [9]. However, the following problem remained open.

Problem 1.3 If $\widehat{R}$ is a Roberts ring, is R so?

It is proved in Proposition 6.2 of [7] that Problem 1.3 is affirmative if and only if so is Problem 1.1.

The following partial result on Problem 1.1 was given by Theorem 1.5 in [7]:

Theorem 1.4 (Kamoi-Kurano, 2001 [7]) *Let R be a homomorphic image of an excellent regular local ring. Assume that R satisfies one of the following three conditions:*
(i) R is henselian,
(ii) $R = S_\mathfrak{n}$, where S is a standard graded ring over a field and $\mathfrak{n} = \oplus_{n>0} S_n$,
(iii) R has only isolated singularity.
Then, the induced map $G_0(R) \to G_0(\widehat{R})$ is injective.

However, the following example was given by Hochster:

Example 1.5 (Hochster [6]) *Let k be a field. We set*

$$\begin{aligned} T &= k[x, y, u, v]_{(x,y,u,v)}, \\ P &= (x, y), \\ f &= xy - ux^2 - vy^2. \end{aligned}$$

Then, $\mathrm{Ker}(G_0(T/fT) \to G_0(\widehat{T/fT})) \ni [T/P] \neq 0$. *In this case,* $2 \cdot [T/P] = 0$.

The ring T/fT is not normal in the above example. Recently Dao [2] found the following example:

Example 1.6 (Dao [2]) *We set*

$$\begin{array}{rcl} R & = & \mathbb{R}[x,y,z,w]_{(x,y,z,w)}/(x^2+y^2-(w+1)z^2), \\ P & = & (x,y,z). \end{array}$$

Then, $\mathrm{Ker}(G_0(R) \to G_0(\widehat{R})) \ni [R/P] \neq 0$. *In this case,* $2 \cdot [R/P] = 0$. *Here,* R *is a normal local ring.*

The following is the main theorem of this paper:

Theorem 1.7 *There exists a* 2*-dimensional local ring* A*, which is essentially of finite type over* $\mathbb{C}$*, that satisfies*

$$\mathrm{Ker}(G_0(A)_{\mathbb{Q}} \to G_0(\widehat{A})_{\mathbb{Q}}) \neq 0.$$

Remark 1.8

1. By Theorem 1.7, we know that both Problem 1.1 and Problem 1.3 are negative. That is to say, there exists a local ring R such that $\widehat{R}$ is a Roberts ring, but R is not so.
2. Problem 1.2 is still open.
3. In [10], we defined notion of numerical equivalence on $G_0(R)$ and $A_*(R)$. We set $\overline{G_0(R)} = G_0(R)/\sim_{\mathrm{num.}}$ and $\overline{A_*(R)} = A_*(R)/\sim_{\mathrm{num.}}$. Then, we have the following:
   (a) $\overline{G_0(R)} \to \overline{G_0(\widehat{R})}$ is injective for any local ring R.
   (b) The induced map $\overline{\tau_{R/T}} : \overline{G_0(R)}_{\mathbb{Q}} \xrightarrow{\sim} \overline{A_*(R)}_{\mathbb{Q}}$ is independent of T.
   (c) R is a numerically Roberts ring iff so is $\widehat{R}$. (Definition of numerically Roberts rings was given in [10]. The vanishing theorem of intersection multiplicities holds true for numerically Roberts rings.)
4. The ring A constructed in the main theorem is not normal. We do not know any example of a normal local ring that does not satisfy Problem 1.1.

Theorem 1.7 immediately follows from the following two lemmas.

Lemma 1.9 *Let* K *be an algebraically closed field, and let* $S = \oplus_{n \geq 0} S_n$ *be a standard graded ring over* K*, that is, a Noetherian graded ring generated by* S_1 *over* $S_0 = K$*. We set* $X = \mathrm{Proj}\, S$*, and assume that* X *is smooth over* K *with* $d = \dim X \geq 1$*. Let* h *be the very ample divisor on* X *of this embedding. Let* $\pi : Y \to \mathrm{Spec}\, S$ *be the blow-up at* $\mathfrak{n} = \oplus_{n>0} S_n$*.*

Assume the following:

1. *Set* $R = S_{\mathfrak{n}}$ *and let* $\widehat{R}$ *be the completion of* R*. Then, the map* $A_1(R)_{\mathbb{Q}} \to A_1(\widehat{R})_{\mathbb{Q}}$ *induced by completion is an isomorphism.*

2. *There exists a smooth connected curve C in Y that satisfies following two conditions:*
   (i) *C transversally intersects with $\pi^{-1}(\mathfrak{n}) \simeq X$ at two points, namely P_1 and P_2.*
   (ii) *$[P_1] - [P_2] \neq 0$ in $A_0(X)_{\mathbb{Q}}/h \cdot A_1(X)_{\mathbb{Q}}$.*

