On Chow groups of G-graded rings

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Abstract

Let A be a Notherian ring graded by a finitely generated Abelian group G. It is shown that a Chow group $\mathrm{A}_\bullet(A)$ of A is determined by cycles and a rational equivalence with respect to certain G-graded ideals of A. In particular, $\mathrm{A}_\bullet(A)$ is isomorphic to the equivariant Chow group of A if G is torsion free.

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1 Introduction

In this paper, we study a Chow group of a Noetherian ring graded by a finitely generated Abelian group and prove that a Chow group of such a graded ring is determined by its cycles and rational equivalence of graded objects. For certain varieties, it had been proved by Fulton, MacPherson, Sottile and Sturmfels[2]. Precisely speaking, their result is stated that a Chow group of a variety (over $\mathbb{C}$) with an action of a connected solvable linear algebraic group is isomorphic to its equivariant Chow group. The aim of this paper is to give an elementary proof to a similar statement for a Chow group of a graded ring. We do not have to assume that the given ring contains a field.

Let G be a finitely generated Abelian group (not necessary torsion free) and $A = \bigoplus_{g \in G} A_g$ be a Noetherian G-graded ring. We call that a G-graded ideal $\mathfrak{p} \subset A$ is a G-prime ideal, if every homogeneous nonzero element of $A/\mathfrak{p}$ is a nonzero divisor. If G is torsion free, then a G-prime ideal is actually a prime ideal. Otherwise a G-prime ideal is not necessary a prime ideal and these ideals involve much information. We define a group $\mathrm{A}^G_\bullet(A) = \mathrm{Z}^G_\bullet(A)/\mathrm{Rat}^G_\bullet(A)$, where $\mathrm{Z}^G_\bullet(A)$ is the free Abelian group generated by $[A/\mathfrak{p}]$ for all G-prime ideal $\mathfrak{p}$ and $\mathrm{Rat}^G_\bullet(A)$ is a subgroup of $\mathrm{Z}^G_\bullet(A)$ which is a graded analogue of rational equivalence determined by homogeneous elements and G-prime ideals (see (2.5)). Then our main result is stated as follows.

Theorem 1.1 *Put* $W = \{P \in \mathrm{Spec}(A) \mid P \in \mathrm{Min}_A(A/\mathfrak{p})$ *for some* G*-prime ideal* $\mathfrak{p}\} \subset \mathrm{Spec}(A)$. *If* $G \cong \mathbb{Z}^m \oplus T$ *with* $|T| < \infty$, *then there is the natural map* $\varphi : \mathrm{A}^G_\bullet(A) \longrightarrow \mathrm{A}_\bullet(A)$ *satisfying following conditions.*

(1) $\mathrm{A}_\bullet(A)$ *is generated by* $\{[A/P] \mid P \in W\}$.

(2) $|T|\mathrm{Ker}(\varphi) = (0)$.

In particular, $\mathrm{A}^G_\bullet(A)$ *is isomorphic to* $\mathrm{A}_\bullet(A)$ *via* φ *if* G *is torsion free.*

Example 3.4 shows that the map φ in Theorem 1.1 is not always an isomorphism.

2 Definition of $\mathrm{A}^G_\bullet(A)$

Let A be a Noetherian ring essentially of finite type over a regular domain R. We treat a Chow group of A using relative dimension instead of the usual Krull dimension (Chap. 20 in Fulton[1]). Relative dimension $\dim_R(A/P)$ is defined as $\dim_R(A/P) = \mathrm{tr.deg}(k(P)/k(R \cap P)) - \mathrm{ht}_R(R \cap P)$ for each $P \in \mathrm{Spec}(A)$. Note that $\dim_R(A/P) = \dim_R(A/Q) + \mathrm{ht}_{A/P}(Q/P)$ for $P \subset Q \in \mathrm{Spec}(A)$. If S is a multiplicatively closed subset of A with $S \cap P = \phi$ for $P \in \mathrm{Spec}(A)$, then we have $\dim_R(S^{-1}A/PS^{-1}A) = \dim_R(A/P)$. For a finitely generated A-module M, we set

$$\begin{aligned} \dim_R(M) &= \sup\{\dim_R(A/P) \mid P \in \mathrm{Supp}_A(M)\} \\ \mathrm{Assh}_R(M) &= \{P \in \mathrm{Supp}_A(M) \mid \dim_R(M) = \dim_R(A/P)\}. \end{aligned}$$

The i-th cycles $\mathrm{Z}_i(A)$ of A is the free Abelian group generated by $[A/P]$ for all $P \in \mathrm{Spec}(A)$ with $\dim_R(A/P) = i$. $\mathrm{Rat}_i(A)$ is the subgroup of $\mathrm{Z}_i(A)$ generated by $\mathrm{div}_A(Q, a)$ for every $Q \in \mathrm{Spec}(A)$ with $\dim_R(A/Q) = i + 1$ and for every $a \in A \setminus Q$, where

$$\mathrm{div}_A(Q, a) = \sum_{P \in \mathrm{Min}_A(A/(a,Q))} \ell_{A_P}(A_P/(a, Q)A_P)[A/P].$$

If no confusion is possible, we denote $\mathrm{div}_A(Q, a)$ simply by $\mathrm{div}(Q, a)$. The i-th Chow group $\mathrm{A}_i(A)$ is defined to be the quotient group $\mathrm{Z}_i(A)/\mathrm{Rat}_i(A)$. We define the Chow group (resp. cycles, rational equivalence) of A by $\mathrm{A}_\bullet(A) = \oplus_{i \in \mathbb{Z}} \mathrm{A}_i(A)$ (resp. $\mathrm{Z}_\bullet(A) = \oplus_{i \in \mathbb{Z}} \mathrm{Z}_i(A)$, $\mathrm{Rat}_\bullet(A) = \oplus_{i \in \mathbb{Z}} \mathrm{Rat}_i(A)$).

The aim of this section is to define a similar notion to Chow groups for graded rings and to define the natural map from this group to the ordinary Chow group. Let $(G, +)$ be a finitely generated Abelian group. We say that a ring A is a G-graded ring, if there exists a family $\{A_g\}_{g \in G}$ of subgroups of A such that $A = \bigoplus_{g \in G} A_g$ and $A_g A_h \subset A_{g+h}$ for every

$g, h \in G$. Similarly, a G-graded A-module is an A-module M with a family $\{M_g\}_{g\in G}$ of subgroups of M such that $M = \bigoplus_{g\in G} M_g$ and $A_g M_h \subset M_{g+h}$ for every $g, h \in G$. The subgroup M_g is called the component of M of degree g. An element $x \in M_g \setminus \{0\}$ is called a homogeneous element of degree g and we write as $\deg x = g$.

Throughout the paper, we assume that A is a Noetherian G-graded ring such that A_0 is an R-algebra and A is essentially of finite type over R, where R is assumed to be an excellent regular domain.

Definition 2.1 A G-graded ideal $\mathfrak{p}$ of A is said to be a G-prime ideal, if every homogeneous nonzero element of $A/\mathfrak{p}$ is not a divisor of zero. We denote the set of all G-prime ideals by $\mathrm{Spec}^G(A)$.

Remark 2.2 If G is torsion free, then G-prime ideals are nothing but G-graded prime ideals and $\mathrm{Spec}^G(A) \subset \mathrm{Spec}(A)$. However, if G has torsion, then G-prime ideals are not necessary prime ideals. For example, put $A = \mathbb{Q}[X]/(X^2-1)$. We consider A as a $\mathbb{Z}/(2)$-graded ring by $\deg X = \bar{1} \in \mathbb{Z}/(2)$. Then A has no graded prime ideals, but $\mathrm{Spec}^G(A) = \{(0)\}$. G-prime ideals have a lot of information on G-graded rings and G-graded modules. they play roles of prime ideals in the category of G-graded rings (and the category of G-graded modules). See, for example, [4], [5].

Let $\mathfrak{p} \subset A$ be a G-prime ideal and let M be a G-graded A-module. We define a homogeneous localization $M_{(\mathfrak{p})}$ of M at $\mathfrak{p}$ by $M_{(\mathfrak{p})} = S^{-1}M$, where S is the set of all homogeneous elements of $A \setminus \mathfrak{p}$. We define a G-graded module $M(g)$ by $M(g) = M$ as the underlying A-modules, that is graded by $M(g)_h = M_{g+h}$ for $h \in G$.

For an ideal $P \subset A$, we put $P^* = \bigoplus_{g\in G} P \cap A_g$ the maximal graded ideal contained in P. If P is a prime ideal, then P^* is a G-prime ideal. Conversely, if $\mathfrak{p} \in \mathrm{Spec}^G(A)$, then $P^* = \mathfrak{p}$ is satisfied for every $P \in \mathrm{Ass}_A(A/\mathfrak{p})$ since any nonzero homogeneous element of $A/\mathfrak{p}$ is not a zero divisor and, therefore, $\mathrm{Spec}^G(A) = \{P^* \mid P \in \mathrm{Spec}(A)\}$. Furthermore, we have $\mathrm{Ass}_A(A/\mathfrak{p}) = \mathrm{Min}_A(A/\mathfrak{p})$ since $A_{(\mathfrak{p})}/\mathfrak{p}A_{(\mathfrak{p})}$ is G-simple. (Note that $\mathrm{Ass}_{A_{(\mathfrak{p})}}(A_{(\mathfrak{p})}/\mathfrak{p}A_{(\mathfrak{p})}) = \mathrm{Min}_{A_{(\mathfrak{p})}}(A_{(\mathfrak{p})}/\mathfrak{p}A_{(\mathfrak{p})})$ is satisfied since $A_{(\mathfrak{p})}/\mathfrak{p}A_{(\mathfrak{p})}$ is G-simple that will be defined in (2.6).)

