Roberts rings and Dutta multiplicities

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1 INTRODUCTION

This is a survey on the theory of Roberts rings and Dutta multiplicities. Proofs of all statements except for the final one will be omitted.

In 1983, using Frobenius maps, S. Dutta [4] defined the limit multiplicity $\chi_\infty(\mathbb{F}.)$ (that we refer as the *Dutta multiplicity*) for a bounded finite free complex $\mathbb{F}.$ with homology of finite length over a local ring A of characteristic positive, and showed that it was a rational number. In 1989, P. Roberts [24] proved that, for a local ring A of positive characteristic, if a complex $\mathbb{F}.$ is not exact and its length (as a complex) equals the dimension of A, then its Dutta multiplicity is positive. The fact played an essential role in his proof of the new intersection theorem in the mixed-characteristic case. After that, the theory of Dutta multiplicities were established for complexes not only over a ring of characteristic positive but also over a homomorphic image of an arbitrary regular local ring [12]. It is not known whether the positivity above (Conjecture 5.2) is true or not in the case where A is mixed-characteristic. We see that the positivity of Dutta multiplicity is deeply related to Serre's positivity conjecture of intersection multiplicities (Remark 5.3).

In 1985, P. Roberts [21] proved the vanishing theorem of intersection multiplicities for a local ring that satisfies $\tau_A([A]) = [\operatorname{Spec} A]_{\dim A}$, where τ_A is the Riemann-Roch map for $\operatorname{Spec} A$ (Fulton [6]). We refer such a ring as a *Roberts ring* [16]. Over a Roberts ring, the Dutta multiplicity of a complex coincides with the Euler characteristic, that is, the alternating sum of length of homology modules. Therefore the theory of Roberts rings are deeply related to that of Dutta multiplicities. In fact, P. Roberts proved the above vanishing theorem over Roberts rings by proving a corresponding vanishing of Dutta multiplicities (Theorem 5.1 (3)). Complete intersections are Roberts rings, but there is an example of a Gorenstein non-Roberts ring. Normal Roberts rings are $\mathbb{Q}$-Gorenstein. Quotient singularities and Galois extensions of regular local rings are Roberts rings (see Section 6).

For rings of characteristic positive, we can describe Dutta multiplicities and characterize Roberts rings by Frobenius maps as in Section 2. In the general case, we can define and characterize Dutta multiplicites and Roberts rings using the singular Riemann-Roch theory (Fulton [6]) or Adams operations (Gillet-Soulé [8]) as in Section 3. For local domains with field of fractions of characteristic 0, we can characterize Dutta multiplicities and Roberts rings in terms of Galois extensions as in Section 4. In Section 5, we shall explain the origin and some applications of the theory of Roberts rings. In Section 6 we shall give basic properties on Roberts rings.

Here we shall explain terminologies that will used later.

Throughout the paper, we assume that all rings are commutative with 1.

For a ring A, $\mathrm{G}_0(A)$ denotes the *Grothendieck group* of finitely generated A-modules. For an A-module M, $[M]$ denotes the element in $\mathrm{G}_0(A)$ corresponding to M.

If a ring homomorphism $g : A \to B$ is finite (i.e., B is a finitely generated A-module), we have the induced homomorphism $g^* : \mathrm{G}_0(B) \to \mathrm{G}_0(A)$ of additive groups defined by $g^*([M]) = [{}_gM]$ for a B-module M, where ${}_gM$ is an A-module M whose A-module structure is given through g.

$\mathrm{A}_*(A) = \oplus_i \mathrm{A}_i(A)$ is the *Chow group* of the affine scheme $\operatorname{Spec} A$. For a prime ideal $\mathfrak{p}$ of A of $\dim A/\mathfrak{p} = i$, $[\operatorname{Spec} A/\mathfrak{p}]$ denotes the cycle in $\mathrm{A}_i(A)$ corresponding to the closed subscheme $\operatorname{Spec} A/\mathfrak{p}$.

For an additive group N, $N_{\mathbb{Q}}$ denotes $N \otimes_{\mathbb{Z}} \mathbb{Q}$.

A bounded complex of finitely generated free modules is called a *perfect complex*. The *support* of a bounded complex $\mathbb{F}.$ is defined to be the union of the supports of the homology modules, and denoted by $\mathrm{Supp}(\mathbb{F}.)$.

Let (A, m) be a Noetherian local ring. Let $\mathbb{F}.$ be a perfect A-complex with support in $\{m\}$ of the following form:

$$\mathbb{F}. \ : \ 0 \to F_s \to \cdots \to F_0 \to 0$$

If F_s is not 0, we say that the complex $\mathbb{F}.$ is of *length* s. We define the Euler characteristic $\chi_{\mathbb{F}.} : \mathrm{G}_0(A) \to \mathbb{Z}$ (or $\chi_{\mathbb{F}.} : \mathrm{G}_0(A)_{\mathbb{Q}} \to \mathbb{Q}$) to be

$$\chi_{\mathbb{F}.}([M]) = \sum_t (-1)^t \ell_A(H_t(\mathbb{F}. \otimes_A M)),$$

where $H_t(\mathbb{F}. \otimes_A M)$ is the t-th homology module of the complex $\mathbb{F}. \otimes_A M$ and $\ell_A(H_t(\mathbb{F}. \otimes_A M))$ denotes its length as an A-module. We sometimes denote $\chi_{\mathbb{F}.}([A])$ simply by $\chi(\mathbb{F}.)$.

