# On maps of Grothendieck groups induced by completion

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**Abstract**

We prove that the induced map $\mathrm{G}_0(A) \to \mathrm{G}_0(\hat{A})$ by completion is injective if $A$ is an excellent Noetherian local ring that satisfies one of the following three conditions; (i) $A$ is henselian, (ii) $A$ is a local ring at the homogeneous maximal ideal of a homogeneous ring over a field, (iii) $A$ has at most isolated singularity.

## 1 Introduction

In this paper, we discuss the injectivity of the map $\mathrm{G}_0(A) \to \mathrm{G}_0(\hat{A})$ induced by completion $A \to \hat{A}$. The problem is closely related to the theory of *Roberts rings* as follows.

For a scheme $X$ that is of finite type over a regular scheme $S$, we have an isomorphism of $\mathbb{Q}$-vector spaces

$$\tau_{X/S} : \mathrm{G}_0(X)_{\mathbb{Q}} \to \mathrm{A}_*(X)_{\mathbb{Q}}$$

by the singular Riemann-Roch theorem (Chapter 18 and 20 in Fulton [2]), where $\mathrm{G}_0(X)$ (resp. $\mathrm{A}_*(X)$) denotes the Grothendieck group of coherent $\mathcal{O}_X$-modules

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(resp. Chow group of $X$). This is the natural generalization of the Grothendieck-Riemann-Roch theorem to singular schemes. Usually the map $\tau_{X/S}$ depends not only on $X$ but also on $S$ (see Section 6).

Let $T$ be a regular local ring and let $A$ be a homomorphic image of $T$. Since $A$ is of finite type over $T$, we have an isomorphism of $\mathbb{Q}$-vector spaces

$$\tau_{\operatorname{Spec} A/\operatorname{Spec} T} : \mathrm{G}_0(\operatorname{Spec} A)_{\mathbb{Q}} \to \mathrm{A}_*(\operatorname{Spec} A)_{\mathbb{Q}}$$

by the singular Riemann-Roch theorem as above. We denote $\tau_{\operatorname{Spec} A/\operatorname{Spec} T}$, $\mathrm{G}_0(\operatorname{Spec} A)$ and $\mathrm{A}_*(\operatorname{Spec} A)$ simply by $\tau_{A/T}$, $\mathrm{G}_0(A)$ and $\mathrm{A}_*(A)$, respectively.

The construction of the map $\tau_{A/T}$ depends not only on $A$ but also on $T$. However, if $A$ is a complete local ring or $A$ is essentially of finite type over either a field or the ring of integers, it is proved in [7] that the map $\tau_{A/T}$ is independent of the choice of $T$. Furthermore, no example is known where the map $\tau_{A/T}$ actually depends on the choice of $T$. It seems natural to consider the following conjecture:

**Conjecture 1.1** Let $A$ be a local ring that is a homomorphic image of a regular local ring $T$. Then, the Riemann-Roch map $\tau_{A/T}$ as above is independent of the choice of $T$.

In 1985, P. Roberts [14] proved that the vanishing theorem holds for a local ring $A$ that satisfies $\tau_{A/T}([A]) \in \mathrm{A}_{\dim A}(A)_{\mathbb{Q}}$, where we say that the vanishing theorem holds for $A$ if $\sum_i (-1)^i \ell_A(\mathrm{Tor}_i^A(M,N)) = 0$ is satisfied for two finitely generated $A$-modules $M$ and $N$ that satisfy the following three conditions; (1) both of them have finite projective dimension, (2) $\dim M + \dim N < \dim A$, (3) $M \otimes_A N$ is of finite length. (This result contains an affirmative answer to a conjecture proposed by Serre [16]. The conjecture was independently solved by Roberts [14], Gillet and Soulé [3].)

Inspired by the result of Roberts, the second author defined the notion of Roberts rings as below and studied them in [7], [8].

**Definition 1.2** A local ring $A$ is said to be a *Roberts ring* if there is a regular local ring $T$ such that $A$ is a homomorphic image of $T$ and $\tau_{A/T}([A]) \in \mathrm{A}_{\dim A}(A)_{\mathbb{Q}}$ is satisfied.

By the result of Roberts [14], we know that Roberts rings satisfy the vanishing theorem.

The category of Roberts rings contains complete intersections, quotient singularities, and Galois extensions of regular local rings. Normal Roberts rings are $\mathbb{Q}$-Gorenstein. There are examples of Gorenstein normal non-Roberts rings. If $A$ is a Roberts ring, then so is the completion $\hat{A}$. (See [7] or Remark 6.1.) Here, we want to ask the following question:

**Question 1.3** Let $A$ be a local ring that is a homomorphic image of a regular local ring. Assume that the completion $\hat{A}$ is a Roberts ring. Then, is $A$ a Roberts ring, too?

There is a deep connection between Conjecture 1.1 and Question 1.3. As we shall see in Proposition 6.2 in Section 6, Question 1.3 is true for any $A$ if and only if Question 1.4 as below is true for any $A$. Furthermore, if Question 1.4 is true for a local ring $A$, then Conjecture 1.1 is true for the local ring $A$ (see Section 6 (I)).

**Question 1.4** Let $A$ be a local ring that is a homomorphic image of a regular local ring. Then, is the map $\mathrm{G}_0(A)_{\mathbb{Q}} \xrightarrow{f_*} \mathrm{G}_0(\hat{A})_{\mathbb{Q}}$ (induced by the flat map $A \xrightarrow{f} \hat{A}$) injective?

In sections 3, 4 and 5, we shall prove the following theorem that is the main theorem of the paper.

**Theorem 1.5** *Let $A$ be a homomorphic image of an excellent regular local ring. If $A$ satisfies one of the following three conditions, then the natural map $\mathrm{G}_0(A) \xrightarrow{f_*} \mathrm{G}_0(\hat{A})$ is injective;*

*(i) $A$ is a henselian local ring,*

*(ii) $A = S_M$, where $S = \oplus_{n \geq 0} S_n$ is a Noetherian positively graded ring over a henselian local ring $(S_0, m_0)$, and $M = m_0 S_0 + S_+$ (where $S_+ = \oplus_{n>0} S_n$),*

*(iii) $A$ has at most an isolated singularity.*

The three assertions in Theorem 1.5 will be proved in completely different ways. In the proof of Claim 4.3 in [7], the injectivity was announced in case (i) as above without a proof. We shall give a precise proof to Theorem 1.5 in case (i) in Section 3. In the proof, Popescu-Ogoma's approximation theorem ([11], [12]) is used. We remark that rings in (i) are contained in those in (ii). In case (ii), we use a method similar to "deformation to normal cones" in Chapter 5 of Fulton [2]. In case (iii), we use the localization sequence in $K$-theory due to Thomason and Trobaugh [18], that is a generalization of the exact sequences constructed by Quillen [13] or Levine [9].

We shall give some applications of Theorem 1.5 in Section 6. As we shall see in Proposition 6.5, Question 1.4 is equivalent to the statement that the induced map $\mathrm{G}_0(A)_{\mathbb{Q}} \xrightarrow{g_*} \mathrm{G}_0(B)_{\mathbb{Q}}$ is injective for any flat local homomorphism $A \xrightarrow{g} B$ such that its extension of the residue class fields is finitely generated. If $A \xrightarrow{g} B$ is a flat local homomorphism such that $A$ contains a field of characteristic 0, then the induced map $\mathrm{G}_0(A)_{\mathbb{Q}} \xrightarrow{g_*} \mathrm{G}_0(B)_{\mathbb{Q}}$ is injective if Question 1.4 is true. The authors have no example of a flat local homomorphism $A \to B$ such that the induced map $\mathrm{G}_0(A) \longrightarrow \mathrm{G}_0(B)$ is not injective.

The next section is devoted to preliminaries.

## 2 Preliminaries

Throughout the article, a local ring is always assumed to be a homomorphic image of a regular local ring. Remark that such rings are universally catenary.

First of all, let us define the Grothendieck group and the Chow group of a ring $A$.

**Definition 2.1** For a ring $A$, let $\mathrm{G}_0(A)$ be the *Grothendieck group* of finitely generated $A$-modules, i.e.,

$$\mathrm{G}_0(A) = \frac{\bigoplus_{M\ :\ \text{a finitely generated } A\text{-molule}} \mathbb{Z}\cdot[M]}{\langle [M]-[L]-[N] \mid 0 \to L \to M \to N \to 0 \text{ is exact}\rangle}.$$

Let $\mathrm{A}_i(A)$ be the *$i$-th Chow group* of $A$, i.e.,

$$\mathrm{A}_i(A) = \frac{\bigoplus_{P\in \mathrm{Spec}\, A,\ \dim A/P=i} \mathbb{Z}\cdot[\mathrm{Spec}\, A/P]}{\langle \mathrm{div}(Q,x) \mid Q \in \mathrm{Spec}\, A,\ \dim A/Q = i+1,\ x \in A\setminus Q\rangle},$$

where

$$\mathrm{div}(Q,x) = \sum_{P\in \mathrm{Min}_A A/(Q,x)} \ell_{A_P}(A_P/(Q,x)A_P)[\mathrm{Spec}\, A/P],$$

where $\ell_{A_P}(\ )$ denotes the length as an $A_P$-module. The *Chow group* of $A$ is defined to be $\mathrm{A}_*(A) = \oplus_{i=0}^{\dim A} A_i(A)$.

For an abelian group $M$, $M_{\mathbb{Q}}$ denotes $M\otimes_{\mathbb{Z}}\mathbb{Q}$.