We put $\mathrm{Supp}_A^G(M) = \{\mathfrak{p} \in \mathrm{Spec}^G(A) \mid M_{(\mathfrak{p})} \neq 0\}$ and denote by $\mathrm{Min}_A^G(M)$ (resp. $\mathrm{Assh}_R^G(M)$) the set of minimal G-prime ideals in $\mathrm{Supp}_A^G(M)$ (resp. the set of G-prime ideals $\mathfrak{p} \in \mathrm{Supp}_A^G(M)$ with $\dim_R(M) = \dim_R(A/\mathfrak{p})$). Note that $P \in \mathrm{Supp}_A(M)$ is equivalent to $P^* \in \mathrm{Supp}_A^G(M)$. For a G-graded A-module M, M has a finite filtration $(0) = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of G-graded submodules of M with $M_{i+1}/M_i \cong A/\mathfrak{p}_i(g_i)$ for some G-prime ideal $\mathfrak{p}_i$ and for some $g_i \in G$. Note that $\mathrm{Supp}_A^G(M) = \{\mathfrak{p} \in \mathrm{Spec}^G(A) \mid \mathfrak{p} \supset \mathfrak{p}_i \text{ for some } i\}$

and $\mathrm{Min}_A^G(M) \subset \{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\}$. If G is torsion free, then $\mathrm{Min}_A^G(M) = \mathrm{Min}_A(M)$ and $\mathrm{Assh}_R^G(M) = \mathrm{Assh}_R(M)$.

Definition 2.3 We denote by $\mathrm{Z}_i^G(A)$ the free Abelian group with basis $[A/\mathfrak{p}]$ consisting of all G-prime ideals $\mathfrak{p}$ such that $\dim_R(A/\mathfrak{p}) = i$. The G-cycles $\mathrm{Z}_\bullet^G(A)$ of A is defined to be the direct sum of $\mathrm{Z}_i^G(A)$ for all i.

For $\mathfrak{p} \in \mathrm{Min}_A^G(M)$, $\ell_{A_{(\mathfrak{p})}}^G(M_{(\mathfrak{p})})$ denotes the length of the maximal chain of G-graded submodules of $M_{(\mathfrak{p})}$. It is easy to see that $\ell_{A_{(\mathfrak{p})}}^G(M_{(\mathfrak{p})})$ is equal to $\ell_{A_\mathfrak{p}}(M_\mathfrak{p})$, if G is torsion free. We put $[M] = \sum_{\mathfrak{p} \in \mathrm{Assh}_R^G(M)} \ell_{A_{(\mathfrak{p})}}^G(M_{(\mathfrak{p})})\,[A/\mathfrak{p}] \in \mathrm{Z}_{\dim_R(M)}^G(A)$.

Definition 2.4 Suppose $\mathfrak{p} \in \mathrm{Spec}^G(A)$ and let $a \in A \setminus \mathfrak{p}$ be a homogeneous element. We put

$$\mathrm{div}^G(\mathfrak{p}, a) = \sum_{\mathfrak{q} \in \mathrm{Min}_A^G(A/(a,\mathfrak{p}))} \ell_{A_{(\mathfrak{q})}}^G(A_{(\mathfrak{q})}/(a,\mathfrak{p})A_{(\mathfrak{q})})\,[A/\mathfrak{q}]\,.$$

We define $\mathrm{Rat}_i^G(A)$ to be the subgroup of $\mathrm{Z}_\bullet^G(A)$ generated by $\mathrm{div}^G(\mathfrak{p}, a)$ for every $\mathfrak{p} \in \mathrm{Spec}^G(A)$ with $\dim_R(A/\mathfrak{p}) = i + 1$ and for every homogeneous element $a \in A \setminus \mathfrak{p}$. We put $\mathrm{Rat}_\bullet^G(A) = \sum_{i \in \mathbb{Z}} \mathrm{Rat}_i^G(A) \subset \mathrm{Z}_\bullet^G(A)$ and call it the G-rational equivalence of A.

Later, we will see $\mathrm{div}^G(\mathfrak{p}, a) \in \mathrm{Z}_i^G(A)$ if $\dim_R(A/\mathfrak{p}) = i + 1$ (Lemma 2.10). Hence we have $\mathrm{Rat}_i^G(A) \subset \mathrm{Z}_i^G(A)$ for each i and $\mathrm{Rat}_\bullet^G(A) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Rat}_i^G(A) \subset \mathrm{Z}_\bullet^G(A)$.

Definition 2.5 The i-th G-Chow group of A is defined by $\mathrm{A}_i^G(A) = \mathrm{Z}_i^G(A)/\mathrm{Rat}_i^G(A)$. We define the G-Chow group of A to be $\mathrm{A}_\bullet^G(A) = \bigoplus_{i \in \mathbb{Z}} \mathrm{A}_i^G(A) = \mathrm{Z}_\bullet^G(A)/\mathrm{Rat}_\bullet^G(A)$.

In order to compare $\mathrm{A}_\bullet^G(A)$ with $\mathrm{A}_\bullet(A)$, we need some lemmas on relative dimension of graded objects.

Definition-Proposition 2.6 ((1.6) in [4]) A G-graded ring A is said to be G-simple, if A has no proper G-graded ideal. If A is G-simple and $G' = \{g \in G \mid A_g \neq 0\}$, then A_0 is a field and A is ismorphic to a twisted group ring $A_0^t[G']$ of G' over the field A_0. It is proved in [4] that any G-simple ring is complete intersection.

Now, we have the following relative dimension formula for G-prime ideals.

Lemma 2.7 *Let* $\mathfrak{p} \in \mathrm{Spec}^G(A)$ *and* $P \in \mathrm{Spec}(A)$ *with* $P^* = \mathfrak{p}$. *Then we have*

$$\dim_R(A/P) = \dim_R(A/\mathfrak{p}) - \dim A_P/\mathfrak{p}A_P.$$

Particularly, $\dim_R(A/\mathfrak{p}) = \dim_R(A/Q)$ *is satisfied for all* $Q \in \mathrm{Ass}_A(A/\mathfrak{p})$, *i.e.* $\mathrm{Assh}_R(A/\mathfrak{p}) = \mathrm{Ass}_A(A/\mathfrak{p})$ *holds.*

Proof. We put $K = [A_{(\mathfrak{p})}/\mathfrak{p}A_{(\mathfrak{p})}]_0$ and $G(\mathfrak{p}) = \{g \in G \mid [A_{(\mathfrak{p})}/\mathfrak{p}A_{(\mathfrak{p})}]_g \neq 0\}$. Then we have

$$\begin{aligned}
\dim_R(A/P) &= \mathrm{tr.deg}(k(P)/K) + \mathrm{tr.deg}(K/k(R \cap \mathfrak{p}_0)) - \mathrm{ht}_R(R \cap \mathfrak{p}_0) \\
&= \dim(K^t[G(\mathfrak{p})]/PK^t[G(\mathfrak{p})]) + \mathrm{tr.deg}(K/k(R \cap \mathfrak{p}_0)) - \mathrm{ht}_R(R \cap \mathfrak{p}_0) \\
&= \dim(A_{(\mathfrak{p})}/PA_{(\mathfrak{p})}) + \mathrm{tr.deg}(K/k(R \cap \mathfrak{p}_0)) - \mathrm{ht}_R(R \cap \mathfrak{p}_0)
\end{aligned}$$

If $Q \in \mathrm{Ass}_A(A/\mathfrak{p})$, then $\dim A_{(\mathfrak{p})}/QA_{(\mathfrak{p})} = \dim A_{(\mathfrak{p})}/\mathfrak{p}A_{(\mathfrak{p})}$ by (2.6). Hence we have

$$\dim_R(A/\mathfrak{p}) - \dim_R(A/P) = \dim A_{(\mathfrak{p})}/\mathfrak{p}A_{(\mathfrak{p})} - \dim A_{(\mathfrak{p})}/PA_{(\mathfrak{p})} = \dim A_P/\mathfrak{p}A_P.$$

□

Corollary 2.8 *For* $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}^G(A)$ *with* $\mathfrak{p} \subset \mathfrak{q}$, *if we put* $r = \dim_R(A/\mathfrak{p}) - \dim_R(A/\mathfrak{q})$, *then there exists a saturated chain* $\mathfrak{p} = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{q}$ *of G-prime ideals.*

Proof. By (2.7), we have $r = \mathrm{ht}_{A/\mathfrak{p}}(\mathfrak{q}/\mathfrak{p})$. If $r > 0$, then there is a homogeneous element $a \in \mathfrak{q}$ such that $a \notin \mathfrak{p}$. Since a is not a zero divisor of $A/\mathfrak{p}$, we have $\mathfrak{p}_1 \in \mathrm{Min}_A^G(A/(a, \mathfrak{p}))$ with $\mathfrak{p}_1 \subset \mathfrak{q}$ and $\mathrm{ht}_{A/\mathfrak{p}_1}(\mathfrak{q}/\mathfrak{p}_1) = r - 1$. The assertion is proved by induction on r. □

Lemma 2.9 *(1)* $\mathrm{Min}_A(M) = \coprod_{\mathfrak{p} \in \mathrm{Min}_A^G(M)} \mathrm{Ass}_A(A/\mathfrak{p})$.

(2) $\mathrm{Assh}_R(M) = \coprod_{\mathfrak{p} \in \mathrm{Assh}_R^G(M)} \mathrm{Ass}_A(A/\mathfrak{p})$.