Then, there exists a local ring A of dimension $d+1$, which is essentially of finite type over K, such that

$$\mathrm{Ker}(G_0(A)_{\mathbb{Q}} \to G_0(\widehat{A})_{\mathbb{Q}}) \neq 0.$$

Lemma 1.10 *We set $S = \mathbb{C}[x_0, x_1, x_2]/(f)$, where f is a homogeneous cubic polynomial. Assume that $X = \mathrm{Proj}\, S$ is smooth over $\mathbb{C}$.*

Then, R satisfies the assumption in Lemma 1.9 with $d = 1$.

We shall prove the above two lemmas in the following sections.

2 A proof of Lemma 1.9

Here, we shall give a proof of Lemma 1.9.

Let $\mathfrak{p}$ be the prime ideal of S that satisfies $\mathrm{Spec}\, S/\mathfrak{p} = \pi(C)$. Set $R = S_{\mathfrak{n}}$ and $\mathfrak{m} = \mathfrak{n}R$.

Then, C is the normalization of $\mathrm{Spec}\, S/\mathfrak{p}$. We denote by v_i the normalized valuation of the discrete valuation ring at $P_i \in C$ for $i = 1, 2$.

First of all, we shall prove the following:

Claim 2.1 *There exists $s \in \mathfrak{m}/\mathfrak{p}R$ such that*

1. *$v_1(s) = v_2(s) > 0$, and*
2. *$K[s]_{(s)} \hookrightarrow R/\mathfrak{p}R$ is finite.*

Proof. Let C' be the smooth projective connected curve over K that contains C as a Zariski open set. We regard P_1, P_2 as points of C'.

Let $R(C')$ be the field of rational functions on C'. Since P_1 is an ample divisor on C', there exists $t_1 \in R(C')^\times$ such that

- P_1 is the only pole of t_1, and
- P_2 is neither a zero nor a pole of t_1.

Similarly, one can find $t_2 \in R(C')^\times$ such that

- P_2 is the only pole of t_2, and
- P_1 is neither a zero nor a pole of t_2.

Replacing t_1 (resp. t_2) with a suitable powers of t_1 (resp. t_2), we may assume $v_1(t_1) = v_2(t_2) < 0$.

Put $t = 1/t_1t_2 \in R(C')^\times$. Then, $\{P_1, P_2\}$ is the set of zeros of t. Note that $v_1(t) = v_2(t) > 0$.

Let O_{v_i} be the discrete valuation ring at P_i for $i = 1, 2$. Then, $K[t]_{(t)}$ is a subring of

$$O_{v_1} \cap O_{v_2} = \overline{S/\mathfrak{p}} \otimes_{S/\mathfrak{p}} R/\mathfrak{p}R,$$

where $\overline{(\ )}$ is the normalization of the given ring.

Since $\{P_1, P_2\}$ is just the set of zeros of t, $O_{v_1} \cap O_{v_2}$ is the integral closure of $K[t]_{(t)}$ in $R(C')$. In particular, $\overline{S/\mathfrak{p}} \otimes_{S/\mathfrak{p}} R/\mathfrak{p}R$ is finite over $K[t]_{(t)}$.

Let I be the conductor ideal of the normalization

$$R/\mathfrak{p}R \subset \overline{S/\mathfrak{p}} \otimes_{S/\mathfrak{p}} R/\mathfrak{p}R.$$

Let $\mathfrak{m}_i$ be the maximal ideal of $\overline{S/\mathfrak{p}} \otimes_{S/\mathfrak{p}} R/\mathfrak{p}R$ corresponding to P_i for $i = 1, 2$. Since I is contained in $\mathfrak{m}/\mathfrak{p}R$,

$$I \subset \mathfrak{m}_1 \cap \mathfrak{m}_2.$$

Therefore, we have

$$\sqrt{I} = \mathfrak{m}_1 \cap \mathfrak{m}_2 \ni t.$$

Thus, t^n is contained in I for a sufficiently large n. In particular, t^n is in $\mathfrak{m}/\mathfrak{p}R$.

Consider the following commutative diagram:

$$\begin{array}{ccc} K[t^n]_{(t^n)} & \longrightarrow & R/\mathfrak{p}R \\ \downarrow & & \downarrow \\ K[t]_{(t)} & \longrightarrow & \overline{S/\mathfrak{p}} \otimes_{S/\mathfrak{p}} R/\mathfrak{p}R \end{array}$$

The morphism in the left-hand-side, as well as the bottom one is finite. Hence, all morphisms are finite.