**Definition 2.2** (1) Let $g : A \to B$ be a flat ring homomorphism. Then, we have the induced homomorphism $g_* : \mathrm{G}_0(A) \to \mathrm{G}_0(B)$ defined by $g_*([M]) = [M\otimes_A B]$.

(2) Let $(A,m)$ be a local ring and $\hat{A}$ denotes the completion of $A$ in the $m$-adic topology. For each $i$, the natural map $A \xrightarrow{f} \hat{A}$ induces the map $\mathrm{A}_i(A) \xrightarrow{f_*} \mathrm{A}_i(\hat{A})$ defined by

$$f_*([\mathrm{Spec}\, A/P]) = \sum_{\mathfrak{p}} \ell_{\hat{A}_{\mathfrak{p}}}(\hat{A}_{\mathfrak{p}}/P\hat{A}_{\mathfrak{p}})[\mathrm{Spec}\, \hat{A}/\mathfrak{p}],$$

where the sum is taken over all minimal prime ideals $\mathfrak{p}$ of $\hat{A}/P\hat{A}$ as an $\hat{A}$-module. Here, remark that $\hat{A}/P\hat{A} = \widehat{A/P}$ is equi-dimensional since $A$ is universally catenary (Theorem 31.7 in Matsumura [10]). See Remark 6.4 for induced maps by general flat local homomorphisms.

**Remark 2.3** Assume that $A$ is a $d$-dimensional excellent normal local ring. Then $\hat{A}$ is also normal and the natural map $\mathrm{Cl}(A) \to \mathrm{Cl}(\hat{A})$ is injective, where $\mathrm{Cl}(A)$ is the divisor class group of $A$.

On the other hand, it is well known that $\mathrm{A}_{d-1}(A)$ coincides with $\mathrm{Cl}(A)$. Thus, we know that $f_* : \mathrm{A}_{d-1}(A) \to \mathrm{A}_{d-1}(\hat{A})$ is injective if $A$ is an excellent normal ring of dimension $d$.

Then, we have the following:

**Proposition 2.4** *Let $A$ be a local ring. Then, the following conditions are equivalent;*

1. $\mathrm{G}_0(A)_{\mathbb{Q}} \xrightarrow{f_*} \mathrm{G}_0(\hat{A})_{\mathbb{Q}}$ *is injective,*
2. $\mathrm{A}_i(A)_{\mathbb{Q}} \xrightarrow{f_*} \mathrm{A}_i(\hat{A})_{\mathbb{Q}}$ *is injective for all $i$.*

*Proof.* Take a regular local ring $T$ such that $A$ is a homomorphic image of $T$. Then, by the singular Riemann-Roch theorem, we have isomorphisms of $\mathbb{Q}$-vector spaces $\tau_{A/T} : \mathrm{G}_0(A)_{\mathbb{Q}} \to \mathrm{A}_*(A)_{\mathbb{Q}}$ and $\tau_{\hat{A}/\hat{T}} : \mathrm{G}_0(\hat{A})_{\mathbb{Q}} \to \mathrm{A}_*(\hat{A})_{\mathbb{Q}}$ such that the following diagram is commutative (Lemma 4.1 (c) in [7]):

$$\begin{array}{ccc} \mathrm{G}_0(A)_{\mathbb{Q}} & \xrightarrow{\tau_{A/T}} & \mathrm{A}_*(A)_{\mathbb{Q}} \\ f_* \downarrow & & \downarrow f_* \\ \mathrm{G}_0(\hat{A})_{\mathbb{Q}} & \xrightarrow{\tau_{\hat{A}/\hat{T}}} & \mathrm{A}_*(\hat{A})_{\mathbb{Q}} \end{array} \tag{2.5}$$

Here $f_* : \mathrm{A}_*(A)_{\mathbb{Q}} \to \mathrm{A}_*(\hat{A})_{\mathbb{Q}}$ is the direct sum of $\{f_* : \mathrm{A}_i(A)_{\mathbb{Q}} \to \mathrm{A}_i(\hat{A})_{\mathbb{Q}} \mid i = 0, 1, \ldots, \dim A\}$. Therefore we know immediately that two conditions in the proposition are equivalent. **q.e.d.**

By the proposition, Question 1.4 is a natural generalization of the injectivity of divisor class groups (Remark 2.3) in a sense. In Proposition 6.2, we shall know that Question 1.4 is true if and only if $\mathrm{A}_{\dim A-1}(A)_{\mathbb{Q}} \to \mathrm{A}_{\dim A-1}(\hat{A})_{\mathbb{Q}}$ is injective for any reduced equi-dimensional local ring $A$.

# 3 The proof of Theorem 1.5 in the case of (i)

We shall prove the injectivity of $\mathrm{G}_0(A) \xrightarrow{f_*} \mathrm{G}_0(\hat{A})$ in the case where $A$ is a henselian local ring in the section.

By the assumption, there is an excellent regular local ring $T$ such that $A$ is a homomorphic image of $T$. Replacing $T$ with its henselization, we may asuume that $T$ is an excellent henselian regular local ring. Set $A = T/I$ for an ideal $I$ of $T$.

Before proving theorem, we need some preliminaries.

The key point of the proof is to apply the following Popescu-Ogoma's approximation theorem [11], [12]:

**Theorem 3.1** *Let $s$ and $t$ be positive integers. Let $(R, n)$ be an excellent henselian local ring. Consider the polynomial ring $R[X_1, \ldots, X_t]$ with variables $X_1, \ldots, X_t$. Let $f_1, \ldots, f_s$ be polynomials in $R[X_1, \ldots, X_t]$. If there are elements $a_1, \ldots, a_t$ of the $n$-adic completion $\hat{R}$ that satisfy*

$$f_1(a_1, \ldots, a_t) = \cdots = f_s(a_1, \ldots, a_t) = 0 \quad \text{in } \hat{R},$$

*then there exist elements $b_1, \ldots, b_t$ of $R$ that satisfy*

$$f_1(b_1, \ldots, b_t) = \cdots = f_s(b_1, \ldots, b_t) = 0 \quad \text{in } R.$$

If $a_1, \ldots, a_t$ satisfy

$$f_1(a_1, \ldots, a_t) = \cdots = f_s(a_1, \ldots, a_t) = 0,$$

we say that $a_1, \ldots, a_t$ is a *solution* of the polynomial equations $f_1 = \cdots = f_s = 0$.

The following lemma will play an essential role in the proof of Theorem 1.5 in the case of (i):

**Lemma 3.2** *Let $T$ be a regular local ring and $\hat{T}$ denotes its completion. Let*

$$0 \longrightarrow \hat{T}^{r_n} \xrightarrow{(a_{nij})} \hat{T}^{r_{n-1}} \xrightarrow{(a_{n-1,i,j})} \cdots \xrightarrow{(a_{2ij})} \hat{T}^{r_1} \xrightarrow{(a_{1ij})} \hat{T}^{r_0} \tag{3.3}$$

*be an exact sequence of free $\hat{T}$-modules, where $(a_{kij})$ is an $r_{k-1} \times r_k$ matrix with entries in $\hat{T}$. Then there exist variables $\{A_{kij} \mid k, i, j\}$ corresponding to $\{a_{kij} \mid k, i, j\}$, some variables $Y_1, \ldots, Y_t$, and polynomials*

$$f_1, \ldots, f_s \in T[\{A_{kij} \mid k, i, j\}, Y_1, \ldots, Y_t]$$

*which satisfy the following two conditions:*

*(a) There are elements $y_1, \ldots, y_t$ of $\hat{T}$ such that $\{a_{kij} \mid k, i, j\}, y_1, \ldots, y_t$ is a solution of $f_1 = \cdots = f_s = 0$.*

*(b) Let $\{b_{kij} \mid k, i, j\}, z_1, \ldots, z_t$ be elements in $T$. If $\{b_{kij} \mid k, i, j\}, z_1, \ldots, z_t$ is a solution of $f_1 = \cdots = f_s = 0$, then the sequence of $T$-linear maps*

$$0 \longrightarrow T^{r_n} \xrightarrow{(b_{nij})} T^{r_{n-1}} \xrightarrow{(b_{n-1,i,j})} \cdots \xrightarrow{(b_{2ij})} T^{r_1} \xrightarrow{(b_{1ij})} T^{r_0}$$

*is exact*

*Proof.* Since the sequence (3.3) is a chain complex, $\{a_{kij} \mid k, i, j\}$ is a solution of the polynomial equations

$$\sum_{q=1}^{r_{k-1}} A_{k-1,i,q} A_{kqj} = 0 \quad (\forall k, i, j).$$

If a system of elements $\{b_{kij} \mid k, i, j\}$ in $T$ is a solution of the equations as above, then the sequence

$$0 \longrightarrow T^{r_n} \xrightarrow{(b_{nij})} T^{r_{n-1}} \xrightarrow{(b_{n-1,i,j})} \cdots \xrightarrow{(b_{2ij})} T^{r_1} \xrightarrow{(b_{1ij})} T^{r_0} \tag{3.4}$$

is a chain complex of $T$-linear maps. We shall argue when it is exact.

Put

$$e_k = r_k - r_{k+1} + r_{k+2} - \cdots$$

for each $k > 0$, where we consider $r_m = 0$ if $m > n$. Remark that each $e_k$ is a non-negative integer for $k = 1, \ldots, n$ since the sequence (3.3) is exact. Thanks to a theorem of Buchsbaum-Eisenbud [1], the complex (3.4) is exact if and only if the following two conditions are satisfied;

1. the rank of the matrix $(b_{kij})$ is equal to $e_k$ for $k = 1, \ldots, n$,
2. the grade of the ideal $I_{e_k}((b_{kij}))$ of $T$ is at least $k$ for $k = 1, \ldots, n$, where $I_{e_k}((b_{kij}))$ denotes the ideal generated by all the $e_k \times e_k$ minors of the matrix $(b_{kij})$. Here, we think that the grade of $I_0((b_{kij}))$ is infinity.