Proof. (1) It is easy to see $\coprod_{\mathfrak{p} \in \mathrm{Min}_A^G(M)} \mathrm{Ass}_A(A/\mathfrak{p}) = \bigcup_{\mathfrak{p} \in \mathrm{Min}_A^G(M)} \mathrm{Ass}_A(A/\mathfrak{p})$, since $\mathrm{Ass}_A(A/\mathfrak{p}) \cap \mathrm{Ass}_A(A/\mathfrak{q}) = \phi$ for $\mathfrak{p} = \mathfrak{q}$. Take $P \in \mathrm{Min}_A(M)$. Since $P \in \mathrm{Supp}_A(M)$, we know $\mathfrak{p} := P^* \in \mathrm{Supp}_A^G(M)$. If $\mathfrak{q}$ is a minimal G-prime ideal of M contained in $\mathfrak{p}$, then $\mathfrak{q} \subset P$. Since $\mathrm{Ass}_A(A/\mathfrak{q}) \subset \mathrm{Supp}_A(M)$, we have $P \in \mathrm{Min}_A(A/\mathfrak{q}) = \mathrm{Ass}_A(A/\mathfrak{q})$ and $\mathfrak{q} = P^* = \mathfrak{p}$. Thus $\mathrm{Min}_A(M) \subset \bigcup_{\mathfrak{p} \in \mathrm{Min}_A^G(M)} \mathrm{Ass}_A(A/\mathfrak{p})$ is satisfied. Conversely, take

$\mathfrak{p} \in \mathrm{Min}_A^G(M)$ and $P \in \mathrm{Ass}_A(A/\mathfrak{p})$. If $Q \in \mathrm{Min}_A(M)$ with $Q \subset P$, then $Q^* \in \mathrm{Min}_A^G(M)$ as above. We have $Q^* \subset P^* = \mathfrak{p}$ and, by the minimality of $\mathfrak{p}$, $Q^* = \mathfrak{p}$ is satisfied. Since P is a minimal prime ideal of $A/\mathfrak{p}$, we have $Q = P$. This completes the proof of (1).

(2) Take $P \in \mathrm{Min}_A(M)$ and put $\mathfrak{p} := P^*$. Then $P \in \mathrm{Assh}_R(M)$ if and only if $\dim_R(A/P) = \dim_R(M)$. On the other hand, by (2.7), we have $\dim_R(A/P) = \dim_R(A/\mathfrak{p})$. Hence $P \in \mathrm{Assh}_R(M)$ iff $\mathfrak{p} \in \mathrm{Assh}_R^G(M)$. □

Lemma 2.10 *Take $\mathfrak{p} \in \mathrm{Spec}^G(A)$ and a homogeneous element $a \in A \setminus \mathfrak{p}$. Then we have the following.*

(1) $\mathrm{Assh}_R^G(A/(a,\mathfrak{p})) = \mathrm{Min}_A^G(A/(a,\mathfrak{p}))$ *and* $\mathrm{Assh}_R(A/(a,\mathfrak{p})) = \mathrm{Min}_A(A/(a,\mathfrak{p}))$ *are satisfied.*

(2) $\dim_R(A/\mathfrak{q}) = \dim_R(A/\mathfrak{p}) - 1$ *for each* $\mathfrak{q} \in \mathrm{Min}_A^G(A/(a,\mathfrak{p}))$*, and* $\dim_R(A/Q) = \dim_R(A/\mathfrak{p}) - 1$ *for each* $Q \in \mathrm{Min}_A(A/(a,\mathfrak{p}))$ *are satisfied.*

Proof. The assertion (1) follows from the assertion (2).

(2) Let $\mathfrak{q} \in \mathrm{Min}_A^G(A/(a,\mathfrak{p}))$ and $Q \in \mathrm{Ass}_A(A/\mathfrak{q})$. Remark that, by (2.9), Q is a minimal prime ideal of $A/(a,\mathfrak{p})$. If $P \in \mathrm{Ass}_A(A/\mathfrak{p})$ with $Q \supset P$, then Q is a minimal prime ideal of $A/(a,P)$. Indeed, if we assume $Q_1 \in \mathrm{Min}_A(A/(a,P))$ such that $Q \supset Q_1$, then $\mathfrak{q} = Q^* \supset Q_1^* \supset (a,P)^* \supset (a,\mathfrak{p})$. Then, by the minimality of $\mathfrak{q}$, we have $\mathfrak{q} = Q^* = Q_1^*$ and, by the minimality of Q, we have $Q = Q_1$. Since $P^* = \mathfrak{p}$, we have $a \notin P$ and $\dim_R(A/\mathfrak{q}) = \dim_R(A/Q) = \dim_R(A/P) - \mathrm{ht}_{A/P}(Q/P) = \dim_R(A/\mathfrak{p}) - 1$. This completes the proof of Lemma. □

Now we define a group homomorphism from $\mathrm{Z}_\bullet^G(A)$ to $\mathrm{Z}_\bullet(A)$ as follows;

$$\begin{array}{rccl} \varphi & : & \mathrm{Z}_\bullet^G(A) & \longrightarrow & \mathrm{Z}_\bullet(A) \\ & & [A/\mathfrak{p}] & \longmapsto & \sum_{P\in\mathrm{Ass}_A(A/\mathfrak{p})} \ell_{A_P}(A_P/\mathfrak{p}A_P)[A/P]. \end{array}$$

Then, by (2.7), φ is a graded group homomorphism, namely, $\varphi(\mathrm{Z}_i^G(A)) \subset \mathrm{Z}_i(A)$ for each i. Sometimes, we consider that $\mathrm{Z}_\bullet^G(A)$ is a subgroup of $\mathrm{Z}_\bullet(A)$ via φ since φ is injective. Take $\mathfrak{p} \in \mathrm{Spec}^G(A)$ and $a \in \bigcup_{g\in G} A_g \setminus \mathfrak{p}$. By (2.10), we have

$$\begin{array}{rl} & \varphi(\mathrm{div}^G(\mathfrak{p},a)) \\ = & \sum_{\mathfrak{q}\in\mathrm{Min}_A^G(A/(a,\mathfrak{p}))} \ell_{A_{(\mathfrak{q})}}^G(A_{(\mathfrak{q})}/(a,\mathfrak{p})A_{(\mathfrak{q})})\varphi([A/\mathfrak{q}]) \\ = & \sum_{\mathfrak{q}\in\mathrm{Min}_A^G(A/(a,\mathfrak{p}))} \sum_{Q\in\mathrm{Ass}_A(A/\mathfrak{q})} \ell_{A_{(\mathfrak{q})}}^G(A_{(\mathfrak{q})}/(a,\mathfrak{p})A_{(\mathfrak{q})})\ell_{A_Q}(A_Q/\mathfrak{q}A_Q)[A/Q] \\ = & \sum_{\mathfrak{q}\in\mathrm{Min}_A^G(A/(a,\mathfrak{p}))} \sum_{Q\in\mathrm{Ass}_A(A/\mathfrak{q})} \ell_{A_Q}(A_Q/(a,\mathfrak{p})A_Q)[A/Q] \\ = & \sum_{Q\in\mathrm{Min}_A(A/(a,\mathfrak{p}))} \ell_{A_Q}(A_Q/(a,\mathfrak{p})A_Q)[A/Q] \\ = & \sum_{Q\in\mathrm{Assh}_R(A/(a,\mathfrak{p}))} \ell_{A_Q}(A_Q/(a,\mathfrak{p})A_Q)[A/Q] \\ = & [A/(a,\mathfrak{p})]\,. \end{array}$$

Then $[A/(a,\mathfrak{p})]$ belongs to $\mathrm{Rat}_{\bullet}(A)$, since a is a nonzero divisor of $A/\mathfrak{p}$ ((1.2.2) of [6]). Hence φ induces a graded group homomorphism $\varphi : \mathrm{A}^G_{\bullet}(A) \longrightarrow \mathrm{A}_{\bullet}(A)$. Henceforth we call this φ the natural homomorphism from $\mathrm{A}^G_{\bullet}(A)$ to $\mathrm{A}_{\bullet}(A)$.

In the same way as ordinary Chow groups, we have the following. (See, for example Ch. 1 of [6].)

Lemma 2.11 *(1) If $f : A \longrightarrow B$ is a flat G-graded ring homomorphism that is essentially of finite type of relative dimension k, then the map $\mathrm{Z}^G_i(A) \xrightarrow{f^*} \mathrm{Z}^G_{i+k}(B)$ that sends $[A/\mathfrak{p}]$ to $[B/\mathfrak{p}B]$ induces a map of G-Chow groups $\mathrm{A}^G_i(A) \xrightarrow{f^*} \mathrm{A}^G_{i+k}(B)$.*

(2) Let S be a multiplicatively closed subset of A consisting of homogeneous elements. Let $\mathrm{Z}^G_{\bullet}(S,A)$ denote the subgroup of $\mathrm{Z}^G_{\bullet}(A)$ generated by all $[A/\mathfrak{p}]$ such that $\mathfrak{p} \cap S \neq \phi$. Then the inclusion $\mathrm{Z}^G_{\bullet}(S,A) \hookrightarrow \mathrm{Z}^G_{\bullet}(A)$ induces an exact sequence

$$\mathrm{Z}^G_{\bullet}(S,A) \longrightarrow \mathrm{A}^G_{\bullet}(A) \longrightarrow \mathrm{A}^G_{\bullet}(S^{-1}A) \longrightarrow 0.$$

(3) A module finite G-graded ring homomorphism $g : A \longrightarrow B$ induces the map $g_* : \mathrm{A}^G_{\bullet}(B) \longrightarrow \mathrm{A}^G_{\bullet}(A)$ such that $g_*([B/\mathfrak{P}]) = \ell^G_{A_{(\mathfrak{p})}}(B_{(\mathfrak{P})}/\mathfrak{P}B_{(\mathfrak{P})})[A/\mathfrak{p}]$, where $\mathfrak{p} = A \cap \mathfrak{P}$. (Note that $\dim_R(B/\mathfrak{P}) = \dim_R(A/A \cap \mathfrak{P})$ for $\mathfrak{P} \in \mathrm{Spec}^G(B)$.)

$\square$

The following lemma is an easy consequence of (2.11) which will be used in the proof of the main result.