Put $s = t^n$. Then, s satisfies all the requirements. **q.e.d.**

Let $R \xrightarrow{\xi} R/\mathfrak{p}R$ be the natural surjective morphism. We set $A = \xi^{-1}(K[s]_{(s)})$.

$$\begin{array}{ccc} R & \xrightarrow{\xi} & R/\mathfrak{p}R \\ \uparrow & \square & \uparrow \\ A & \rightarrow & K[s]_{(s)} \end{array}$$

In the rest of this section, we shall prove that the ring A satisfies the required condition.

Next we shall prove the following:

Claim 2.2 *The morphism $A \to R$ is finite birational, and A is essentially of finite type over K of dimension $d + 1$.*

Proof. Remark that
$$A \supset \mathfrak{p}R \neq 0$$
since the dimension of R is at least 2. Take $0 \neq a \in \mathfrak{p}R$. Since $A[a^{-1}] = R[a^{-1}]$, $A \to R$ is birational.

One can prove that A is a Noetherian ring by Eakin-Nagata's theorem. However, here, we shall prove that A is essentially of finite type over K without using Eakin-Nagata's theorem.

Let B be the integral closure of $K[s]$ in $R/\mathfrak{p}R$. Remark that B is of finite type over K.

Since $R/\mathfrak{p}R$ is finite over $K[s]_{(s)}$, $B \otimes_{K[s]} K[s]_{(s)} = R/\mathfrak{p}R$.

$$\begin{array}{ccccc} R & \xrightarrow{\xi} & R/\mathfrak{p}R & \longleftarrow & B \\ \uparrow & & \uparrow & & \uparrow \\ S & & K[s]_{(s)} & \longleftarrow & K[s] \end{array}$$

Take an element $s' \in R$ that satisfies $\xi(s') = s$. Suppose $S = K[s_1, \ldots, s_n]$. Since $B \otimes_{K[s]} K[s]_{(s)} = R/\mathfrak{p}R$, there exist $g_i \in B$ and $f_i \in K[s] \setminus (s)$ such that $\xi(s_i) = g_i/f_i$ for $i = 1, \ldots, n$. Take an element $f'_i \in K[s']$ such that $\xi(f'_i) = f_i$ for $i = 1, \ldots, n$. Put
$$S' = K[s', s_1 f'_1, \ldots, s_n f'_n].$$
Remark that R is a localization of S', and $\xi(S') \subset B$. Since B is of finite type over K, there exists a ring D that satisfies

- $S' \subset D \subset R$
- D is of finite type over K,
- R is a localization of D, and
- $\xi(D) = B$.

Put $\phi = \xi|_D$ and $E = \phi^{-1}(K[s])$. Then, D is finite over E.

$$\begin{array}{ccc} D & \xrightarrow{\phi} & B \\ \uparrow & \square & \uparrow \\ E & \to & K[s] \end{array}$$

Since $B \otimes_{K[s]} K[s]_{(s)} = R/\mathfrak{p}R$, there is only one prime ideal N of B lying over $(s) \subset K[s]$. Therefore, $\phi^{-1}(N)$ is the only one prime ideal lying over the prime ideal $\phi^{-1}((s))$ of E. Localizing all the rings in the above diagram, we have the following diagram:

$$\begin{array}{ccc} D \otimes_E E_{\phi^{-1}((s))} & \longrightarrow & B \otimes_E E_{\phi^{-1}((s))} \\ \uparrow & \square & \uparrow \\ E_{\phi^{-1}((s))} & \to & K[s] \otimes_E E_{\phi^{-1}((s))} \end{array}$$

Remark that $D \otimes_E E_{\phi^{-1}((s))} = R$, $K[s] \otimes_E E_{\phi^{-1}((s))} = K[s]_{(s)}$ and $B \otimes_E E_{\phi^{-1}((s))} = R/\mathfrak{p}R$. Therefore, A coincides with $E_{\phi^{-1}((s))}$.

Since D is finite over E and D is of finite type over K, E is also of finite type over K.

Therefore, we know that A is essentially of finite type over K and R is finite over A. It is easy to see

$$\dim A = \dim R = \dim S = d + 1.$$

q.e.d.

In particular, A is a homomorphic image of a regular local ring T. Therefore, we have the commutative diagram

$$\begin{array}{ccc} G_0(A)_{\mathbb{Q}} & \stackrel{\tau_{A/T}}{\longrightarrow} & A_*(A)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ G_0(\widehat{A})_{\mathbb{Q}} & \stackrel{\tau_{\widehat{A}/\widehat{T}}}{\longrightarrow} & A_*(\widehat{A})_{\mathbb{Q}} \end{array}$$

by the singular Riemann-Roch theorem (Chapter 18, 20 in [3]). Remark that the horizontal maps in the above diagram are isomorphisms. Therefore, in order to prove that $\mathrm{Ker}(G_0(A)_{\mathbb{Q}} \to G_0(\widehat{A})_{\mathbb{Q}})$ is not 0, it is sufficient to prove that $\mathrm{Ker}(A_1(A)_{\mathbb{Q}} \to A_1(\widehat{A})_{\mathbb{Q}})$ is not 0.