It is easy to see that there are polynomials such that, if $\{b_{kij} \mid k, i, j\}$ is a solution of the polynomial equations, then the rank of the matrix $(b_{kij})$ is at most $e_k$ for each $k$.

Therefore we have only to find polynomials to keep grade high. Remark that the grade of an ideal coincides with the height of it, because $T$ is a regular local ring. Then, by Lemma 3.7 in [5], we can find polynomials with coefficients in $T$ which preserve height of ideals. **q.e.d.**

Now we start to prove Theorem 1.5 (i).

Assume that $\alpha \in \mathrm{G}_0(A)$ satisfies $f_*(\alpha) = 0$. We want to show $\alpha = 0$. There are finitely generated $A$-modules $M$ and $N$ such that $\alpha = [M] - [N]$. By definition, $f_*(\alpha) = [M \otimes_A \hat{A}] - [N \otimes_A \hat{A}]$. Therefore, $[M \otimes_A \hat{A}] = [N \otimes_A \hat{A}]$ is satisfied in $\mathrm{G}_0(\hat{A})$. Under the situation, we want to prove $[M] = [N]$ in $\mathrm{G}_0(A)$.

The category of finitely generated $\hat{A}$-modules must contain some short exact sequences which give the relation $[M \otimes_A \hat{A}] = [N \otimes_A \hat{A}]$ in $\mathrm{G}_0(\hat{A})$. For example, if there are short exact sequences

$$\begin{array}{c} 0 \to M \otimes_A \hat{A} \to L_1 \to L_2 \to 0 \\ 0 \to L_2 \to L_1 \to N \otimes_A \hat{A} \to 0 \end{array} \tag{3.5}$$

of finitely generated $\hat{A}$-modules, then

$$[M \otimes_A \hat{A}] = [L_1] - [L_2] = [N \otimes_A \hat{A}] \quad \text{in } \mathrm{G}_0(\hat{A})$$

is satisfied.

For the simplicity, we assume that there exist short exact sequences as in (3.5). (The general case would be proved in completely the same way.)

Let $\mathbb{F}.$ and $\mathbb{G}.$ be finite $T$-free resolutions of $M$ and $N$, respectively. Since there are short exact sequences as in (3.5), we have exact sequences of chain complexes of free $\hat{T}$-modules

$$
\begin{aligned}
&0 \to \mathbb{F}. \otimes_T \hat{T} \to \hat{\mathbb{P}}. \to \hat{\mathbb{R}}. \to 0\\
&0 \to \hat{\mathbb{S}}. \to \hat{\mathbb{Q}}. \to \mathbb{G}. \otimes_T \hat{T} \to 0,
\end{aligned}
\tag{3.6}
$$

where both $\hat{\mathbb{P}}.$ and $\hat{\mathbb{Q}}.$ are finite $\hat{T}$-free resolutions of $L_1$, and both $\hat{\mathbb{R}}.$ and $\hat{\mathbb{S}}.$ are finite $\hat{T}$-free resolutions of $L_2$. In particular,

$$
\hat{\mathbb{P}}., \hat{\mathbb{Q}}., \hat{\mathbb{R}}. \text{ and } \hat{\mathbb{S}}. \text{ are finite free resolutions.} \tag{3.7}
$$

Furthermore, there exist exact sequences of chain complexes of free $\hat{T}$-modules

$$
\begin{aligned}
&0 \to \hat{\mathbb{T}}. \to \hat{\mathbb{P}}. \to \hat{\mathbb{Q}}. \to \hat{\mathbb{U}}. \to 0\\
&0 \to \hat{\mathbb{V}}. \to \hat{\mathbb{R}}. \to \hat{\mathbb{S}}. \to \hat{\mathbb{W}}. \to 0
\end{aligned}
\tag{3.8}
$$

such that

$$
\hat{\mathbb{T}}., \hat{\mathbb{U}}., \hat{\mathbb{V}}. \text{ and } \hat{\mathbb{W}}. \text{ are bounded split exact sequences.} \tag{3.9}
$$

Using Lemma 3.2, we know that there is a set of polynomial equations with coefficients in $T$ that preserves conditions (3.6), (3.7), (3.8) and (3.9). Since $H_0(\hat{\mathbb{P}}.)$, $H_0(\hat{\mathbb{Q}}.)$, $H_0(\hat{\mathbb{R}}.)$ and $H_0(\hat{\mathbb{S}}.)$ are $\hat{A} = \hat{T}/I\hat{T}$-modules,

$$
H_0(\hat{\mathbb{P}}.), H_0(\hat{\mathbb{Q}}.), H_0(\hat{\mathbb{R}}.) \text{ and } H_0(\hat{\mathbb{S}}.) \text{ are annihilated by } I. \tag{3.10}
$$

It is easy to see that there is a set of polynomial equations with coefficients in $T$ that preserves the condition (3.10).

The polynomial equations which we found as above has a solution in $\hat{T}$. Then, appplying Popescu-Ogoma's approximation theorem (see Theorem 3.1), the set of polynomial equations has a solution in $T$, because $T$ is an excellent henselian local ring. Then we have exact sequences of finite $T$-free resolutions

$$
\begin{aligned}
&0 \to \mathbb{F}. \to \mathbb{P}. \to \mathbb{R}. \to 0\\
&0 \to \mathbb{S}. \to \mathbb{Q}. \to \mathbb{G}. \to 0\\
&0 \to \mathbb{T}. \to \mathbb{P}. \to \mathbb{Q}. \to \mathbb{U}. \to 0\\
&0 \to \mathbb{V}. \to \mathbb{R}. \to \mathbb{S}. \to \mathbb{W}. \to 0
\end{aligned}
\tag{3.11}
$$

such that

- $\mathbb{T}.$, $\mathbb{U}.$, $\mathbb{V}.$ and $\mathbb{W}.$ are split exact,
- $H_0(\mathbb{P}.)$, $H_0(\mathbb{Q}.)$, $H_0(\mathbb{R}.)$ and $H_0(\mathbb{S}.)$ are annihilated by $I$, and

- $\mathbb{P}.$, $\mathbb{Q}.$, $\mathbb{R}.$ and $\mathbb{S}.$ are finite $T$-free resolutions of $H_0(\mathbb{P}.)$, $H_0(\mathbb{Q}.)$, $H_0(\mathbb{R}.)$ and $H_0(\mathbb{S}.)$, respectively.

Then, by the first two exact sequences in (3.11), we have exact sequences of $A$-modules as

$$0 \to M \to H_0(\mathbb{P}.) \to H_0(\mathbb{R}.) \to 0$$
$$0 \to H_0(\mathbb{S}.) \to H_0(\mathbb{Q}.) \to N \to 0.$$

Furthermore, by the last two exact sequences in (3.11), we know that $H_0(\mathbb{P}.)$ (resp. $H_0(\mathbb{R}.)$) is ismomorphic to $H_0(\mathbb{Q}.)$ (resp. $H_0(\mathbb{S}.)$) as a $T$-module. Since modules are annihilated by $I$, they are isomorphisms as $A$-modules.

Hence, we have

$$[M] = [H_0(\mathbb{P}.)] - [H_0(\mathbb{R}.)] = [H_0(\mathbb{Q}.)] - [H_0(\mathbb{S}.)] = [N]$$

in $\mathrm{G}_0(A)$.

We have completed the proof of Theorem 1.5 (i).

# 4 The proof of Theorem 1.5 in the case of (ii)

We shall prove Theorem 1.5 in the case of (ii) in the section.

Put $A = S_M$, where $S = \oplus_{n\geq 0} S_n$ is a Noetherian positively graded ring over a henselian local ring $(S_0, m_0)$, and $M = m_0 S_0 + S_+$.

Put $B = \prod_{n\geq 0} S_n$. It is the $S_+A$-adic completion of $A$. Here, $B$ is flat over $A$ and $\hat{A} = \hat{B}$ is satisfied. Let $g : A \to B$ denote the natural map. Since $A$ is an excellent local ring, so is $B$ by Theorem 3 in Rotthaus [15]. Furthermore, since $S_0$ is a henselian local ring, so is $B$. Since $B$ is an excellent henselian local ring, $\mathrm{G}_0(B) \to \mathrm{G}_0(\hat{B}) = \mathrm{G}_0(\hat{A})$ is injective by (i) in Theorem 1.5. Therefore, in order to show the injectivity of $\mathrm{G}_0(A) \to \mathrm{G}_0(\hat{A})$, we have only to prove that $\mathrm{G}_0(A) \to \mathrm{G}_0(B)$ is injective.