Lemma 2.12 *Let $A[x]$ be a polynomial ring over A. We regard $A[x]$ as a G-graded ring by $\deg(x) = g$ for some $g \in G$. Then we have an isomorphism $\mathrm{A}^G_{\bullet}(A[x]) \xrightarrow{\cong} \mathrm{A}^G_{\bullet}(A[x,x^{-1}])$ induced by $A[x] \longrightarrow A[x,x^{-1}]$.*

Proof. By (2.11), (2), it is enough to show that $\mathrm{A}^G_{\bullet}(A[x]) \longrightarrow \mathrm{A}^G_{\bullet}(A[x,x^{-1}])$ is injective. By (2.11), (2), the kernel of this map is generated by G-cycles $[A[x]/\mathfrak{P}]$ such that $x \in \mathfrak{P}$. For each such $\mathfrak{P}$, there is a G-prime ideal $\mathfrak{p} \subset A$ such that $\mathfrak{P} = (\mathfrak{p}, x)A[x]$ and $[A[x]/\mathfrak{P}] = \mathrm{div}^G(\mathfrak{p}A[x], x)$. Hence we have $[A[x]/\mathfrak{P}] = 0$ in $\mathrm{A}^G_{\bullet}(A[x])$. This completes the proof of Lemma 2.12. $\square$

3 Proof of Theorem 1.1

Proof of (1). Take $P \in \mathrm{Spec}(A)$ with $\dim A_P/P^*A_P = d$. We shall prove that $[A/P] \in \mathrm{A}_\bullet(A)$ comes from $\mathrm{Z}_\bullet(W)$ by induction on d. Note that the image of $\mathrm{Z}_\bullet(W)$ coincides with $\mathrm{Z}_\bullet(W)/\mathrm{Z}_\bullet(W) \cap \mathrm{Rat}_\bullet(A)$. Suppose $d > 0$ and put $Y = W \cap \mathrm{Spec}(A/P^*)$. Then the diagram

$$\begin{array}{ccc} \frac{\mathrm{Z}_\bullet(W)}{\mathrm{Z}_\bullet(W)\cap\mathrm{Rat}_\bullet(A)} & \hookrightarrow & \mathrm{A}_\bullet(A) \\ \uparrow & \circlearrowleft & \uparrow \\ \frac{\mathrm{Z}_\bullet(Y)}{\mathrm{Z}_\bullet(Y)\cap\mathrm{Rat}_\bullet(A/P^*)} & \hookrightarrow & \mathrm{A}_\bullet(A/P^*) \end{array}$$

is commutative. To prove our assertion, it is enough to show that $[A/P]$ is contained in $\mathrm{Z}_\bullet(Y)/\mathrm{Z}_\bullet(Y) \cap \mathrm{Rat}_\bullet(A/P^*)$. Thus we may assume $P^* = (0)$, (namely, any nonzero homogeneous elements are nonzero divisors). Consider the localization sequence

$$\mathrm{Z}_{\dim_R(A/P)}(S, A) \longrightarrow \mathrm{A}_{\dim_R(A/P)}(A) \longrightarrow \mathrm{A}_{\dim_R(A/P)}(A_{(0)}) \to 0$$

where S is the set of all nonzero homogeneous elements of A. Since $A_{(0)}$ is G-simple (or $A_{(0)}$ is a twisted group ring over the field $[A_{(0)}]_0$), $\mathrm{A}_\bullet(A_{(0)}) \cong \mathrm{A}_{\dim_R(A_{(0)})}(A_{(0)}) = \mathrm{A}_{\dim_R(A)}(A_{(0)})$ is satisfied. On the other hand, we have $\dim_R(A) > \dim_R(A/P)$ since $d = \dim A_P = \dim_R(A) - \dim_R(A/P) > 0$. Hence $[A/P]$ is rationally equivalent to a cycle $\sum_s n_s[A/P_s]$ such that $\dim_R(A/P_s) = \dim_R(A/P)$ and P_s contains a nonzero homogeneous element for each s. Since $P_s^* \neq 0$ and $\dim_R(A/P_s^*) < \dim_R(A)$, we have $\dim A_{P_s}/P_s^*A_{P_s} = \dim_R(A/P_s^*) - \dim_R(A/P_s) < \dim_R(A) - \dim_R(A/P) = d$. Then, by induction hypothesis, $[A/P_s]$ is in $\mathrm{Z}_\bullet(W)/\mathrm{Z}_\bullet(W) \cap \mathrm{Rat}_\bullet(A)$ for each s and so is $[A/P]$. □

Remark 3.1 If G is torsion free, then a G-prime ideal is a G-graded prime ideal and $\mathrm{Spec}^G(A) = W \subset \mathrm{Spec}(A)$. Hence $\mathrm{A}_\bullet(A)$ is generated by cycles $[A/\mathfrak{p}]$ such that $\mathfrak{p} \in \mathrm{Spec}^G(A)$. Therefore φ is surjective.

To prove the statement (2), we need some lemmas.

Lemma 3.2 *Suppose that G is torsion free and put $\dim_R(A) = d$. Then $\varphi : \mathrm{A}^G_{d-1}(A) \longrightarrow \mathrm{A}_{d-1}(A)$ is an isomorphism.*

Proof. By (1.1), (1), it is enough to show that φ is injective.

First, we assume that A is an itegral domain and denote by $\bar{A}$ the normalization of A. Note that $\bar{A}$ is a G-graded ring since G is torsion free, and the inclusion $i : A \hookrightarrow \bar{A}$ is finite since A is excellent. Consider the following commutative diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathrm{Z}^G_{d-1}(\bar{A}) & \xrightarrow{i_*} & \mathrm{Z}^G_{d-1}(A) \\ {}^{\varphi}\downarrow & & \downarrow^{\varphi} \\ \mathrm{Z}_{d-1}(\bar{A}) & \xrightarrow{i_*} & \mathrm{Z}_{d-1}(A). \end{array}$$

By the definition of the map i_* (cf. (2.11), (3)), we have $\mathrm{Rat}^G_{d-1}(A) = i_*(\mathrm{Rat}^G_{d-1}(\bar{A}))$ and $\mathrm{Rat}_{d-1}(A) = i_*(\mathrm{Rat}_{d-1}(\bar{A}))$ (cf. Prop 1.4. of [1]). Now, we show that $\mathrm{Rat}_{d-1}(A) \cap \mathrm{Z}^G_{d-1}(A) \subset \mathrm{Rat}^G_{d-1}(A)$. Let $D \in \mathrm{Rat}_{d-1}(A) \cap \mathrm{Z}^G_{d-1}(A)$. If we denote by $D = \sum_{i=1}^r \mathrm{div}_A((0), a_i)$, then $D = \sum_{i=1}^r \mathrm{div}_A((0), a_i) = \sum_{i=1}^r i_*(\mathrm{div}_{\bar{A}}((0), a_i)) = i_*(\mathrm{div}_{\bar{A}}((0), \prod_i a_i))$. Hence there is an element a of the quotient field of A such that $D = \mathrm{div}_A((0), a) = i_*(\mathrm{div}_{\bar{A}}((0), a))$. Put $\mathrm{div}_{\bar{A}}((0), a) = \sum_i n_i \left[\bar{A}/P_i\right]$ with $P_i \neq P_j$ for $i \neq j$. Assume that P_i is not a G-graded prime ideal for some i and put $Q = A \cap P_i$. Then the prime ideal Q is not G-graded. Since $QA_{(0)}$ is prime and $\bar{A} \subset A_{(0)}$, P_i is the unique prime ideal lying over Q. Thus the coefficient of $[A/Q]$ in D is also n_i. On the other hand, D is a linear combination of cycles corresponding to G-graded prime ideals of A since $D \in \mathrm{Z}^G_{\bullet}(A)$. Therefore, we have $n_i = 0$ for each non graded prime ideal P_i and $\mathrm{div}_{\bar{A}}((0), a) \in \mathrm{Z}^G_{\bullet}(\bar{A})$. Once $\mathrm{Rat}_{d-1}(\bar{A}) \cap \mathrm{Z}^G_{d-1}(\bar{A}) \subset \mathrm{Rat}^G_{d-1}(\bar{A})$ is proved, then we have $D \in i_*(\mathrm{Rat}^G_{d-1}(\bar{A})) = \mathrm{Rat}^G_{d-1}(A)$. Hence we may assume that A is normal. Since $\mathrm{Z}^G_{d-1}(A)$ (resp. $\mathrm{Z}_{d-1}(A)$) is the group of G-graded divisorial ideals (resp. divisorial ideal) and $\mathrm{Rat}^G_{d-1}(A)$ (resp. $\mathrm{Rat}_{d-1}(A)$) is the group of G-graded principal divisors (resp. principal divisors), our problem is described in terms of ideal theory. Namely, if $I \subset A_{(0)}$ is a G-graded divisorial ideal (i.e. a divisorial ideal and a G-graded A-submodule of $A_{(0)}$) such that I is isomorphic to a principal divisor of A, then there exists a G-homogeneous element x of $A_{(0)}$ such that $I = Ax$. Indeed, if a G-graded fractional ideal $I \subset A_{(0)}$ is principal, it must be generated by a homogeneous element since G is torsion free.

Now, we prove the assertion in general. Let L be the kernel of $\bigoplus_{\mathfrak{p} \in \mathrm{Min}_A(A)} \mathrm{Z}^G_{d-1}(A/\mathfrak{p}) \longrightarrow \mathrm{Z}^G_{d-1}(A)$. Note that $\mathrm{Min}_A(A) = \mathrm{Min}^G_A(A)$, since G is torsion free, and $\bigoplus_{\mathfrak{p} \in \mathrm{Min}_A(A)} \mathrm{Z}^G_{d-1}(A/\mathfrak{p}) \longrightarrow \mathrm{Z}^G_{d-1}(A)$ is surjective. For $\mathfrak{q} \in \mathrm{Min}_A(A)$ and a G-cycle $C \in \mathrm{Z}^G_{d-1}(A/\mathfrak{q})$, $C_{\mathfrak{q}}$ denotes the element in $\bigoplus_{\mathfrak{p} \in \mathrm{Min}_A(A)} \mathrm{Z}^G_{d-1}(A/\mathfrak{p})$ where component corresponding to $\mathrm{Z}^G_{d-1}(A/\mathfrak{p})$ is C (resp. 0) if $\mathfrak{p}$ is equal to $\mathfrak{q}$ (resp. otherwise). Then L is generated by elements $[A/\mathfrak{q}]_{\mathfrak{p}} - [A/\mathfrak{q}]_{\mathfrak{p}'}$ such that $\dim_R(A/\mathfrak{q}) = d-1$ and $\mathfrak{q} \supset \mathfrak{p} \cup \mathfrak{p}'$ for $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Assh}_R(A)$ with $\mathfrak{p} \neq \mathfrak{p}'$. Similarly, we denote by L' the kernel of $\bigoplus_{\mathfrak{p} \in \mathrm{Min}_A(A)} \mathrm{Z}_{d-1}(A/\mathfrak{p}) \longrightarrow \mathrm{Z}_{d-1}(A)$ and, for $\mathfrak{q} \in \mathrm{Min}_A(A)$ and $C' \in \mathrm{Z}_{d-1}(A/\mathfrak{p})$, $C'_{\mathfrak{q}}$ denotes the element in $\bigoplus_{\mathfrak{p} \in \mathrm{Min}_A(A)} \mathrm{Z}_{d-1}(A/\mathfrak{p})$ where component corresponding to $\mathrm{Z}_{d-1}(A/\mathfrak{p})$ is C' (resp. 0) if $\mathfrak{p}$ is equal to $\mathfrak{q}$ (resp. otherwise). Then L' is also generated by elements $[A/Q]_{\mathfrak{p}} - [A/Q]_{\mathfrak{p}'}$ such that $\dim_R(A/Q) = d-1$, $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Assh}_R(A)$ with $\mathfrak{p} \neq \mathfrak{p}'$ and $Q \supset \mathfrak{p} \cup \mathfrak{p}'$.