The diagram

$$\begin{array}{ccc} R & \longrightarrow & \widehat{R} \\ \uparrow & & \uparrow \\ A & \longrightarrow & \widehat{A} \end{array}$$

induces the commutative diagram

$$\begin{array}{ccc} A_1(R)_{\mathbb{Q}} & \longrightarrow & A_1(\widehat{R})_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ A_1(A)_{\mathbb{Q}} & \longrightarrow & A_1(\widehat{A})_{\mathbb{Q}} \end{array} \qquad (1)$$

where the vertical maps are induced by the finite morphisms $A \to R$ and $\widehat{A} \to \widehat{R}$, and the horizontal maps are induced by the completions $A \to \widehat{A}$ and $R \to \widehat{R}$.

The top map in the diagram (1) is an isomorphism by assumption 1 of Lemma 1.9.

Here we shall show, for each prime ideal of A, there exists only one prime ideal of R lying over it. Let $\mathfrak{q}$ be a prime ideal of A. Recall that the conductor ideal $\mathfrak{p}R$ is a prime ideal of both A and R. If $\mathfrak{q}$ does not contain $\mathfrak{p}R$, then $A_{\mathfrak{q}}$ coincides with $R \otimes_A A_{\mathfrak{q}}$. Therefore there exists only one prime ideal of R lying over $\mathfrak{q}$ in this case. Next suppose that $\mathfrak{q}$ contains $\mathfrak{p}R$. Then $\mathfrak{q}$ is either $\mathfrak{p}R$ or the unique maximal ideal of A. In any cases, there exists only one prime ideal of R lying over $\mathfrak{q}$.

Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Rat_1(R) & \longrightarrow & Z_1(R) & \longrightarrow & A_1(R) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Rat_1(A) & \longrightarrow & Z_1(A) & \longrightarrow & A_1(A) & \longrightarrow & 0 \end{array}$$

We refer the reader to Chapter 1 in [3] for definition of Rat_* and Z_*. Since the morphism $A \to R$ is finite injective, the cokernel of $Rat_1(R) \to Rat_1(A)$ is torsion by Proposition 1.4 of Chapter 1 in [3]. Since, for each prime ideal of A, there is only one prime ideal of R lying over it, the map $Z_1(R) \to Z_1(A)$ is injective and the cokernel of it is a torsion module $\mathbb{Z}/(2v)$, where $v = v_1(s) = v_2(s)$. Therefore the map in the left-hand-side in diagram (1) is also an isomorphism.

By the commutativity of diagram (1), we know that, in order to prove that $\mathrm{Ker}(A_1(A)_{\mathbb{Q}} \to A_1(\widehat{A})_{\mathbb{Q}})$ is not 0, it is sufficient to show that

$$\mathrm{Ker}(A_1(\widehat{R})_{\mathbb{Q}} \to A_1(\widehat{A})_{\mathbb{Q}}) = \mathbb{Q}.$$

Since $\widehat{A}/(\mathfrak{p}R)\widehat{A} = \widehat{K[s]_{(s)}} = K[[s]]$, $(\mathfrak{p}R)\widehat{A}$ is a prime ideal of $\widehat{A}$ of height d. We have the following bijective correspondences:

$$\begin{aligned} & \text{the set of prime ideals of } \widehat{R} \text{ lying over } (\mathfrak{p}R)\widehat{A} \\ \longleftrightarrow\ & \text{the set of minimal prime ideals of } \widehat{R/\mathfrak{p}R} \\ \longleftrightarrow\ & \text{the set of maximal ideals of } \overline{S/\mathfrak{p}} \otimes_{S/\mathfrak{p}} R/\mathfrak{p}R \\ \longleftrightarrow\ & \{P_1, P_2\}, \end{aligned}$$

where $\overline{S/\mathfrak{p}} \otimes_{S/\mathfrak{p}} R/\mathfrak{p}R$ is the normalization of $R/\mathfrak{p}R$. Therefore, there are just two prime ideals of $\widehat{R}$ lying over $(\mathfrak{p}R)\widehat{A}$. We denote them by $\mathfrak{p}_1$ and $\mathfrak{p}_2$.