Put

$$F_i = \begin{cases} (\oplus_{n\geq i} S_n)A & \text{if } i \geq 0 \\ A & \text{if } i < 0. \end{cases}$$

Then, $F = \{F_i\}_{i\in\mathbb{Z}}$ is a filtration of ideals of $A$, that is, it satisfies the following three conditions; (1) $F_i \supseteq F_{i+1}$ for any $i \in \mathbb{Z}$, (2) $F_0 = A$, (3) $F_i F_j \subseteq F_{i+j}$ for any $i, j \in \mathbb{Z}$. Similarly put

$$\hat{F}_i = \begin{cases} \prod_{n\geq i} S_n & \text{if } i \geq 0 \\ B & \text{if } i < 0. \end{cases}$$

Then, $\hat{F} = \{\hat{F}_i\}_{i\in\mathbb{Z}}$ is a filtration of ideals of $B$. Remark that $\hat{F}_i$ coincides with $F_i B = F_i \otimes_A B$ for each $i$. Put

$$R(F) \;=\; \oplus_{i\in\mathbb{Z}} F_i t^i \subseteq A[t, t^{-1}]$$

$$\begin{aligned} G(F) &= R(F)/t^{-1}R(F) = \oplus_{i\geq 0}F_i/F_{i+1} \\ R(\hat{F}) &= \oplus_{i\in\mathbb{Z}}\hat{F}_i t^i \subseteq B[t,t^{-1}] \\ G(\hat{F}) &= R(\hat{F})/t^{-1}R(\hat{F}) = \oplus_{i\geq 0}\hat{F}_i/\hat{F}_{i+1}, \end{aligned}$$

where $t$ is an indeterminate. Remark that $R(F)\otimes_A B = R(\hat{F})$, $G(F)\otimes_A B = G(\hat{F})$ and $S = G(F) = G(\hat{F})$.

Flat homomorphisms $A \xrightarrow{\alpha} A[t,t^{-1}]$ and $R(F) \xrightarrow{\beta} R(F)[(t^{-1})^{-1}] = A[t,t^{-1}]$ induce the maps $\mathrm{G}_0(A) \xrightarrow{\alpha_*} \mathrm{G}_0(A[t,t^{-1}])$ and $\mathrm{G}_0(R(F)) \xrightarrow{\beta_*} \mathrm{G}_0(A[t,t^{-1}])$, respectively (see Definition 2.2). Since the natural projection $R(F) \xrightarrow{\gamma} G(F)$ is finite, we have the induced map $\mathrm{G}_0(G(F)) \xrightarrow{\gamma^*} \mathrm{G}_0(R(F))$ defined by $\gamma^*([M]) = [M]$ for each finitely generated $G(F)$-module $M$. Thus we have the following diagram:

$$\begin{array}{ccccccc} & & & & \mathrm{G}_0(A) & & \\ & & & & {\scriptstyle\alpha_*}\downarrow & & \\ \mathrm{G}_0(G(F)) & \xrightarrow{\gamma^*} & \mathrm{G}_0(R(F)) & \xrightarrow{\beta_*} & \mathrm{G}_0(A[t,t^{-1}]) & \longrightarrow & 0 \end{array}$$

It is known that the horizontal sequence in the above diagram is exact. We refer the basic facts on algebraic $K$-theory to Quillen [13] or Srinivas [17]. (The horizontal exact sequence is called the *localization sequence* induced by a localization of a category.)

On the other hand, we have a map $\gamma_* : \mathrm{G}_0(R(F)) \to \mathrm{G}_0(G(F))$ satisfying $\gamma_*([N]) = [N/t^{-1}N] - [0 :_N t^{-1}]$ for each finitely generated $R(F)$-module $N$. It is easy to see $\gamma_*\gamma^* = 0$. Hence, we obtain the induced map $\overline{\gamma_*} : \mathrm{G}_0(A[t,t^{-1}]) \to \mathrm{G}_0(G(F))$ that satisfies $\overline{\gamma_*}\beta_* = \gamma_*$ because of the exactness of the localization sequence. It is easy to see that, for an ideal $I$ of $A$,

$$\overline{\gamma_*}\alpha_*([A/I]) = \left[\bigoplus_{i\geq 0} \frac{F_i}{F_i \cap I + F_{i+1}}\right] \tag{4.1}$$

is satisfied.

Similarly we have the diagram

$$\begin{array}{ccccccc} & & & & \mathrm{G}_0(B) & & \\ & & & & {\scriptstyle\hat{\alpha}_*}\downarrow & & \\ \mathrm{G}_0(G(\hat{F})) & \xrightarrow{\hat{\gamma}^*} & \mathrm{G}_0(R(\hat{F})) & \xrightarrow{\hat{\beta}_*} & \mathrm{G}_0(B[t,t^{-1}]) & \longrightarrow & 0 \\ & & {\scriptstyle\hat{\gamma}_*}\downarrow & & & & \\ & & \mathrm{G}_0(G(\hat{F})) & & & & \end{array}$$

and the induced map $\overline{\hat{\gamma}_*} : \mathrm{G}_0(B[t,t^{-1}]) \to \mathrm{G}_0(G(\hat{F}))$.

Let $h : S \to A$ be the localization. Put $g_1 = 1\otimes g : A[t,t^{-1}] \to A[t,t^{-1}]\otimes_A B = B[t,t^{-1}]$ and $g_2 = 1 \otimes g : G(F) \to G(F) \otimes_A B = G(\hat{F})$. Remark that $g_2$ is an isomorphism.

Then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
G_0(S) & \xrightarrow{h_*} & G_0(A) & \xrightarrow{\alpha_*} & G_0(A[t,t^{-1}]) & \xrightarrow{\overline{\gamma_*}} & G_0(G(F)) & & \\
 & & \downarrow g_* & & \downarrow g_{1*} & & \downarrow g_{2*} & & \\
 & & G_0(B) & \xrightarrow{\hat{\alpha}_*} & G_0(B[t,t^{-1}]) & \xrightarrow{\overline{\hat{\gamma}_*}} & G_0(G(\hat{F})) & = & G_0(S)
\end{array}$$

We denote by $\varphi : G_0(S) \to G_0(S)$ the composite map as above.

We need to show the following claim:

**Claim 4.2** *$\varphi$ is the identity map.*

We shall finish the proof of Theorem 1.5 (ii).

Is is easy to see that $h_*$ is surjective since $h$ is a localization. Then, by the claim, we know that $h_*$ is an isomorphism and $g_* : G_0(A) \to G_0(B)$ is injective.

*Proof of Claim 4.2.* It is easily verified that $G_0(S)$ is generated by

$$\{[S/P] \mid P \text{ is a homogeneous prime ideal of } S\}.$$

Therefore, we have only to show $\varphi([S/P]) = [S/P]$ for any homogeneous prime ideal $P$ of $S$. Put $P = \oplus_{i\geq 0} P_i$. Then we have

$$\begin{aligned}
\varphi([S/P]) &= \overline{\gamma_*}\alpha_* h_*([S/P]) = \overline{\gamma_*}\alpha_*([A/PA]) \\
&= \left[\bigoplus_{i\geq 0} \frac{F_i}{F_i \cap PA + F_{i+1}}\right] = \left[\bigoplus_{i\geq 0} S_i/P_i\right] = [S/P]\,.
\end{aligned}$$

(The third equality is obtained by the equality (4.1) as above.)

We have completed the proof in the case (ii).

# 5 The proof of Theorem 1.5 in the case of (iii)

We shall prove Theorem 1.5 in the case of (iii) in the section.

It is enough to show the following claim:

**Claim 5.1** *Let $A$ be a Noetherian local ring and let $I$ be an ideal of $A$. Let $B$ be the $I$-adic completion of $A$. Assume that both* $\operatorname{Spec} A \setminus (\operatorname{Spec} A/I)$ *and* $\operatorname{Spec} B \setminus (\operatorname{Spec} B/IB)$ *are regular schemes. Then, the induced map* $G_0(A) \to G_0(B)$ *is injective.*

Remark that, if $A$ is an excellent local ring of isolated singularity, then $\hat{A}$ is also isolated singularity. Therefore, applying the claim, $G_0(A) \to G_0(\hat{A})$ is proved to be injective in the case.

Now we start to prove Claim 5.1.

We put $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} A/I$, $U = X \setminus Y$, $\hat{X} = \operatorname{Spec} B$, $\hat{Y} = \operatorname{Spec} B/IB$, $\hat{U} = \hat{X} \setminus \hat{Y}$. Then the natural map $\hat{Y} \to Y$ is an isomorphism and we have the following fibre squares:

$$\begin{array}{ccccc} \hat{Y} & \longrightarrow & \hat{X} & \longleftarrow & \hat{U} \\ \| & & \downarrow & & \downarrow \\ Y & \longrightarrow & X & \longleftarrow & U \end{array}$$

If $U$ is empty, the assertion is obvious. Suppose that $U$ is not empty.

We have the following commutative diagram:

$$\begin{array}{ccccccccccc} \mathrm{G}_1(X) & \xrightarrow{p} & \mathrm{G}_1(U) & \longrightarrow & \mathrm{G}_0(Y) & \longrightarrow & \mathrm{G}_0(X) & \longrightarrow & \mathrm{G}_0(U) & \longrightarrow & 0 \\ \downarrow & & \downarrow r & & \downarrow s & & \downarrow t & & \downarrow u & & \\ \mathrm{G}_1(\hat{X}) & \xrightarrow{q} & \mathrm{G}_1(\hat{U}) & \longrightarrow & \mathrm{G}_0(\hat{Y}) & \longrightarrow & \mathrm{G}_0(\hat{X}) & \longrightarrow & \mathrm{G}_0(\hat{U}) & \longrightarrow & 0 \end{array}$$

Here, for a scheme $W$, $\mathrm{G}_i(W)$ denotes the $i$-th $K$-group of the exact category of coherent $\mathcal{O}_W$-modules. Horizontal sequences are exact (see Quillen [13] or Srinivas [17]). Vertical maps are induced by flat morphisms.

We denote by $C$ (resp. $\hat{C}$) the cokernel of $p$ (resp. $q$). Let $v : C \to \hat{C}$ be the induced map by $r$. In order to prove the injectivity of $t : \mathrm{G}_0(X) \to \mathrm{G}_0(\hat{X})$, we have only to show the following;

- $u : \mathrm{G}_0(U) \to \mathrm{G}_0(\hat{U})$ is injective,
- $v : C \to \hat{C}$ is surjective.