Claim. The natural map $L \longrightarrow L'$ is an isomorphism.

Proof of Claim. Take $Q \in \mathrm{Spec}(A)$ such that $\dim_R(A/Q) = d-1$ and $Q \supset \mathfrak{p} \cup \mathfrak{p}'$ for some $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Assh}_R(A)$ with $\mathfrak{p} \neq \mathfrak{p}'$. Suppose that Q is not homogeneous. Since Q^* is a prime ideal that is properly contained in Q and $\dim_R(A/Q) = d-1$, we have $Q^* \in \mathrm{Assh}_R(A)$. On the other hand, Q^* is the maximal G-graded ideal contained in Q and, thus, $Q^* \supset \mathfrak{p}, \mathfrak{p}'$. This implies $Q^* = \mathfrak{p} = \mathfrak{p}'$ and contradicts to $\mathfrak{p} \neq \mathfrak{p}'$. Hence Q is homogeneous. The surjectivity of $L \longrightarrow L'$ is proved. The injectivity is easy. The proof of Claim is completed.

We put $\bar{L} = L/L \cap \bigoplus_{\mathfrak{p} \in \mathrm{Assh}_R(A)} \mathrm{Rat}^G_{d-1}(A/\mathfrak{p})$ and $\bar{L}' = L/L \cap \bigoplus_{\mathfrak{p} \in \mathrm{Assh}_R(A)} \mathrm{Rat}_{d-1}(A/\mathfrak{p})$. Then we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \bar{L} & \longrightarrow & \bigoplus_{\mathfrak{p} \in \mathrm{Min}_A(A)} \mathrm{A}^G_{d-1}(A/\mathfrak{p}) & \longrightarrow & \mathrm{A}^G_{d-1}(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bar{L}' & \longrightarrow & \bigoplus_{\mathfrak{p} \in \mathrm{Min}_A(A)} \mathrm{A}_{d-1}(A/\mathfrak{p}) & \longrightarrow & \mathrm{A}_{d-1}(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Note that the middle vertical map is an isomorphism since $A/\mathfrak{p}$ is an integral domain and $\dim_R(A/\mathfrak{p}) \leq d$. The map $\bar{L} \to \bar{L}'$ is surjective by the previous claim. Then our assertion follows from the snake lemma. $\square$

Lemma 3.3 *Suppose that G is torsion free. Let B be a G-graded ring and let b be a homogeneous element of B. We define a homomorphism* $\mathrm{div}^G(b)\cap : \mathrm{Z}^G_i(B) \longrightarrow \mathrm{Z}^G_{i-1}(B/(b))$ *by* $\mathrm{div}^G(b) \cap ([B/\mathfrak{p}]) = [B/(b, \mathfrak{p})]$ *if* $b \notin \mathfrak{p}$, *and* $\mathrm{div}^G(b) \cap ([B/\mathfrak{p}]) = 0$ *if* $b \in \mathfrak{p}$. *Then this map induces a homomorphism* $\mathrm{div}^G(b)\cap : \mathrm{A}^G_i(B) \longrightarrow \mathrm{A}^G_{i-1}(B/(b))$.

Proof. We show $\mathrm{div}^G(b) \cap (\mathrm{Rat}^G_i(B)) \subset \mathrm{Rat}^G_{i-1}(B/(b))$. Take $\mathrm{div}^G(\mathfrak{q}, c) \in \mathrm{Rat}^G_i(B)$. To prove $\mathrm{div}^G(b) \cap (\mathrm{div}^G(\mathfrak{q}, c)) \in \mathrm{Rat}^G_{i-1}(B/(b))$, we have only discuss the case $b \notin \mathfrak{q}$. Hence assume $b \notin \mathfrak{q}$. Since the diagram

$$
\begin{array}{ccc}
\mathrm{Z}^G_i(B) & \xrightarrow{\mathrm{div}^G(b)\cap} & \mathrm{Z}^G_{i-1}(B/(b)) \\
\uparrow & & \uparrow \\
\mathrm{Z}^G_i(B/\mathfrak{q}) & \xrightarrow{\mathrm{div}^G(b)\cap} & \mathrm{Z}^G_{i-1}(B/(\mathfrak{q}, b))
\end{array}
$$

is commutative, it is enough to show that $\mathrm{div}^G(b) \cap (\mathrm{div}^G(\mathfrak{q}, c)) \in \mathrm{Rat}^G_{i-1}(B/(\mathfrak{q}, b))$. Hence we may assume that $\mathfrak{q} = (0)$ and $i = \dim_R(B) - 1$ by replacing B with $B/\mathfrak{q}$. Consider the commutative diagram

$$\begin{array}{ccc}
\mathrm{Z}_i^G(B) & \xrightarrow{\mathrm{div}^G(b)\cap} & \mathrm{Z}_{i-1}^G(B/(b)) \\
\downarrow & & \downarrow \\
\mathrm{A}_i^G(B) & & \mathrm{A}_{i-1}^G(B/(b)) \\
\downarrow & & \downarrow \\
\mathrm{A}_i(B) & \xrightarrow{\mathrm{div}(b)\cap} & \mathrm{A}_{i-1}(B/(b))
\end{array}$$

where the bottom line is defined in (2.4.1) of Fulton[1]. By (3.2), we have isomorphisms $\mathrm{A}_i^G(B) \cong \mathrm{A}_i(B)$ and $\mathrm{A}_{i-1}^G(B/(b)) \cong \mathrm{A}_{i-1}(B/(b))$. This implies that $\mathrm{div}^G(b) \cap (\mathrm{Rat}_i^G(B))$ is contained in $\mathrm{Rat}_{i-1}^G(B/(b))$. □

Proof of (2). First, we suppose $G \cong \mathbb{Z}^m$. We want to prove $\mathrm{A}_\bullet^G(A) \cong \mathrm{A}_\bullet(A)$. Let $A[G] = \bigoplus_{g\in G} Ae_g$ be a group ring over A and we regard $A[G]$ as a G-graded ring by $\deg(ae_g) = \deg(a) + g$ for each homogeneous element $a \in A$ and for each $g \in G$. We define a flat ring homomorphism $f : A \longrightarrow A[G]$ by $f(\sum_{g\in G} a_g) = \sum_{g\in G} a_g e_{-g}$, where a_g is the homogeneous component of $\sum_{g\in G} a_g$ of degree g. Then $A = \bigoplus_{g\in G} A_g$ is isomorphic to $A[G]_0 = \bigoplus_{g\in G} A_g e_{-g}$ via f. Since $A[G] = \bigoplus_{g\in G} A[G]_0 e_g$ is also a group ring over $A[G]_0$, we have the following bijective correspondence between $\mathrm{Spec}(A[G]_0)$ and $\mathrm{Spec}^G(A[G])$:

$$\begin{array}{ccc}
\mathrm{Spec}(A[G]_0) & \longleftrightarrow & \mathrm{Spec}^G(A[G]) \\
P & \longrightarrow & PA[G] \\
\mathfrak{P}_0 & \longleftarrow & \mathfrak{P}
\end{array}$$

This bijection gives isomorphisms $\mathrm{Z}_i(A[G]_0) \cong \mathrm{Z}_{i+m}^G(A[G])$, $\mathrm{Rat}_i(A[G]_0) \cong \mathrm{Rat}_{i+m}^G(A[G])$ and $\mathrm{A}_i(A[G]_0) \cong \mathrm{A}_{i+m}^G(A[G])$. Consequently, $\mathrm{A}_i(A) \xrightarrow{f^*} \mathrm{A}_{i+m}^G(A[G])$ is an isomorphism. Note that, for $P \in \mathrm{Spec}(A)$, $f(P)A[G]$ may not be equal to $PA[G]$. However it is easy to see $f(\mathfrak{p})A[G] = \mathfrak{p}A[G]$ and $f^*([A/\mathfrak{p}]) = [A[G]/\mathfrak{p}A[G]]$ for each $\mathfrak{p} \in \mathrm{Spec}^G(A)$. (See Remark 3.7).