It is easy to see that $\mathfrak{p}R$ is the conductor ideal of the ring extension $A \subset R$, that is,

$$\mathfrak{p}R = A :_A R.$$

Then, $(\mathfrak{p}R)\widehat{A} = \widehat{A} :_{\widehat{A}} \widehat{R}$ is satisfied. Therefore, $(\mathfrak{p}R)\widehat{A}$ is the conductor ideal of the ring extension $\widehat{A} \subset \widehat{R}$. Consider the map

$$\varphi : Z_1(\widehat{R}) \longrightarrow Z_1(\widehat{A}).$$

Let $\mathfrak{q}$ be a prime ideal of $\widehat{A}$ of height d. If $\mathfrak{q}$ does not contain the conductor ideal $(\mathfrak{p}R)\widehat{A}$, then there exists only one prime ideal $\mathfrak{q}'$ of $\widehat{R}$ lying over $\mathfrak{q}$. Furthermore, $\widehat{A}/\mathfrak{q}$ is birational to $\widehat{R}/\mathfrak{q}'$. Therefore,

$$\varphi([\mathrm{Spec}\, \widehat{R}/\mathfrak{q}']) = [\mathrm{Spec}\, \widehat{A}/\mathfrak{q}].$$

Here, we shall show

$$\varphi([\mathrm{Spec}\, \widehat{R}/\mathfrak{p}_1]) = \varphi([\mathrm{Spec}\, \widehat{R}/\mathfrak{p}_2]) = v[\mathrm{Spec}\, \widehat{A}/(\mathfrak{p}R)\widehat{A}],$$

where $v = v_1(s) = v_2(s)$. Recall that

$$\widehat{O_{v_1}} \times \widehat{O_{v_2}} = (\overline{R/\mathfrak{p}R})^{\wedge} = \overline{\widehat{R}/\mathfrak{p}\widehat{R}} = \overline{\widehat{R}/\mathfrak{p}_1} \times \overline{\widehat{R}/\mathfrak{p}_2}.$$

Therefore, we may assume $\widehat{O_{v_i}} \simeq \overline{\widehat{R}/\mathfrak{p}_i}$ for $i = 1, 2$. Then, we have

$$\begin{aligned} & \operatorname{rank}_{\widehat{A}/(\mathfrak{p}R)\widehat{A}} \widehat{R}/\mathfrak{p}_i = \operatorname{rank}_{\widehat{A}/(\mathfrak{p}R)\widehat{A}} \overline{\widehat{R}/\mathfrak{p}_i} = \operatorname{rank}_{\widehat{A}/(\mathfrak{p}R)\widehat{A}} \widehat{O_{v_i}} = \operatorname{rank}_{K[[s]]} \widehat{O_{v_i}} \\ = \; & \dim_K \widehat{O_{v_i}}/s\widehat{O_{v_i}} = \dim_K O_{v_i}/sO_{v_i} = v \end{aligned}$$

for $i = 1, 2$. Here, $\dim_K$ means the dimension of the given K-vector space.

Thus, we have the following exact sequence

$$0 \longrightarrow \mathbb{Z} \cdot ([\operatorname{Spec} \widehat{R}/\mathfrak{p}_1] - [\operatorname{Spec} \widehat{R}/\mathfrak{p}_2]) \longrightarrow Z_1(\widehat{R}) \longrightarrow Z_1(\widehat{A}) \longrightarrow \mathbb{Z}/(v) \longrightarrow 0.$$

Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Rat_1(\widehat{R}) & \longrightarrow & Z_1(\widehat{R}) & \longrightarrow & A_1(\widehat{R}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Rat_1(\widehat{A}) & \longrightarrow & Z_1(\widehat{A}) & \longrightarrow & A_1(\widehat{A}) & \longrightarrow & 0 \end{array}$$

Since the morphism $\widehat{A} \to \widehat{R}$ is finite injective, the cokernel of $Rat_1(\widehat{R}) \to Rat_1(\widehat{A})$ is torsion (c.f. Proposition 1.4 in [3]). Thus, we have the following exact sequence

$$0 \longrightarrow \mathbb{Q} \cdot ([\operatorname{Spec} \widehat{R}/\mathfrak{p}_1] - [\operatorname{Spec} \widehat{R}/\mathfrak{p}_2]) \longrightarrow A_1(\widehat{R})_{\mathbb{Q}} \longrightarrow A_1(\widehat{A})_{\mathbb{Q}} \longrightarrow 0.$$

Therefore, we have only to prove

$$[\operatorname{Spec} \widehat{R}/\mathfrak{p}_1] - [\operatorname{Spec} \widehat{R}/\mathfrak{p}_2] \neq 0$$

in $A_1(\widehat{R})_{\mathbb{Q}}$.