(Recall that $s : \mathrm{G}_0(Y) \to \mathrm{G}_0(\hat{Y})$ is an isomorphism.)

On the other hand, thanks to Thomason and Trobaugh [18], we have the localization sequence in $K$-theory, that is, we have the following commutative diagram:

$$\begin{array}{ccccccccccc} \mathrm{K}_1(X) & \xrightarrow{p'} & \mathrm{K}_1(U) & \to & \mathrm{K}_0(X \text{ on } Y) & \to & \mathrm{K}_0(X) & \xrightarrow{\alpha} & \mathrm{K}_0(U) & \to & \mathrm{K}_{-1}(X \text{ on } Y) \\ \downarrow & & \downarrow r' & & \downarrow s' & & \downarrow t' & & \downarrow u' & & \downarrow w' \\ \mathrm{K}_1(\hat{X}) & \xrightarrow{q'} & \mathrm{K}_1(\hat{U}) & \to & \mathrm{K}_0(\hat{X} \text{ on } \hat{Y}) & \to & \mathrm{K}_0(\hat{X}) & \xrightarrow{\beta} & \mathrm{K}_0(\hat{U}) & \to & \mathrm{K}_{-1}(\hat{X} \text{ on } \hat{Y}) \end{array}$$

Here, for a scheme $W$, $\mathrm{K}_i(W)$ denotes the $i$-th $K$-group of the exact category of locally free $\mathcal{O}_W$-modules of finite rank. We denote by $\mathrm{K}_i(X \text{ on } Y)$ the $i$-th $K$-group of the derived category of perfect $\mathcal{O}_X$-complexes with support in $Y$. By Thomason and Trobaugh [18], horizontal sequences in the above diagram are exact.

Let $W$ be a scheme. Then, by definition, we have the natural map $\xi_W : \mathrm{K}_i(W) \to \mathrm{G}_i(W)$ for each $i \geq 0$. Furthermore, $\xi_W$ is an isomorphism for each $i \geq 0$ if $W$ is a regular scheme (see 27p in Quillen [13]).

We have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{K}_0(U) & \stackrel{\xi_U}{\longrightarrow} & \mathrm{G}_0(U) \\
u' \downarrow & & \downarrow u \\
\mathrm{K}_0(\hat{U}) & \stackrel{\xi_{\hat{U}}}{\longrightarrow} & \mathrm{G}_0(\hat{U})
\end{array}$$

Since both of $U$ and $\hat{U}$ are regular schemes, both of $\xi_U$ and $\xi_{\hat{U}}$ are isomorphisms. Thus, $u$ is injective if and only if so is $u'$.

On the other hand, since $\hat{X} \to X$ is a flat map with $Y = \hat{Y}$, we know that the natural map $\mathrm{K}_i(X \text{ on } Y) \to \mathrm{K}_i(\hat{X} \text{ on } \hat{Y})$ is an isomorphism for each $i \in \mathbb{Z}$ by Theorem 7.1 in Thomason and Trobaugh [18]. In particular, $s'$ and $w'$ are isomorphisms. Furthermore, since $A$ and $B$ are local rings, we have $\mathrm{K}_0(X) = \mathrm{K}_0(\hat{X}) = \mathbb{Z}$ and $t'$ is an isomorphism. Since neither $U$ nor $\hat{U}$ is empty, both $\alpha$ and $\beta$ are injective. Therefore, $u'$ is injective.

Since $\alpha$ and $\beta$ are injective, the cokernel of $p'$ (resp. $q'$) coincides with $\mathrm{K}_0(X \text{ on } Y)$ (resp. $\mathrm{K}_0(\hat{X} \text{ on } \hat{Y})$). Since $s'$ is an isomorphism, we have

$$\mathrm{K}_1(\hat{U}) = \mathrm{Im}(q') + \mathrm{Im}(r'), \tag{5.2}$$

where Im( ) denotes the image of the given map. On the other hand, we have the following commutative diagram:

$$\begin{array}{ccccc}
 & & \mathrm{K}_1(U) & & \\
 & & \downarrow r' & \searrow \xi_U & \\
\mathrm{K}_1(\hat{X}) & \stackrel{q'}{\longrightarrow} & \mathrm{K}_1(\hat{U}) & & \mathrm{G}_1(U) \\
 & \searrow \xi_{\hat{X}} & & \searrow \xi_{\hat{U}} & \downarrow r \\
 & & \mathrm{G}_1(\hat{X}) & \stackrel{q}{\longrightarrow} & \mathrm{G}_1(\hat{U})
\end{array}$$

Since $\hat{U}$ is a regular scheme, $\xi_{\hat{U}}$ is an isomorphism. By (5.2), we immediately obtain

$$\mathrm{G}_1(\hat{U}) = \mathrm{Im}(q) + \mathrm{Im}(r).$$

Therefore, the map $v : C \to \hat{C}$ is surjective. We have completed the proof of Theorem 1.5.

**Remark 5.3** If a local ring satisfies one of (i), (ii) and (iii) in Theorem 1.5, we know, by Proposition 2.4, that $\mathrm{A}_i(A)_{\mathbb{Q}} \stackrel{f_*}{\longrightarrow} \mathrm{A}_i(\hat{A})_{\mathbb{Q}}$ is injective for all $i$.

If a local ring satisfies either (i) or (ii) in Theorem 1.5, we can prove that $\mathrm{A}_i(A) \stackrel{f_*}{\longrightarrow} \mathrm{A}_i(\hat{A})$ is injective for all $i$ using the same method as in the proof of Theorem 1.5.

## 6 Applications

We shall give applications of Theorem 1.5 in the section.

Recall that local rings are assumed to be homomorphic images of regular local rings throughout the paper. Therefore, remark that they are universally catenary.

(I) Let $X$ be a scheme of finite type over a regular scheme $S$. Then, the singular Riemann-Roch theorem says that there exists an isomorphism of $\mathbb{Q}$-vector spaces

$$\tau_{X/S} \ : \ \mathrm{G}_0(X)_{\mathbb{Q}} \longrightarrow \mathrm{A}_*(X)_{\mathbb{Q}}$$

satisfying several good properties (Chapter 18 in Fulton [2]). Remark that the construction of the map $\tau_{X/S}$ depends not only on $X$ but also on $S$.

In fact, there are examples that the map $\tau_{X/S}$ actually depends on the choice of a regular base scheme $S$. Let $k$ be an arbitrary field. Put $X = \mathbb{P}^1_k$ and $S = \operatorname{Spec} k$. Then, we have $\tau_{X/X}(\mathcal{O}_X) = [X]$ by the construction of $\tau_{X/X}$. On the other hand, by Hirzebruch-Riemann-Roch theorem, we obtain $\tau_{X/S}(\mathcal{O}_X) = [X] + \chi(\mathcal{O}_X)[t]$, where $t$ is a rational point of $X$. It is well known that $\chi(\mathcal{O}_X) = 1$ and $[t] \neq 0$ in $\mathrm{A}_*(\mathbb{P}^1_k)_{\mathbb{Q}}$.

Let $T$ be a regular local ring and let $A$ be a homomorphic image of $T$. Then, by the singular Riemann-Roch theorem as above, we have an isomorphism of $\mathbb{Q}$-vector spaces

$$\tau_{A/T} \ : \ \mathrm{G}_0(A)_{\mathbb{Q}} \longrightarrow \mathrm{A}_*(A)_{\mathbb{Q}}$$

determined by both of $A$ and $T$.

It seems to be natural to consider Conjecture 1.1. In fact, for many important local rings, the conjecture is true. (Conjecture 1.1 is affirmatively solved in [7] if $A$ is a complete local ring or $A$ is essentially of finite type over either a field or the ring of integers.)

Look at the diagram (2.5). The bottom of the diagram (2.5) is independent of the choice of $\hat{T}$ since $\hat{A}$ is complete. Therefore, if vertical maps in the diagram (2.5) are injective, $\tau_{A/T}$ is independent of the choice of $T$. Hence, *if Question 1.4 is true for a local ring $A$, then Conjecture 1.1 is true for the local ring $A$.*

In particular, Conjecture 1.1 is true for a local ring $A$ that satisfies one of the three conditions in Theorem 1.5.

(II) Let $A$ and $T$ be rings as above and put $d = \dim A$. Set

$$\tau_{A/T}([A]) = \tau_d + \tau_{d-1} + \cdots + \tau_0, \quad (\tau_i \in \mathrm{A}_i(A)_{\mathbb{Q}}).$$

These $\tau_i$'s satisfy interesting properties as follows (see Proposition 3.1 in [8]):

(a) If $A$ is a Cohen-Macaulay ring, then

$$\tau_{A/T}([\omega_A]) = \tau_d - \tau_{d-1} + \tau_{d-2} - \cdots + (-1)^i \tau_{d-i} + \cdots$$

is satisfied, where $\omega_A$ denotes the canonical module of $A$.

(b) If $A$ is a Gorenstein ring, then we have $\tau_{d-i} = 0$ for each odd $i$.

(c) If $A$ is a complete intersection, then we have $\tau_i = 0$ for $i < d$.

(d) We have $\tau_d \neq 0$, since $\tau_d$ is equal to $[\mathrm{Spec}\, A]_d$, where

$$[\mathrm{Spec}\, A]_d = \sum_{\substack{P \in \mathrm{Spec}\, A \\ \dim A/P = d}} \ell_{A_P}(A_P)[\mathrm{Spec}\, A/P] \in \mathrm{A}_d(A)_{\mathbb{Q}}.$$

(e) Assume that $A$ is normal. Let $\mathrm{cl}(\omega_A) \in \mathrm{Cl}(A)$ be the isomorphism class containing $\omega_A$. Then, we have $\tau_{d-1} = \mathrm{cl}(\omega_A)/2$ in $\mathrm{A}_{d-1}(A)_{\mathbb{Q}} = \mathrm{Cl}(A)_{\mathbb{Q}}$.