Next, we regard $A[G]$ as a Laurent polynomial ring $A[G] = A[x_1^{\pm 1}, \cdots, x_m^{\pm 1}]$ with homogeneous variables $x_1, \cdots, x_m$. Then the map $g^* : \mathrm{A}_\bullet^G(A[x_1, \cdots, x_m]) \cong \mathrm{A}_\bullet^G(A[x_1^{\pm 1}, \cdots, x_m^{\pm 1}]) = \mathrm{A}_\bullet^G(A[G])$ induced by $A[x_1, \cdots, x_m] \longrightarrow A[x_1^{\pm 1}, \cdots, x_m^{\pm 1}]$ is an isomorphism by (2.12). Furthermore, by (3.3), there are homomorphisms

$$\mathrm{A}_{i+m}^G(A[x_1, \cdots, x_m]) \xrightarrow{\mathrm{div}^G(x_1)\cap} \mathrm{A}_{i+m-1}^G(A[x_2, \cdots, x_m]) \longrightarrow \cdots \xrightarrow{\mathrm{div}^G(x_m)\cap} \mathrm{A}_i^G(A).$$

Denote the composition by $\eta : \mathrm{A}_{i+m}^G(A[x_1, \cdots, x_m]) \longrightarrow \mathrm{A}_i^G(A)$. If $\mathfrak{p}$ is a G-prime ideal of A, then $\eta([A[x_1, \cdots, x_m]/\mathfrak{p}A[x_1, \cdots, x_m]]) = [A/\mathfrak{p}]$ by the definition of maps (cf. (3.3)). Finally, we know that the composition of

$$\mathrm{A}_i^G(A) \xrightarrow{\varphi} \mathrm{A}_i(A) \xrightarrow{f^*} \mathrm{A}_{i+m}^G(A[G]) \xrightarrow{(g^*)^{-1}} \mathrm{A}_{i+m}^G(A[x_1, \ldots, x_m]) \xrightarrow{\eta} \mathrm{A}_i^G(A)$$

is identity on $\mathrm{A}_i^G(A)$. This shows the injectivity of φ and, therefore, φ is an isomorphism if G is torsion free.

Suppose that $G \cong \mathbb{Z}^m \oplus T$ with $|T| < \infty$. In order to prove $|T|\mathrm{Ker}(\varphi) = (0)$, we construct $\psi : \mathrm{A}_\bullet(A) \longrightarrow \mathrm{A}_\bullet^G(A)$ such that $\psi\varphi([A/\mathfrak{p}]) = |T|[A/\mathfrak{p}]$ for every $\mathfrak{p} \in \mathrm{Spec}^G(A)$. Henceforth, we identify G with $\mathbb{Z}^m \oplus T$. If we put $A_\alpha = \bigoplus_{t\in T} A_{(\alpha,t)}$ for $\alpha \in \mathbb{Z}^m$, then the family $\{A_\alpha\}_{\alpha\in\mathbb{Z}^m}$ gives a $\mathbb{Z}^m$-grading on A, that is, $A = \bigoplus_{\alpha\in\mathbb{Z}^m} A_\alpha$. We have a homomorphism $\varphi' : \mathrm{A}_\bullet^G(A) \xrightarrow{\varphi} \mathrm{A}_\bullet(A) \cong \mathrm{A}_\bullet^{\mathbb{Z}^m}(A)$ such that $\varphi'([A/\mathfrak{p}]) = [A/\mathfrak{p}]$ for $\mathfrak{p} \in \mathrm{Spec}^G(A)$, by definition of $\mathrm{A}_\bullet(A) \cong \mathrm{A}_\bullet^{\mathbb{Z}^m}(A)$. Here the right hand side of the equality $\varphi'([A/\mathfrak{p}]) = [A/\mathfrak{p}]$ is the class of $\mathbb{Z}^m$-graded module $A/\mathfrak{p}$ in $\mathrm{A}_\bullet^{\mathbb{Z}^m}(A)$.

We use the same argument as the previous part of the proof of (2). We consider a group ring $A[T] = \bigoplus_{t\in T} Ae_t$ and regard it as a G-graded ring by $\deg(ae_t) = \deg(a) + t$ for each G-homogeneous element $a \in A$ and for each $t \in T$. Then $A[T]^{(\mathbb{Z}^m)} := \bigoplus_{\alpha\in\mathbb{Z}^m} A[T]_{(\alpha,0)} = \bigoplus_{\alpha\in\mathbb{Z}^m} \left(\oplus_{t\in T} A_{(\alpha,t)}e_{-t}\right)$ and $A[T]$ can be regarded as a group ring of T over $A[T]^{(\mathbb{Z}^m)}$. Hence we have a bijective correspondence between $\mathrm{Spec}^{\mathbb{Z}^m}(A[T]^{(\mathbb{Z}^m)})$ and $\mathrm{Spec}^G(A[T])$, and the natural isomorphism $\mathrm{A}_\bullet^{\mathbb{Z}^m}(A[T]^{(\mathbb{Z}^m)}) \cong \mathrm{A}_\bullet^G(A[T])$. If we define a flat homomorphism $h : A \longrightarrow A[T]$ by $h(\sum_{(\alpha,t)\in G} a_{(\alpha,t)}) = \sum_{(\alpha,t)\in G} a_{(\alpha,t)}e_{-t}$, then A is isomorphic to $A[T]^{(\mathbb{Z}^m)}$ (as $\mathbb{Z}^m$-graded rings) and, thus, we have an isomorphism $h^* : \mathrm{A}_\bullet^{\mathbb{Z}^m}(A) \cong \mathrm{A}_\bullet^{\mathbb{Z}^m}(A[T]^{(\mathbb{Z}^m)}) \cong \mathrm{A}_\bullet^G(A[T])$. By definition of h, $h^*([A/\mathfrak{p}]) = [A[T]/\mathfrak{p}A[T]]$ is satisfied for each $\mathfrak{p} \in \mathrm{Spec}^G(A)$. Furthermore, the inclusion $i : A \longrightarrow A[T]$, (i.e. A maps to Ae_0) determines $i_* : \mathrm{A}_\bullet^G(A[T]) \longrightarrow \mathrm{A}_\bullet^G(A)$, since T is finite. We denote by $\psi : \mathrm{A}_\bullet(A) \longrightarrow \mathrm{A}_\bullet^G(A)$ the composite map of

$$\mathrm{A}_\bullet(A) \xrightarrow{\cong} \mathrm{A}_\bullet^{\mathbb{Z}^m}(A) \xrightarrow{h^*} \mathrm{A}_\bullet^G(A[T]) \xrightarrow{i_*} \mathrm{A}_\bullet^G(A)$$

and claim that ψ is the desired homomorphism. Recall that $[A/\mathfrak{p}]$ maps to $[A[T]/\mathfrak{p}A[T]]$ under $\mathrm{A}_\bullet^G(A) \xrightarrow{\varphi} \mathrm{A}_\bullet(A) \xrightarrow{\cong} \mathrm{A}_\bullet^{\mathbb{Z}^m}(A) \xrightarrow{h^*} \mathrm{A}_\bullet^G(A[T])$ for $\mathfrak{p} \in \mathrm{Spec}^G(A)$ by the definition of each maps. Hence we have $\psi(\varphi([A/\mathfrak{p}])) = i_*([A[T]/\mathfrak{p}A[T]]) = i_*([(A/\mathfrak{p})[T]]) = |T|[A/\mathfrak{p}]$ for every $\mathfrak{p} \in \mathrm{Spec}^G(A)$. This implies that $|T|\mathrm{Ker}(\varphi) = \psi\varphi(\mathrm{Ker}(\varphi)) = 0$. $\square$

In the proof of (1.1), we showed that φ is an isomorphism if G is torsion free. However, it is not true in general.

Example 3.4 Let $A = k[x, y]/(x^2 - y^2)$ with field k. We consider A as a $G := \mathbb{Z}/(2)$-graded ring with $\deg(x) = 0 \in G$ and $\deg(y) = \bar{1} \in G$. Then we have

$$\mathrm{A}_\bullet(A) = \mathrm{A}_1(A) = \begin{cases} \mathbb{Z} & (ch(k) = 2) \\ \mathbb{Z}^2 & (ch(k) \neq 2). \end{cases}$$

On the other hand, $\mathrm{A}_\bullet^G(A)$ is generated by $[A]$ and $[A/(x, y)]$, and $\mathrm{A}_\bullet^G(A) = \mathrm{A}_1^G(A) \oplus \mathrm{A}_0^G(A) \cong \mathbb{Z} \oplus \mathbb{Z}/(2)$. On can show that the map φ is neither injective nor surjective. Remark that the cokernel of φ is not torsion if $ch(k) \neq 2$.

Corollary 3.5 *Let H be a subgroup of G such that G/H is torsion. Then $\mathrm{A}^H_\bullet(A^{(H)})_\mathbb{Q}$ is isomorphic to $\mathrm{A}^G_\bullet(A)_\mathbb{Q}$, where $A^{(H)} = \bigoplus_{h\in H} A_h$ and $\mathrm{A}^G_\bullet(-)_\mathbb{Q} = \mathrm{A}^G_\bullet(-)\otimes_\mathbb{Z} \mathbb{Q}$.*

Proof. Since G/H is torsion, we have the following bijective correspondence between $\mathrm{Spec}^H(A^{(H)})$ and $\mathrm{Spec}^G(A)$:

$$\begin{array}{ccc} \mathrm{Spec}^H(A^{(H)}) & \longleftrightarrow & \mathrm{Spec}^G(A) \\ \mathfrak{p} & \longrightarrow & (\sqrt{\mathfrak{p}A})^* \\ \mathfrak{P}^{(H)} & \longleftarrow & \mathfrak{P} \end{array}$$

Then we have the following exact sequence

$$0 \to \mathrm{Z}^G_\bullet(A) \to \mathrm{Z}^H_\bullet(A^{(H)}) \to D \to 0$$

such that D is torsion. The cokernel of $\mathrm{Rat}^G_\bullet(A) \to \mathrm{Rat}^H_\bullet(A^{(H)})$ is also torsion, (cf. Prop. 1.4. of [1]). This completes the proof of the corollary. □

Corollary 3.6 *Let $A = \bigoplus_{n\geq 0} A_n$ be a positively graded ring. Then $\mathrm{A}_\bullet(A^{(d)})_\mathbb{Q}$ of the d-th Veronese subring $A^{(d)}$ of A is isomorphic to $\mathrm{A}_\bullet(A)_\mathbb{Q}$ for any integer $d > 0$.*

Proof. This is a direct consequence of (1.1) and (3.5). □

With notation as in (3.6), $\mathrm{A}_\bullet(A)$ is not always isomorphic to $\mathrm{A}_\bullet(A^{(d)})$. For example, if A is a polynomial ring with two variables over a field, then $\mathrm{A}_\bullet(A) = \mathrm{A}_2(A)$ and $\mathrm{A}_1(A^{(2)}) = Cl(A^{(2)}) \neq 0$.