Let $\widehat{\pi} : \widehat{Y} \to \operatorname{Spec} \widehat{R}$ be the blow-up at $\mathfrak{m}\widehat{R}$. Since $\widehat{\pi}^{-1}(\mathfrak{m}\widehat{R}) \simeq X$,

$$A_1(X)_{\mathbb{Q}} \xrightarrow{i_*} A_1(\widehat{Y})_{\mathbb{Q}} \xrightarrow{\widehat{\pi}_*} A_1(\widehat{R})_{\mathbb{Q}} \to 0$$

is exact and

$$\widehat{\pi}_* \left([\operatorname{Spec} \overline{\widehat{R}/\mathfrak{p}_1}] - [\operatorname{Spec} \overline{\widehat{R}/\mathfrak{p}_2}] \right) = [\operatorname{Spec} \widehat{R}/\mathfrak{p}_1] - [\operatorname{Spec} \widehat{R}/\mathfrak{p}_2],$$

where $i : X \to \widehat{Y}$ is the inclusion. Consider the following commutative diagram:

$$\begin{array}{ccccc} P_i & \longrightarrow & \{P_1, P_2\} & \longrightarrow & X \\ \downarrow & \square & \downarrow & \square & \downarrow \\ \operatorname{Spec} O_{v_i} & \longrightarrow & \operatorname{Spec} \overline{R/\mathfrak{p}} & \longrightarrow & Y \\ & & \downarrow & & \downarrow \\ & & \operatorname{Spec} R/\mathfrak{p} & \longrightarrow & \operatorname{Spec} R \end{array}$$

Take the fibre product with $\operatorname{Spec}\widehat{R}$ over $\operatorname{Spec}R$. We may assume that $\operatorname{Spec}\overline{\widehat{R}/\mathfrak{p}_i}$ coincides with $\operatorname{Spec}\widehat{O_{v_i}}$ for $i = 1, 2$ so that the following diagram commutes:

$$
\begin{array}{ccccccccc}
 & & P_i & = & P_i & \longrightarrow & \{P_1, P_2\} & \longrightarrow & X \\
 & & \downarrow & \square & \downarrow & \square & \downarrow & \square & \downarrow \\
\operatorname{Spec}\overline{\widehat{R}/\mathfrak{p}_i} & = & \operatorname{Spec}\widehat{O_{v_i}} & \longrightarrow & \operatorname{Spec}O_{v_i}\otimes_R \widehat{R} & \longrightarrow & \operatorname{Spec}\overline{R/\mathfrak{p}}\otimes_R \widehat{R} & \longrightarrow & \widehat{Y} \\
\downarrow & & & & & & \downarrow & & \downarrow \\
\operatorname{Spec}\widehat{R}/\mathfrak{p}_i & & & & \longrightarrow & & \operatorname{Spec}\widehat{R}/\mathfrak{p}\widehat{R} & \longrightarrow & \operatorname{Spec}\widehat{R}
\end{array}
$$

Assume that

$$[\operatorname{Spec}\widehat{R}/\mathfrak{p}_1] - [\operatorname{Spec}\widehat{R}/\mathfrak{p}_2] = 0$$

in $A_1(\widehat{R})_{\mathbb{Q}}$. Then, there exists $\delta \in A_1(X)_{\mathbb{Q}}$ such that

$$i_*(\delta) = [\operatorname{Spec}\overline{\widehat{R}/\mathfrak{p}_1}] - [\operatorname{Spec}\overline{\widehat{R}/\mathfrak{p}_2}].$$

Here, consider the map

$$A_1(\widehat{Y})_{\mathbb{Q}} \xrightarrow{i^!} A_0(X)_{\mathbb{Q}},$$

that is taking the intersection with $\widehat{\pi}^{-1}(\mathfrak{m}\widehat{R}) = X$. Since $i^! i_*(\delta) = -h \cdot \delta$ and

$$i^!\left([\operatorname{Spec}\overline{\widehat{R}/\mathfrak{p}_1}] - [\operatorname{Spec}\overline{\widehat{R}/\mathfrak{p}_2}]\right) = i^!\left([\operatorname{Spec}\widehat{O_{v_1}}] - [\operatorname{Spec}\widehat{O_{v_2}}]\right) = [P_1] - [P_2],$$

we have

$$[P_1] - [P_2] = -h \cdot \delta.$$

It contradicts to

$$[P_1] - [P_2] \neq 0$$

in $A_0(X)_{\mathbb{Q}}/h \cdot A_1(X)_{\mathbb{Q}}$.

We have completed the proof of Lemma 1.9.

3 A proof of Lemma 1.10

We shall give a proof of Lemma 1.10 in this section.