We define the notion of *Roberts rings* as in Definition 1.2.

The category of Roberts rings contains complete intersections (see (c) as above), quotient singularities and Galois extensions of regular local rings. There are examples of Gorenstein non-Roberts rings. (It is proved in [6] that

$$\frac{k[x_{ij} \mid i = 1, \dots, m \ ; \ j = 1, \dots, n]_{(\{x_{ij}\})}}{I_t(x_{ij})}$$

is a Roberts ring if and only if it is a complete intersection. Therefore, if $m = n > 2$, the ring is a Gorenstein ring that is not a Roberts ring.) In 1985, P. Roberts [14] proved that the vanishing property of intersection multiplicity is satisfied for Roberts rings. We refer the reader to basic facts and examples of Roberts rings to [7] and [8].

**Remark 6.1** By the diagram (2.5), we immediately obtain that, if $A$ is a Roberts ring, then so is the completion $\hat{A}$. (By the commutativity of the diagram (2.5), we have $\tau_{\hat{A}/\hat{T}}([\hat{A}]) = f_*(\tau_{A/T}([A]))$. Remark that the map $\mathrm{A}_*(A)_{\mathbb{Q}} \xrightarrow{f_*} \mathrm{A}_*(\hat{A})_{\mathbb{Q}}$ in the diagram (2.5) is graded.)

On the other hand, assume that $\hat{A}$ is a Roberts ring. As in (I), the Riemann-Roch map $\tau_{\hat{A}/S}$ is independent of a regular local ring $S$. Hence, we may assume $\tau_{\hat{A}/\hat{T}}([\hat{A}]) \in \mathrm{A}_{\dim A}(\hat{A})_{\mathbb{Q}}$. Therefore, if $f_*$ is injective, then $A$ is a Roberts ring, too. In a sense, the converse is also true as we shall see in the following proposition.

We give some equivalent conditions to Question 1.4.

**Proposition 6.2** *All local rings are assumed to be homomorphic images of excellent regular local rings. Then, the following conditions are equivalent:*

*(1) The induced map* $\mathrm{G}_0(A)_{\mathbb{Q}} \to \mathrm{G}_0(\hat{A})_{\mathbb{Q}}$ *is injective for any local ring $A$, that is, Question 1.4 is true.*

*(2) The induced map* $\mathrm{A}_{\dim A-1}(A)_{\mathbb{Q}} \to \mathrm{A}_{\dim \hat{A}-1}(\hat{A})_{\mathbb{Q}}$ *is injective for any reduced equi-dimensional local ring $A$.*

(3) *For any local ring $A$, $A$ is a Roberts ring if so is $\hat{A}$. (That is to say, Question 1.3 is true.)*

*Proof.* We have already seen in Proposition 2.4 and Remark 6.1 that the condition (1) implies both of (2) and (3).

We first prove (2) $\Longrightarrow$ (1). Let ${}^{h}\!A$ denote the henselization of $A$. Since ${}^{h}\!A \to \hat{A}$ induces the injection $\mathrm{G}_0({}^{h}\!A)_{\mathbb{Q}} \to \mathrm{G}_0(\hat{A})_{\mathbb{Q}}$ by Theorem 1.5 (i), it is sufficient to show the injectivity of $g_* : \mathrm{G}_0(A)_{\mathbb{Q}} \to \mathrm{G}_0({}^{h}\!A)_{\mathbb{Q}}$. It is equvalent to the injectivity of $g_* : \mathrm{A}_*(A)_{\mathbb{Q}} \to \mathrm{A}_*({}^{h}\!A)_{\mathbb{Q}}$ since the diagram

$$\begin{array}{ccc} \mathrm{G}_0(A)_{\mathbb{Q}} & \xrightarrow{\tau_{A/T}} & \mathrm{A}_*(A)_{\mathbb{Q}} \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ \mathrm{G}_0({}^{h}\!A)_{\mathbb{Q}} & \xrightarrow{\tau_{{}^{h}\!A/{}^{h}\!T}} & \mathrm{A}_*({}^{h}\!A)_{\mathbb{Q}} \end{array}$$

is commutative by Lemma 4.1 (c) in [7]. Let $P_1, \ldots, P_s$ be prime ideals of $A$ of coheight $l$, and let $n_1, \ldots, n_r$ be integers such that $g_*(\overline{\sum_i n_i[\mathrm{Spec}\, A/P_i]}) = 0$, where $\overline{\sum_i n_i[\mathrm{Spec}\, A/P_i]}$ denotes the image of $\sum_i n_i[\mathrm{Spec}\, A/P_i]$ in $\mathrm{A}_l(A)_{\mathbb{Q}}$. We want to prove $\overline{\sum_i n_i[\mathrm{Spec}\, A/P_i]} = 0$ in $\mathrm{A}_l(A)_{\mathbb{Q}}$. We may assume $l < \dim A$. There are prime ideals $Q_1, \ldots, Q_t$ of ${}^{h}\!A$ of coheight $l+1$, elements $a_j \in {}^{h}\!A \setminus Q_j$ for $j = 1, \ldots, t$, and integers $b \neq 0$, $b_1, \ldots, b_t$ such that

$$b \sum_i n_i[\mathrm{Spec}\, {}^{h}\!A/P_i{}^{h}\!A] = \sum_j b_j \mathrm{div}(Q_j, a_j).$$

On the other hand, $\dim A/(Q_j \cap A) = l + 1$ holds for each $j$, since any fibre of a henselization has dimension 0. Set $I = (Q_1 \cap A) \cap \cdots \cap (Q_t \cap A)$ and $g' : A/I \to {}^{h}\!A/I{}^{h}\!A$. We may assume that $I$ is contained in all of $P_i$'s. Then we have $g'_*(\overline{\sum_i n_i[\mathrm{Spec}\, A/P_i]}) = 0$. Since the diagram

$$\begin{array}{ccc} \mathrm{A}_l(A)_{\mathbb{Q}} & \xrightarrow{g_*} & \mathrm{A}_l({}^{h}\!A)_{\mathbb{Q}} \\ \uparrow & & \uparrow \\ \mathrm{A}_l(A/I)_{\mathbb{Q}} & \xrightarrow{g'_*} & \mathrm{A}_l({}^{h}\!A/I{}^{h}\!A)_{\mathbb{Q}} \end{array}$$

is commutative, we have only to show that $g'_*$ is injective. Here, vertical maps of the diagram are induced by proper morphisms $\mathrm{Spec}\, A/I \to \mathrm{Spec}\, A$ and $\mathrm{Spec}\, {}^{h}\!A/I{}^{h}\!A \to \mathrm{Spec}\, {}^{h}\!A$. Remark that ${}^{h}\!A/I{}^{h}\!A$ coincides with the henselization of $A/I$. Replacing $A/I$ with $A$, it is sufficient to show that, for a reduced equi-dimensional local ring $A$, the map $\mathrm{A}_{\dim A-1}(A)_{\mathbb{Q}} \to \mathrm{A}_{\dim {}^{h}\!A-1}({}^{h}\!A)_{\mathbb{Q}}$ is injective. Since $\mathrm{A}_{\dim A-1}(A)_{\mathbb{Q}} \to \mathrm{A}_{\dim \hat{A}-1}(\hat{A})_{\mathbb{Q}}$ is injective by (2), so is $\mathrm{A}_{\dim A-1}(A)_{\mathbb{Q}} \to \mathrm{A}_{\dim {}^{h}\!A-1}({}^{h}\!A)_{\mathbb{Q}}$.

Next we shall prove (3) $\Longrightarrow$ (2). Assume that $\mathrm{A}_{\dim A-1}(A)_{\mathbb{Q}} \xrightarrow{f_*} \mathrm{A}_{\dim \hat{A}-1}(\hat{A})_{\mathbb{Q}}$ is not injective for an equi-dimensional local ring $A$. Let $c$ be a non-zero element of $\mathrm{A}_{\dim A-1}(A)_{\mathbb{Q}}$ such that $f_*(c) = 0$. Put $d = \dim A$. Since $\tau_{A/T} : \mathrm{G}_0(A)_{\mathbb{Q}} \to$

$\mathrm{A}_*(A)_{\mathbb{Q}}$ is an isomorphism, there exist finitely generated $A$-modules $M$ and $N$ with dimension less than $d$ such that

$$\tau_{A/T}\left(\frac{[M]-[N]}{n}\right) = c - (\tau_{d-1} + \tau_{d-2} + \cdots + \tau_0)$$

for some positive integer $n$. Since $A$ is equi-dimensional, we may assume that $[N] = 0$. Let $B$ denote the idealization $A \ltimes (A^{n-1} \oplus M)$. Then we have the following commutative diagram of isomorphisms:

$$\begin{array}{ccc} \mathrm{G}_0(B)_{\mathbb{Q}} & \stackrel{\tau_{B/T}}{\longrightarrow} & \mathrm{A}_*(B)_{\mathbb{Q}} \\ \downarrow & & \downarrow \alpha \\ \mathrm{G}_0(A)_{\mathbb{Q}} & \stackrel{\tau_{A/T}}{\longrightarrow} & \mathrm{A}_*(A)_{\mathbb{Q}} \end{array}$$

Here, vertical maps are induced by the proper map $\operatorname{Spec} B \to \operatorname{Spec} A$. Then we have

$$\alpha\tau_{B/T}([B]) = \tau_{A/T}([B]) = \tau_{A/T}(n[A] + [M]) = n\tau_d + nc.$$