Remark 3.7 (5.2 of [1]) We are able to describe the inverse map $\eta(g^*)^{-1}f^*$ of φ in the proof of (1.1) explicitly, if the given graded ring is standard. If $A = \bigoplus_{n\geq 0} A_n$, then the map $\eta(g^*)^{-1}f^*$ is determined by

$$\begin{array}{ccccc} \mathrm{A}_\bullet(A) & \xrightarrow{(g^*)^{-1}f^*} & \mathrm{A}^G_\bullet(A[x_1]) & \xrightarrow{\eta} & \mathrm{A}^G_\bullet(A) \\ [A/P] & \longmapsto & \left[A[x_1]/{}^hP\right] & \longmapsto & [A/in(P)]\,, \end{array}$$

where ${}^hP$ is the homogenization of P and $in(P)$ is the initial ideal of P for $P \in \mathrm{Spec}(A)$.

4 Appendix

The most important point of Theorem 1.1 is Lemma 3.3, where we discussed a graded version of the intersection operator defined in (2.3) of Fulton[1]. Beside we assumed that G is torsion free in Lemma 3.3, but it is still true for any finitely generated Abelian group, that is the purpose of the appendix. Using this general statement, we can prove (1.1) directly. However, our proof of the general statement is different from the proof of (3.3). The proof will be done in the completely parallel way to Corollary 2.4.1 of Fulton[1]. Precisely speaking, the general statement follows from a commutativity of the intersection with divisors, that is, $\mathrm{div}^G(\mathfrak{p}, b) \cap [A/(a)] = \mathrm{div}^G(\mathfrak{p}, a) \cap [A/(b)]$ in $\mathrm{A}^G_{\bullet}(A/(a, b, \mathfrak{p}))$ for homogeneous elements a, b and $\mathfrak{p} \in \mathrm{Spec}^G(A)$. The corresponding equality in the ordinary Chow group was proved in Chapter 2 of Fulton[1]. In order to prove this property, we have to argue not only in the category of graded rings, but also in the category of "graded schemes". Hence the important problem is to define a concept of graded schemes by which we can prove an analogue statements of [1]. In this appendix, we give such a notion of G-graded objects in a category of schemes.

In this section, we do not assume that an Abelian group is finitely generated for a little while. A graded ring (A, G) is a pair of a ring A and an Abelian group G such that A is a G-graded ring. If no confusion is possible, we say that A is a graded ring. Let A be a G-graded ring and B be a G'-graded ring. We say that a graded homomorphism $f : A \longrightarrow B$ is a ring homomorphism f together with a group homomorphism $\tilde{f} : G \longrightarrow G'$ such that $f(A_g) \subset B_{\tilde{f}(g)}$ for each $g \in G$. In particular, f is called G-graded, if $G = G'$ and $\tilde{f} = 1_G$. We denote by $gr\mathbf{Ring}$ the category of graded rings and graded homomorphisms and denote by $gr^G\mathbf{Ring}$ the category of G-graded rings and G-graded homomorphisms.

A G-graded ringed space $(X, \mathcal{O}_X)$ is a ringed space such that $\mathcal{O}_X$ is a sheaf on X with objects in $gr^G\mathbf{Ring}$. Note that each stalk of a G-graded ringed space is also a G-graded ring. A G-graded ringed space $(X, \mathcal{O}_X)$ is said to be a G-graded locally ringed space, if $\mathcal{O}_{X,x}$ has a unique maximal G-graded ideal for each $x \in X$ and we denote this maximal G-graded ideal of $\mathcal{O}_{X,x}$ by $\mathfrak{m}_x$ for $x \in X$. A graded (locally) ringed space means a G-graded (locally) ringed space for some Abelian group G. Let $(X, \mathcal{O}_X)$ be a G-graded ringed space and $(Y, \mathcal{O}_Y)$ be a G'-graded ringed space. A graded homomorphism $f : X \longrightarrow Y$ between graded ringed spaces is defined by a homomorphism $f : X \longrightarrow Y$ of ringed spaces such that a ring homomorphism $\mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}(U))$ induced by f is graded for each open subset $U \subset Y$. A morphism of (G-)graded locally ringed spaces is a (G-)graded homomorphism $f : X \longrightarrow Y$ as (G-)graded ringed spaces such that $f_x(\mathfrak{m}_{f(x)}) \subset \mathfrak{m}_x$ for each $x \in X$. We denote by $gr\mathbf{Local^{Sp}}$ (resp. $gr^G\mathbf{Local^{Sp}}$) the category of graded (resp. G-graded) locally ringed spaces and graded (resp. G-graded) local homomorphisms.

Definition 4.1 (Affine G-graded Schemes) Let A be a G-graded ring. We put $D_A^G(f) = \{\mathfrak{p} \in \mathrm{Spec}^G(A) \mid f \notin \mathfrak{p}\}$ for each homogeneous element $f \in A$. Then $\mathrm{Spec}^G(A)$ can be regarded as a topological space with open basis $\{D_A^G(f) \mid f \in A \text{ is homogeneous}\}$. Actually, $\mathrm{Spec}^G(A)$ can be identified with the quotient space $\mathrm{Spec}(A)/\sim$ of $\mathrm{Spec}(A)$ with an equivalence relation defined by $P \sim Q$ iff $P^* = Q^*$ for $P, Q \in \mathrm{Spec}(A)$. Thus a map $\varphi : \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}^G(A)$ defined by $\varphi(P) = P^*$ ($P \in \mathrm{Spec}(A)$) is continuous (the quotient map) and it determines a ringed space $(\mathrm{Spec}^G(A), \mathcal{O}_{\mathrm{Spec}^G(A)})$ by $\mathcal{O}_{\mathrm{Spec}^G(A)} = \varphi_* \mathcal{O}_{\mathrm{Spec}(A)}$. Then $(\mathrm{Spec}^G(A), \mathcal{O}_{\mathrm{Spec}^G(A)})$ is a G-graded locally ringed space such that

- $\mathcal{O}_{\mathrm{Spec}^G(A)}(D_A^G(f)) = A_f$ for each homogeneous element $f \in A$
- $\mathcal{O}_{\mathrm{Spec}^G(A),\mathfrak{p}} = A_{(\mathfrak{p})}$ for $\mathfrak{p} \in \mathrm{Spec}^G(A)$
- φ is a graded homomorphism from the 0-graded ringed space $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ to the G-graded ringed space $(\mathrm{Spec}^G(A), \mathcal{O}_{\mathrm{Spec}^G(A)})$.

We denote by $(\mathrm{Spec}^G(A), A)$ instead of $(\mathrm{Spec}^G(A), \mathcal{O}_{\mathrm{Spec}^G(A)})$ and call it an affine G-graded scheme. We call φ the natural map of the affine G-graded scheme $(\mathrm{Spec}^G(A), A)$.

Definition 4.2 (G-graded Schemes) A G-graded locally ringed space $(X, \mathcal{O}_X)$ is said to be a G-graded scheme, if it has an open covering $\{U_i\}$ of X such that $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic to an affine G-graded scheme in $gr^G\mathbf{Local}^{\mathbf{Sp}}$. We call that a graded locally ringed space is a graded scheme, if it is a G-graded scheme for some G. We denote by $gr\mathbf{Sch}$ (resp. $gr^G\mathbf{Sch}$) the full subcategory of $gr\mathbf{Local}^{\mathbf{Sp}}$ (resp. $gr^G\mathbf{Local}^{\mathbf{Sp}}$) consisting of all graded schemes (resp. G-graded schemes).

It is easy to reword the statements of schemes to those of (G-)graded schemes. For example, the following property holds for graded schemes, (see, for example [3]).

(Glueing Lemma) Let $\{X_i\}_{i \in I}$ be a family of G-graded schemes. Suppose that there are given open subsets $U_{ij} \subset X_i$ and isomorphisms $\varphi_{ij} : (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \longrightarrow (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$ in $gr^G\mathbf{Sch}$ for each $i, j \in I$ such that (1) $\varphi_{ij}^{-1} = \varphi_{ji}$ for each i, j, (2) $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ and $\varphi_{ik} = \varphi_{jk}\varphi_{ij}$ on $U_{ij} \cap U_{ik}$ for each i, j, k. Then there exist a G-graded scheme X and morphism $\psi_i : X_i \longrightarrow X$ in $gr^G\mathbf{Sch}$ for each $i \in I$ such that (1) ψ_i is an isomorphism of X_i onto a G-graded open subscheme of X, (2)$X = \bigcup_{i \in I} \psi_i(X_i)$, (3) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ and (4) $\psi_i = \psi_j \varphi_{ij}$ on U_{ij}.

(Fibre Product) $gr\mathbf{Sch}$ has a fibre products. Namely, if $f : X \longrightarrow S$ and $g : Y \longrightarrow S$ are morphisms in $gr\mathbf{Sch}$, then there exists a graded scheme $X \times_S Y$ together with morphisms

$p : X \times_S Y \longrightarrow X$, $q : X \times_S Y \longrightarrow Y$ in $gr\mathbf{Sch}$ such that

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{q} & Y \\ {}^{p}\downarrow & & \downarrow^{g} \\ X & \xrightarrow{f} & S \end{array}$$

is commutative, and such that for any graded scheme Z and for any morphisms $p' : Z \longrightarrow X$, $q' : Z \longrightarrow Y$ in $gr\mathbf{Sch}$ with $fp' = gq'$, there is a unique morphism $h : Z \longrightarrow X \times_S Y$ such that $p' = ph$ and $q' = qh$. Particularly, if grading of S, X and Y have value in G, G' and G'', respectively, then $X \times_S Y$ is a $G' \coprod_{(\tilde{f},\tilde{g})} G''$-graded scheme where $G' \coprod_{(\tilde{f},\tilde{g})} G''$ is a pushout of $G' \xleftarrow{\tilde{f}} G \xrightarrow{\tilde{g}} G''$. We remark that if $S = \mathrm{Spec}^G(A)$, $X = \mathrm{Spec}^{G'}(B)$ and $Y = \mathrm{Spec}^{G''}(C)$, then $X \times_S Y \cong \mathrm{Spec}^{G' \coprod_{(\tilde{f},\tilde{g})} G''}(B \otimes_A C)$.