Suppose that $S = \mathbb{C}[x_0, x_1, x_2]/(f)$ and $X = \operatorname{Proj}S$ satisfy the assumption in Lemma 1.10. Let Z be the projective cone of X, that is, $Z = \operatorname{Proj}\mathbb{C}[x_0, x_1, x_2, x_3]/(f)$.

Let $W \xrightarrow{\xi} Z$ be the blow-up at $(0, 0, 0, 1)$. We set $X_\infty = V_+(x_3)$ and $X_0 = \xi^{-1}((0, 0, 0, 1))$. Remark that both of X_0 and X_∞ are isomorphic to X. Then, $W \xrightarrow{\eta} X$ is a $\mathbb{P}^1$-bundle.

Take any two closed points $Q_1, Q_2 \in X$. We set $L_i = \eta^{-1}(Q_i)$ for $i = 1, 2$. Consider the Weil divisor $L_1 + L_2 + X_\infty$ on W. Here we shall prove the following:

Claim 3.1 *The complete linear system* $|L_1 + L_2 + X_\infty|$ *is base-point free, and the induced morphism* $W \xrightarrow{f} \mathbb{P}^n$ *satisfies that* $\dim f(W) \geq 2$.

Proof. Since the complete linear system $|Q_1 + Q_2|$ on X is base-point free, so is $|L_1{+}L_2|$. Since the complete linear system $|X_\infty|$ is base-point free, so is $|L_1{+}L_2{+}X_\infty|$.

In order to show $\dim f(W) \geq 2$, we have only to show that the set

$$\{a \in R(W)^\times \mid \operatorname{div}(a) + L_1 + L_2 + X_\infty \geq 0\}$$

contains two algebraically independent elements over $\mathbb{C}$.

Note that, since $W \xrightarrow{\eta} X$ is a surjective morphism, $R(X)$ is contained in $R(W)$. Consider

$$\begin{aligned} H^0(W, \mathcal{O}_W(L_1 + L_2 + X_\infty)) &= \{a \in R(W)^\times \mid \operatorname{div}(a) + L_1 + L_2 + X_\infty \geq 0\} \cup \{0\} \\ H^0(X, \mathcal{O}_X(Q_1 + Q_2)) &= \{a \in R(X)^\times \mid \operatorname{div}(a) + Q_1 + Q_2 \geq 0\} \cup \{0\}. \end{aligned}$$

It is easy to see

$$H^0(W, \mathcal{O}_W(L_1 + L_2 + X_\infty)) \supset H^0(X, \mathcal{O}_X(Q_1 + Q_2)) \supset \mathbb{C}.$$

The set $H^0(X, \mathcal{O}_X(Q_1 + Q_2))$ contains a transcendental element over $\mathbb{C}$. Since $R(X)$ is algebraically closed in $R(W)$ and

$$H^0(W, \mathcal{O}_W(L_1 + L_2 + X_\infty)) \neq H^0(X, \mathcal{O}_X(Q_1 + Q_2)),$$

$H^0(W, \mathcal{O}_W(L_1 + L_2 + X_\infty))$ contains two algebraically independent elements over $\mathbb{C}$. **q.e.d.**

Since $|L_1 + L_2 + X_\infty|$ is base-point free as in Claim 3.1,

$$\operatorname{div}(a) + L_1 + L_2 + X_\infty$$

is smooth for a general element $a \in H^0(W, \mathcal{O}_W(L_1 + L_2 + X_\infty)) \setminus \{0\}$ (e.g., III Corollary 10.9 in [5]). Since $\dim f(W) \geq 2$ as in Claim 3.1,

$$\operatorname{div}(a) + L_1 + L_2 + X_\infty$$

is connected for any $a \in H^0(W, \mathcal{O}_W(L_1 + L_2 + X_\infty)) \setminus \{0\}$ (e.g., III Exercise 11.3 in [5]).

Let $\{a_1, \cdots, a_n\}$ be a $\mathbb{C}$-basis of $H^0(W, \mathcal{O}_W(L_1 + L_2 + X_\infty))$. Let α_i be the local equation defining the Cartier divisor $\operatorname{div}(a_i) + L_1 + L_2 + X_\infty$ for $i = 1, \ldots, n$. For $c = (c_1, \ldots, c_n) \in \mathbb{C}^n \setminus \{(0, \ldots, 0)\}$, D_c denotes the Cartier divisor on W defined by $c_1\alpha_1 + \cdots + c_n\alpha_n$.

For a general point $c \in \mathbb{C}^n$, D_c does not contain X_0 as a component and D_c intersect with X_0 at two distinct points. Recall that X_0 is isomorphic to X. Set $D_c \cap X_0 = \{Q_{c1}, Q_{c2}\} \subset X$.