Hence $B$ is not a Roberts ring since $\alpha$ is a graded isomorphism. On the other hand, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{A}_*(B)_{\mathbb{Q}} & \stackrel{\gamma}{\longrightarrow} & \mathrm{A}_*(\hat{B})_{\mathbb{Q}} \\ \alpha \downarrow & & \downarrow \beta \\ \mathrm{A}_*(A)_{\mathbb{Q}} & \stackrel{f_*}{\longrightarrow} & \mathrm{A}_*(\hat{A})_{\mathbb{Q}} \end{array}$$

where $\beta$ is the map induced by the proper morphism $\operatorname{Spec} \hat{B} \to \operatorname{Spec} \hat{A}$. Using Lemma 4.1 (c) in [7], we have $\gamma\tau_{B/T}([B]) = \tau_{\hat{B}/\hat{T}}([\hat{B}])$. Therefore, we have

$$\beta\tau_{\hat{B}/\hat{T}}([\hat{B}]) = \beta\gamma\tau_{B/T}([B]) = f_*\alpha\tau_{B/T}([B]) = f_*(n\tau_d + nc) = nf_*(\tau_d) \in \mathrm{A}_d(\hat{A})_{\mathbb{Q}}.$$

We have $\tau_{\hat{B}/\hat{T}}([\hat{B}]) \in \mathrm{A}_d(\hat{B})_{\mathbb{Q}}$ since $\beta$ is a graded isomorphism. Hence, $\hat{B}$ is a Roberts ring. **q.e.d.**

We give some remarks.

**Remark 6.3** Note that the henselization ${}^hA$ of a Noetherian local ring $A$ is the direct limit of rings $B$ as below. Therefore, in completely the same way as the proof of Proposition 6.2, it will be proved that, under the same situation, the following conditions are also equivalent to the conditions (1), (2) and (3) in Propostion 6.2:

(4) Let $(A, m)$ be a reduced equi-dimensional local ring. Let $n$ be a positive integer and let $a_1, \ldots, a_n$ be elements in $A$. Assume that $a_n \in m$ and $a_{n-1} \notin m$. Put $B = A[x]_{(m,x)}/(x^n + a_1x^{n-1} + \cdots + a_n)$, where $x$ is a variable. Then, for any rings $A$ and $B$ that satisfy the assumption as above, the induced map $\mathrm{G}_0(A)_{\mathbb{Q}} \to \mathrm{G}_0(B)_{\mathbb{Q}}$ is injective.

(5) For any rings $A$ and $B$ that satisfy the same assumption as in (4), the induced map $\mathrm{A}_{\dim A-1}(A)_{\mathbb{Q}} \to \mathrm{A}_{\dim B-1}(B)_{\mathbb{Q}}$ is injective.

(6) For any rings $A$ and $B$ that satisfy the same assumption as in (4), $A$ is a Roberts ring if so is $B$.

(7) Let $(A, m)$ be a reduced equi-dimensional local ring. Let $n$ be a positive integer and let $a_1, \ldots, a_n$ be elements in $A$. Assume that $a_n \in m$ and $a_{n-1} \notin m$. Let $h(x)$ be in $A[x]$ such that $h(0) \notin m$, where $x$ is a variable. Put $C = A[x, h(x)^{-1}]/(x^n + a_1 x^{n-1} + \cdots + a_n)$. Then, for any rings $A$ and $C$ that satisfy the assumption as above, the induced map $\mathrm{G}_0(A)_{\mathbb{Q}} \to \mathrm{G}_0(C)_{\mathbb{Q}}$ is injective.

**Remark 6.4** Let $f : (A, m) \to (B, n)$ be a flat local homomorphism of Noetherian local rings.

For an ideal $I$ of $A$, set

$$[\mathrm{Spec}\, A/I] = \sum_{P \in \mathrm{Assh}_A A/I} \ell_{A_P}(A_P/IA_P)[\mathrm{Spec}\, A/P],$$

where $\mathrm{Assh}_A A/I = \{P \in \mathrm{Min}_A A/I \mid \dim A/P = \dim A/I\}$. Then, we obtain a graded morphism $f_* : \mathrm{A}_*(A) \to \mathrm{A}_*(B)$ defined by $f_*([\mathrm{Spec}\, A/Q]) = [\mathrm{Spec}\, B/QB]$.

If $B/PB$ is equi-dimensional for any minimal prime ideal $P$ of $A$, then $B/QB$ is equi-dimensional for any prime ideal $Q$ of $A$. In the case, $f_*$ satisfies

$$f_*([\mathrm{Spec}\, A/Q]) = \sum_{\mathfrak{q}} \ell_{B_\mathfrak{q}}(B_\mathfrak{q}/QB_\mathfrak{q})[\mathrm{Spec}\, B/\mathfrak{q}],$$

where the sum is taken over all minimal prime ideals of $B/QB$.

If the closed fibre $B/mB$ is Cohen-Macaulay, then $B/QB$ is equi-dimensional for any prime ideal $Q$ of $A$. It is easily verified since all fibres are Cohen-Macaulay if so is the closed fibre (that is proved using Macaulayfication due to Kawasaki [4]). We remark that flat local homomorphisms in (2) and (5) in Proposition 6.2 and Remark 6.3 satisfy the condition.

For a local ring $(A, m)$, we set $A_s = A[x_1, \ldots, x_s]_{mA[x_1,\ldots,x_s]}$, where $x_1, \ldots, x_s$ are variables.

**Proposition 6.5** *Let $f : (A, m) \to (B, n)$ be a flat local homomorphism of Noetherian local rings.*

*(a) Assume that $B/n$ is finitely generated over $A/m$ as a field and $\mathrm{G}_0(A_s)_{\mathbb{Q}} \to \mathrm{G}_0(\widehat{A_s})_{\mathbb{Q}}$ is injective for $s = \mathrm{tr.deg}_{A/m} B/n$, that is the transcendence degree of $B/n$ over $A/m$. Then, both $f_* : \mathrm{G}_0(A)_{\mathbb{Q}} \to \mathrm{G}_0(B)_{\mathbb{Q}}$ and $f_* : \mathrm{A}_*(A)_{\mathbb{Q}} \to \mathrm{A}_*(B)_{\mathbb{Q}}$ are injective.*

(b) *Suppose that $A$ contains a field of characteristic* 0. *If Question 1.4 is true for any local ring, then both $f_* : \mathrm{G}_0(A)_{\mathbb{Q}} \to \mathrm{G}_0(B)_{\mathbb{Q}}$ and $f_* : \mathrm{A}_*(A)_{\mathbb{Q}} \to \mathrm{A}_*(B)_{\mathbb{Q}}$ are injective.*

*Proof.* We shall only prove the injectivity of the maps between Grothendieck groups. The injectivity of the maps between Chow groups will be proved in completely the same way.

First we shall prove (a). Take $t_1, \ldots, t_s \in B$ such that $\overline{t_1}, \ldots, \overline{t_s} \in B/n$ is a transcendence basis over $A/m$. Consider the homomorphisms

$$A \xrightarrow{g} D = A[x_1, \ldots, x_s]_{mA[x_1,\ldots,x_s]} \xrightarrow{h} B,$$

where $h$ is defined by $h(x_i) = t_i$ for each $i$. By the local flatness criterion (e.g., Theorem 22.3 in [10]), we know that $h$ is flat. Since $f = hg$, we have $f_* = h_* g_*$. We shall prove that both $g_*$ and $h_*$ are injective.

We first prove that $g_*$ is injective. We may assume $s = 1$. We have only to prove that a flat map $A \to A[x, p(x)^{-1}]$ induces the injective map $\mathrm{G}_0(A)_{\mathbb{Q}} \to \mathrm{G}_0(A[x, p(x)^{-1}])_{\mathbb{Q}}$ for $p(x) \in A[x] \setminus mA[x]$. Take a monic polynomial $q(x) \in A[x]$ of positive degree such that $\overline{p(x)}$ and $\overline{q(x)}$ (in $(A/m)[x]$) are relatively prime. Since $(p(x), q(x), m)A[x] = A[x]$, $p(x)$ is a unit in $A[x]/(q(x))$. Therefore,

$$A[x, p(x)^{-1}]/(q(x)) = A[x]/(q(x)) \tag{6.6}$$

is satisfied. The commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & A[x, p(x)^{-1}] \\ & \searrow & \downarrow \\ & & A[x, p(x)^{-1}]/(q(x)) \end{array}$$

induces the following commutative diagram:

$$\begin{array}{ccc} \mathrm{G}_0(A)_{\mathbb{Q}} & \longrightarrow & \mathrm{G}_0(A[x, p(x)^{-1}])_{\mathbb{Q}} \\ & \searrow & \downarrow \\ & & \mathrm{G}_0(A[x, p(x)^{-1}]/(q(x)))_{\mathbb{Q}} \end{array}$$

Here, the vertical map sends $[M]$ to $[M/q(x)M] - [0 :_M q(x)]$ for each finitely generated $A[x, p(x)^{-1}]$-module $M$. Since $A[x, p(x)^{-1}]/(q(x))$ is a finitely generated $A$-free module by (6.6), the map $\mathrm{G}_0(A)_{\mathbb{Q}} \to \mathrm{G}_0(A[x, p(x)^{-1}]/(q(x)))_{\mathbb{Q}}$ is injective. Hence $\mathrm{G}_0(A)_{\mathbb{Q}} \to \mathrm{G}_0(A[x, p(x)^{-1}])_{\mathbb{Q}}$ is injective.