For any Abelian group G, we regard $\mathbb{Z}$ as a G-graded ring with $\mathbb{Z}_0 = \mathbb{Z}$ and $\mathbb{Z}_g = 0$ for all $0 \neq g \in G$. Then $\mathrm{Spec}^G(\mathbb{Z})$ is the terminal object of $gr^G\mathbf{Sch}$. If $\tilde{u} : G \longrightarrow G'$ is a group homomorphism, then it determines a morphism $\mathrm{Spec}^{G'}(\mathbb{Z}) \xrightarrow{u} \mathrm{Spec}^G(\mathbb{Z})$ in $gr\mathbf{Sch}$ such that $u^\# = 1_{\mathbb{Z}}$. Hence we have a functor $(-) \times_{\mathrm{Spec}^G(\mathbb{Z})} \mathrm{Spec}^{G'}(\mathbb{Z})$ from $gr^G\mathbf{Sch}$ to $gr^{G'}\mathbf{Sch}$. In particular, if $\tilde{u}$ is injective, then this functor gives a fully faithful embedding $gr^G\mathbf{Sch} \hookrightarrow gr^{G'}\mathbf{Sch}$. Moreover, an arbitrary group homomorphism $\tilde{u}$ gives a natural bijection between $Hom_{gr^{G'}\mathbf{Sch}}(X, Y \times_{\mathrm{Spec}^G(\mathbb{Z})} \mathrm{Spec}^{G'}(\mathbb{Z}))$ and $\{f \in Hom_{gr\mathbf{Sch}}(X, Y) \mid \tilde{f} = \tilde{u}\}$ for all object X of $gr^{G'}\mathbf{Sch}$ and all object Y of $gr^G\mathbf{Sch}$. In particular, $(-) \times_{\mathrm{Spec}^G(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}) : gr^G\mathbf{Sch} \longrightarrow \mathbf{Sch}$ induces a natural bijection

$$Hom_{gr\mathbf{Sch}}(X, Y) \cong Hom_{\mathbf{Sch}}(X, Y \times_{\mathrm{Spec}^G(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}))$$

for all scheme X and all G-graded scheme Y. From this property, a G-graded scheme is associated to a scheme which determines a structure of a G-graded scheme. The following statement is just a translation of the functor in terms of a universal arrow, but it explains the relationship of graded schemes with schemes.

Proposition 4.3 *Let $(Y, \mathcal{O}_Y)$ be a G-graded scheme. Then there exists a scheme $(X, \mathcal{O}_X)$ together with a graded homomorphism $\varphi : X \longrightarrow Y$ satisfying the following condition; for any scheme Z and any graded homomorphism $f : Z \longrightarrow Y$, there is a unique homomorphism $g : Z \longrightarrow X$ in* **Sch** *such that $f = \varphi g$. In particular, if $Y = \bigcup_i \mathrm{Spec}^G(A_i)$ is an affine G-graded open covering and $\{\mathrm{Spec}(A_i) \xrightarrow{\varphi_i} \mathrm{Spec}^G(A_i)\}_i$ is a family of natural maps, then X and φ are obtained by gluing $\{\mathrm{Spec}(A_i) \xrightarrow{\varphi_i} \mathrm{Spec}^G(A_i)\}_i$ (and, thus, $\mathcal{O}_Y \cong \varphi_* \mathcal{O}_X$).*

We denote by $(X^G, \mathcal{O}_{X^G})$ the G-graded scheme $(Y, \mathcal{O}_Y)$ as above. In general, we have the following.

Proposition 4.4 *Let $(X^G, \mathcal{O}_{X^G})$ be a G-graded scheme and let $\tilde{\varphi} : G \longrightarrow G'$ be a group homomorphism. Then there exists a G'-graded scheme $(X^{G'}, \mathcal{O}_{X^{G'}})$ together with a graded homomorphism $\varphi : X^{G'} \longrightarrow X^G$ satisfying the following condition; for any G'-graded scheme Y and any graded homomorphism $f : Y \longrightarrow X^G$ with $\tilde{f} = \tilde{\varphi}$, there is a unique homomorphism $g : Y \longrightarrow X^{G'}$ in $gr^{G'}$**Sch** such that $f = \varphi g$. In particular, if $X^G = \bigcup_i \mathrm{Spec}^G(A_i)$ is an affine G-graded open covering, then $X^{G'}$ and φ are obtained by gluing $\{\mathrm{Spec}^{G'}(A_i) \xrightarrow{\varphi_i} \mathrm{Spec}^G(A_i)\}_i$.*

As we mentioned first, any notion and any argument of scheme theory can be replateced by a graded version. In particular, we are able to argue G-Chow groups in the category of G-graded schemes of finite type over a regular scheme as in the same way as [1]. Such arguments conclude the following.

Theorem 4.5 *Let G be a finitely generated Abelian group and let A be a G-graded Noetherian ring. For a homogeneous element $a \in A$, we define a homomorphism $\mathrm{div}^G(a)\cap : \mathrm{Z}_i^G(A) \longrightarrow \mathrm{Z}_{i-1}^G(A/(a))$ by $\mathrm{div}^G(a)\cap([A/\mathfrak{p}]) = [A/(a,\mathfrak{p})]$ if $a \notin \mathfrak{p}$, and $\mathrm{div}^G(a)\cap([A/\mathfrak{p}]) = 0$ if $a \in \mathfrak{p}$. Then this map induces a homomorphism $\mathrm{div}^G(a)\cap : \mathrm{A}_i^G(A) \longrightarrow \mathrm{A}_{i-1}^G(A/(a))$.*

Corollary 4.6 *Let A be a G-graded ring and let $A[x]$ be a G-graded polynomial ring with a homogeneous variable x. Then $\mathrm{div}^G(x)\cap : \mathrm{A}_{i+1}^G(A[x]) \longrightarrow \mathrm{A}_i^G(A)$, and the flat pull-back map $\mathrm{A}_i^G(A) \longrightarrow \mathrm{A}_{i+1}^G(A[x])$ are isomorphisms for each $i \in \mathbb{Z}$.*

At last, we give an important example of a G-graded scheme. Suppose that G is finitely generated and A is a Noetherian G-graded ring. We denote by $\Gamma(A)$ the set of all homogeneous unit and put $\deg(A) = \{g \in G \mid A_g \neq (0)\}$. If u is in $\Gamma(A)$ of $\deg(u) = g$ and H is a subgroup of G, then $A^{(g,H)} := \bigoplus_{h\in H} A_{g+h} = A^{(H)}u$. We call that A is free over $A^{(H)}$ by homogeneous unit, if there exists a set of homogeneous unit $\{u_j\}_{j\in J}$ of A such that $\{\deg(u_j)\}_{j\in J}$ is a representatives of G/H (and, thus, $A = \bigoplus_{j\in J} A^{(H)}u_j$). For a subgroup $H \subset G$, we put $X_H = \{\mathfrak{p} \in \mathrm{Spec}^G(A) \mid G(\mathfrak{p})_{\mathbb{Q}} + H_{\mathbb{Q}} = G_{\mathbb{Q}}\}$, (cf. the proof of (2.7)), and $I_H(A) = \{a \in \bigcup_{g\in G} A_g \setminus \{0\} \mid \deg(\Gamma(A[a^{-1}]))_{\mathbb{Q}} + H_{\mathbb{Q}} = G_{\mathbb{Q}}\}$. Then $X_H = \bigcup_{a\in I_H(A)} D_A^G(a)$, since G is finitely generated.

Remark 4.7 For an element a of $I_H(A)$, we put $N = \deg(\Gamma(A[a^{-1}]))$. Then $A[a^{-1}]^{(N+H)}$ is free over $A[a^{-1}]^{(H)}$ by homogeneous unit, namely there exists a set of homogeneous unit $\{u_j\}_{j\in J}$ of $A[a^{-1}]$ such that $\{\deg(u_j)\}_{j\in J}$ is a representatives of $N+H/H$ and $A[a^{-1}]^{(N+H)} = \bigoplus_{j\in J} A[a^{-1}]^{(H)}u_j$. Furthermore, if $b \in A$ is a homogeneous element with $\deg(b)-\deg(u_j) \in H$ for some $j \in J$, then there is a homogeneous element $b_0 \in A[a^{-1}]^{(H)}$ such that $b = b_0u_j$ in $A[a^{-1}]$ and $A[(ab)^{-1}]^{(H)} = (A[a^{-1}]^{(H)})[b_0^{-1}]$.

By (4.7), there is a bijective correspondence between $D_A^G(a)$ and $\mathrm{Spec}^H(A[a^{-1}]^{(H)})$ for any $a \in I_H(A)$. Then, by gluing $\{\mathrm{Spec}^H(A[a^{-1}]^{(H)})\}_{a \in I_H(A)}$, we can define a H-graded scheme structure on X_H and put $Proj_H(A) = X_H \times_{\mathrm{Spec}^H(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z})$. Then $Proj_H(A)$ is obtained by glueing $\{\mathrm{Spec}(A[a^{-1}]^{(H)})\}_{a \in I_H(A)}$.

Example 4.8 Let $\mathfrak{a}$ be a G-graded ideal of A and let $R(\mathfrak{a}) = \bigoplus_{n \geq 0} \mathfrak{a}^n t^n \subset A[t]$ be the Rees algebra of $\mathfrak{a}$. We regard $R(\mathfrak{a})$ as $G \oplus \mathbb{Z}$-graded ring with $\deg(t) = (0, 1)$. Then the scheme $Proj(R(\mathfrak{a}))$ coincides with $Proj_G(R(\mathfrak{a}))$, where $Proj(-)$ is the ordinary Proj.

Remark 4.9 $Proj_H(A)$ is a generalization of $Proj$ in Definition 8.2.1 of Roberts[6], and is the almost same as $Proj$ of Rosenberg[7]. If the reader want to check (4.5) quickly, then it is enough to show the similar statement to Theorem 8.9.2 of [6] by replacing $Proj_H(A)$ with $Proj(A)$.

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