Choose $e \in X$ such that the Weil divisor $3e$ coincides with the very ample divisor corresponding to the embedding $X = \operatorname{Proj} S$. We regard the set of closed points of the elliptic curve X as an abelian group with unit e as in the usual way.

Let $\varphi : X \to \mathbb{P}^1_{\mathbb{C}}$ be the morphism defined by $|2e|$.

For a general point $c \in \mathbb{C}^n$, we set

$$\theta(c) = \varphi(Q_{c1} \ominus Q_{c2}) \in \mathbb{P}^1_{\mathbb{C}},$$

where $\ominus$ means the difference in the group X. One can prove that there exists a non-empty Zariski open set U of $\mathbb{C}^n$ such that $\theta|_U : U \to \mathbb{P}^1_{\mathbb{C}}$ is a non-constant morphism and D_c is smooth connected for any $c \in U$. Then, there exists a a non-empty Zariski open set of $\mathbb{P}^1_{\mathbb{C}}$ which is contained in $\operatorname{Im}(\theta|_U)$. Let F be the set of elements of X of order finite. Then, it is well-known that F is a countable set. In particular, $\varphi(F)$ does not contain $\operatorname{Im}(\theta|_U)$. Therefore, there exists $c \in U$ such that $\theta(c) \notin \varphi(F)$. Then, D_c is a smooth connected curve in W such that D_c intersect with $X_0 \simeq X$ at two points $\{P_1, P_2\}$ transversally such that $P_1 \ominus P_2$ has order infinite in X.

Let $\phi : X \to A_0(X)$ be a map defined by $\phi(P) = [P] - [e]$. It is well known that ϕ is a group homomorphism. We have the following exact sequence:

$$0 \longrightarrow X \xrightarrow{\phi} A_0(X) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0$$

Since $\deg(h) = 3$, we have an isomorphism

$$X \otimes_{\mathbb{Z}} \mathbb{Q} \overset{\overline{\phi}}{\simeq} A_0(X)_{\mathbb{Q}}/hA_1(X)_{\mathbb{Q}}.$$

By definition, we have

$$0 \neq \overline{\phi}(P_1 \ominus P_2) = [P_1] - [P_2]$$

in $A_0(X)_{\mathbb{Q}}/hA_1(X)_{\mathbb{Q}}$.

Let Y be the blow-up of $\operatorname{Spec} S$ at the origin. Then, Y is an open subvariety of W. We set $C = D_c \cap Y$. Then C satisfies assumption 2 in Lemma 1.10.

Since $H^1(X, \mathcal{O}_X(n)) = 0$ for $n > 0$, we have $\operatorname{Cl}(R) \simeq \operatorname{Cl}(\widehat{R})$ by Danilov's Theorem (Corollary in 497p and Proposition 8 in [1]). Therefore, R satisfies assumption 1 in Lemma 1.9.

We have completed the proof the Lemma 1.10.

Remark 3.2 Let A be a 2-dimensional local ring constructed using Lemma 1.9 and Lemma 1.10. Since A and $\widehat{A}$ are 2-dimensional excellent local domains, we have the following isomorphisms:

$$\begin{array}{rcl} G_0(A) & \simeq & \mathbb{Z} \oplus A_1(A) \\ G_0(\widehat{A}) & \simeq & \mathbb{Z} \oplus A_1(\widehat{A}) \end{array}$$

Therefore,

$$\operatorname{Ker}(G_0(A) \to G_0(\widehat{A})) \simeq \operatorname{Ker}(A_1(A) \to A_1(\widehat{A})).$$

Using it, we can prove that

$$\mathrm{Ker}(G_0(A) \to G_0(\widehat{A})) \simeq \mathbb{Z}$$

as follows. Consider the following diagram

$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
0 & & \mathbb{Z} \\
\downarrow & & \downarrow \\
A_1(R) & \underset{\sim}{\xrightarrow{f}} & A_1(\widehat{R}) \\
\downarrow i & & \downarrow \\
A_1(A) & \xrightarrow{g} & A_1(\widehat{A}) \\
\downarrow & & \downarrow \\
\mathbb{Z}/(2v) & \longrightarrow & \mathbb{Z}/(v) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Let α_i be the element of $A_1(R)$ such that $f(\alpha_i) = [\mathrm{Spec}\, \widehat{R}/\mathfrak{p}_i]$ for $i = 1, 2$. Then, the kernel of g is generated by

$$i(\alpha_1) - v[\mathrm{Spec}\, A/\mathfrak{p}R].$$

Here, note that

$$2\,(i(\alpha_1) - v[\mathrm{Spec}\, A/\mathfrak{p}R]) = i(\alpha_1) - i(\alpha_2).$$

Since the kernel of g is not torsion, it must be isomorphic to $\mathbb{Z}$.

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