We next prove that $h_*$ is injective. Remark that $(D, P) \to (B, n)$ is a flat local homomorphism such that $B/n$ is a finite algebraic extension of $D/P$. Since $\mathrm{G}_0(D)_{\mathbb{Q}} \to \mathrm{G}_0(\hat{D})_{\mathbb{Q}}$ is injective, we may assume that both $D$ and $B$ are complete. We shall prove the following claim:

**Claim 6.7** *Let $h : (D, P) \to (B, n)$ be a flat local homomorphism of complete local rings such that $B/n$ is a finite algebraic extension of $D/P$. Then, the induced map $h_* : \mathrm{G}_0(D)_{\mathbb{Q}} \to \mathrm{G}_0(B)_{\mathbb{Q}}$ is injective.*

Take $q_1, \ldots, q_t \in n$ such that the image $\overline{q_1}, \ldots, \overline{q_t}$ is a system of parameters of $B/PB$. Put $B' = B/(q_1, \ldots, q_t)$. Let $\alpha : \mathrm{G}_0(B)_{\mathbb{Q}} \to \mathrm{G}_0(B')_{\mathbb{Q}}$ be a map defined by $\alpha([M]) = \sum_i (-1)^i [H_i(\mathbb{K}.\otimes_B M)]$, where $\mathbb{K}.$ is the Koszul complex over $B$ with respect to $q_1, \ldots, q_t$. Remark that $h' : D \to B'$ is finite, because $D$ is complete. We have a map $h'^* : \mathrm{G}_0(B')_{\mathbb{Q}} \to \mathrm{G}_0(D)_{\mathbb{Q}}$ defined by $h'^*([M]) = [M]$.

We denote by $\phi$ the composite map of

$$\mathrm{G}_0(D)_{\mathbb{Q}} \xrightarrow{h_*} \mathrm{G}_0(B)_{\mathbb{Q}} \xrightarrow{\alpha} \mathrm{G}_0(B')_{\mathbb{Q}} \xrightarrow{h'^*} \mathrm{G}_0(D)_{\mathbb{Q}}.$$

We have only to prove that the composite map is an isomorphism. It is enough to show that, for any prime ideal $Q$ of $D$,

$$(6.8) \quad \begin{aligned} & \phi([D/Q]) \\ = \; & [B/n : D/P] e_{(q_1, \ldots, q_t)}(B/PB) \cdot [D/Q] + (\text{lower dimensional terms}) \end{aligned}$$

is satisfied, where $e_{(q_1, \ldots, q_t)}(B/PB)$ denotes the multiplicity of $B/PB$ with respect to $(q_1, \ldots, q_t)$.

We now start to verify the equality as above. Replacing $D/Q$ by $D$, we may assume that $D$ is an integral domain and $Q = 0$. Let $p_1, \ldots, p_d$ be a system of parameters of $D$, and $\mathbb{L}.$ denotes the Koszul complex over $D$ with respect to $p_1, \ldots, p_d$. Let $\beta : \mathrm{G}_0(D)_{\mathbb{Q}} \to \mathbb{Q}$ be a map defined by $\beta([M]) = \sum_i (-1)^i [H_i(\mathbb{L}.\otimes_D M)]$. Then, we have

$$\begin{aligned} \beta\phi([D]) &= [B/n : D/P] e_{(p_1, \ldots, p_d, q_1, \ldots, q_t)}(B) \\ &= [B/n : D/P] e_{(p_1, \ldots, p_d)}(D) e_{(q_1, \ldots, q_t)}(B/PB). \end{aligned}$$

Since $\beta([M]) = \mathrm{rank}_D M \cdot e_{(p_1, \ldots, p_d)}(D)$, the equality (6.8) is proved. We have completed the proof of (a).

We next prove (b). We may assume that both $A$ and $B$ are complete. Since $A$ contains a field of characteristic 0, we can take a coefficient field $K$ (resp. $L$) of $A$ (resp. $B$) such that $K \subseteq L$. Let $M$ be an intermediate field such that $L$ is algebraic over $M$ and $M$ is purely transcendental over $K$. Set

$$A = K[[y_1, \ldots, y_t]]/I.$$

Put

$$C = M[[y_1, \ldots, y_t]]/IM[[y_1, \ldots, y_t]] \quad \text{and} \quad D = L[[y_1, \ldots, y_t]]/IL[[y_1, \ldots, y_t]].$$

Then we have flat local homomorphisms

$$A \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{r} B$$

such that $f = rhg$. We shall prove that $g_*$, $h_*$ and $r_*$ are injective.

The injectivity of $r_*$ follows from Claim 6.7.

Next we prove that $h_*$ is injective. It is easy to see that $h$ coincides with the composite map of

$$C \xrightarrow{h_1} C \otimes_M L \xrightarrow{h_2} D.$$

It is easily verified that $C \otimes_M L$ is a Noetherian local ring and $D$ is the completion of $C \otimes_M L$. Since we are assuming that Question 1.4 is true, $h_{2*}$ is injective. Since $L$ is a direct limit of finite algebraic extensions $M'$ over $M$, $C \otimes_M L$ is a direct limit of rings $C_{M'} = M'[[y_1, \ldots, y_t]]/IM'[[y_1, \ldots, y_t]]$. Since $C_{M'}$ is finitely generated free $C$-module, the map $\mathrm{G}_0(C)_{\mathbb{Q}} \to \mathrm{G}_0(C_{M'})_{\mathbb{Q}}$ is injective. Therefore $h_{1*}$ is injective.

Next we prove that $g_*$ is injective. Let $\{t_\lambda \mid \lambda \in \Lambda\}$ be a transcendence basis of $M$ over $K$ suth that $M = K(\{t_\lambda \mid \lambda \in \Lambda\})$. Set

$$E = A[t_\lambda \mid \lambda \in \Lambda]_{mA[t_\lambda \mid \lambda \in \Lambda]}.$$

It is easy to see that $g$ coincides with the composite map of

$$A \xrightarrow{g_1} E \xrightarrow{g_2} C.$$

It is easy to see that $E$ is a Noetherian local ring and $C$ is the completion of $E$. Since we are assuming that Question 1.4 is true, $g_{2*}$ is injective. It is easy to see that $E$ is the direct limit of rings $A_s = A[t_1, \ldots, t_s]_{mA[t_1,\ldots,t_s]}$. As we have seen in the proof of (a), the map $\mathrm{G}_0(A)_{\mathbb{Q}} \to \mathrm{G}_0(A_s)_{\mathbb{Q}}$ is injective. Hence, $g_{1*}$ is injective. **q.e.d.**

Using Theorem 1.5 and Proposition 6.5, the following Corollary is immediately proved:

**Corollary 6.9** *Let $(A, m) \to (B, n)$ be a flat local homomorphism of excellent local rings. Assume that $A$ has at most isolated singularity.*

1. *If $B/n$ is finitely generated over $A/m$, then $\mathrm{G}_0(A)_{\mathbb{Q}} \to \mathrm{G}_0(B)_{\mathbb{Q}}$ and $\mathrm{A}_*(A)_{\mathbb{Q}} \to \mathrm{A}_*(B)_{\mathbb{Q}}$ are injective.*
2. *If $A$ contains a field of characteristic $0$, then $\mathrm{G}_0(A)_{\mathbb{Q}} \to \mathrm{G}_0(B)_{\mathbb{Q}}$ and $\mathrm{A}_*(A)_{\mathbb{Q}} \to \mathrm{A}_*(B)_{\mathbb{Q}}$ are injective.*

Let $(A, m) \to (B, n)$ be a flat local homomorphism with closed fibre $F$. Then, we can prove the following:

- If $B$ is a Roberts ring, then so is $F$.
- Even if both $A$ and $F$ are Roberts rings, $B$ is not necessary so.

It is natural to ask equivalence between the Robertsness of $A$ and that of $B$ under some strong assumption on $F$.

**Proposition 6.10** *Let $f : (A, m) \to (B, n)$ be a flat local homomorphism of Noetherian local rings such that $B/n$ is finitely generated over $A/m$ as a field.*

*(a) Suppose that $B/n$ is separable over $A/m$ and $B/mB$ is a complete intersection. Assume that $\mathrm{G}_0(B)_{\mathbb{Q}} \to \mathrm{G}_0(\hat{B})_{\mathbb{Q}}$ is injective. Then, if $A$ is a Roberts ring, so is $B$.*

*(b) Suppose that $B/mB$ is Cohen-Macaulay. Assume that $\mathrm{G}_0(A_s)_{\mathbb{Q}} \to \mathrm{G}_0(\widehat{A_s})_{\mathbb{Q}}$ is injective for $s = \mathrm{tr.deg}_{A/m} B/n$. Then, if $B$ is a Roberts ring, so is $A$.*

We omit a proof of the proposition as above. Using Theorem 1.5, we immediately obtain the following corollary:

**Corollary 6.11** *Let $f : (A, m) \to (B, n)$ be a flat local homomorphism of Noetherian local rings.*

1. *Suppose that $B/n$ is a finitely generated separable extension over $A/m$ and $B/mB$ is a complete intersection. Assume that $\mathrm{G}_0(A_s)_{\mathbb{Q}} \to \mathrm{G}_0(\widehat{A_s})_{\mathbb{Q}}$ and $\mathrm{G}_0(B)_{\mathbb{Q}} \to \mathrm{G}_0(\hat{B})_{\mathbb{Q}}$ are injective for $s = \mathrm{tr.deg}_{A/m} B/n$. Then, $A$ is a Roberts ring if and only if so is $B$.*

2. *Suppose that both $A$ and $B$ are excellent and $f$ is étale essentially of finite type. Assume that $A$ has at most isolated singularity. Then, $A$ is a Roberts ring if and only if so is $B$.*

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