2 THE CASE OF POSITIVE CHARACTERISTIC

In the section, we assume that (A, m) is a d-dimensional complete local ring containing a field of characteristic $p > 0$ with perfect residue class field A/m unless otherwise specified.

In the case, the notion of Roberts rings and Dutta multiplicities are defined in an elementary way, i.e., we do not need the singular Riemann-Roch theorem to define them.

Let $f : A \to A$ be the Frobenius map, i.e., $f(x) = x^p$. Since f is finite, we have the induced map $f^* : \mathrm{G}_0(A)_{\mathbb{Q}} \to \mathrm{G}_0(A)_{\mathbb{Q}}$. We put

$$\mathrm{L}_i\mathrm{G}_0\,(A)_{\mathbb{Q}} = \{c \in \mathrm{G}_0(A)_{\mathbb{Q}} \mid f^*(c) = p^i c\}$$

for $i = 0, \dots, d$, that is, $\mathrm{L}_i\mathrm{G}_0(A)_{\mathbb{Q}}$ is the eigen space of $f^* \in \mathrm{End}_{\mathbb{Q}}(\mathrm{G}_0(A)_{\mathbb{Q}})$ of eigen value p^i. Then,

$$\mathrm{G}_0(A)_{\mathbb{Q}} = \oplus_{i=0}^{d} \mathrm{L}_i\mathrm{G}_0(A)_{\mathbb{Q}} \tag{1}$$

is satisfied. (It will be proved elementarily. Using the theory of Adams operations instead of Frobenius maps, an equality like (1) was proved by Gillet-Soulé [8] for a homomorphic image of an arbitrary regular ring.) Put

$$[A] = q_d + q_{d-1} + \cdots + q_0, \quad (q_i \in \mathrm{L}_i\mathrm{G}_0(A)_{\mathbb{Q}}). \tag{2}$$

DEFINITION 2.1 Let $\mathbb{F}.$ be a perfect A-complex with support in $\{m\}$. The rational number $\chi_{\mathbb{F}.}(q_d)$ is called the *Dutta multiplicity* of the complex $\mathbb{F}.$ and denoted by $\chi_\infty(\mathbb{F}.)$.

We say that A is a *Roberts ring* if $q_i = 0$ for $i = 0, 1, \dots, d-1$.

By the equation (2), we have

$$(f^*)^e([A]) = [{}_{f^e}A] = p^{de}q_d + p^{(d-1)e}q_{d-1} + \cdots + p^e q_1 + q_0$$

for each e. Since

$$\chi_{\mathbb{F}.}([{}_{f^e}A]) = \chi(F^e(\mathbb{F}.)),$$

we have

$$\chi_\infty(\mathbb{F}.) = \chi_{\mathbb{F}.}(q_d) = \lim_{e\to\infty} \frac{1}{p^{de}}\chi_{\mathbb{F}.}([{}_{f^e}A]) = \lim_{e\to\infty} \frac{1}{p^{de}}\chi(F^e(\mathbb{F}.)), \tag{3}$$

where $F^e(\mathbb{F}.)$ is the perfect complex whose entries are p^e-th power of those of $\mathbb{F}.$. (S. Dutta [4] defined the multiplicity of the form as above.) Assume that A is a Cohen-Macaulay ring and I is an m-primary ideal of finite projective dimension. Let $\mathbb{F}.$ be the minimal free resolution of A/I. Then, by the equality (3), the Dutta multiplicity $\chi_\infty(\mathbb{F}.)$ coincides with the Hilbert-Kunz multiplicity $e_{HK}(I, A)$, that was introduced by Kunz [11] and Monsky [20].

Note that A is a Roberts ring iff $[{}_fA] = p^d[A]$ in $\mathrm{G}_0(A)_{\mathbb{Q}}$. Therefore, it is easy to see that a regular local ring is a Roberts ring. If A is a Roberts ring, then the Dutta multiplicity coincides with the Euler characteristic, i.e., $\chi_\infty(\mathbb{F}.) = \chi_{\mathbb{F}.}([A])$.

In 1987, P. Roberts [24] proved that, for a perfect A-complex $\mathbb{F}.$ of length at most $d = \dim A$ with support $\{m\}$, $\chi_\infty(\mathbb{F}.) > 0$ holds (see [24]). Using the fact, he proved the new intersection theorem in the mixed characteristic case.

3 THE GENERAL CASE

In the section, we assume that (A, m) is a d-dimensional local ring that is a homomorphic image of a regular local ring unless otherwise specified.

By the singular Riemann-Roch theorem (Fulton [6]), we have an isomorphism of $\mathbb{Q}$-vector spaces $\tau_A : \mathrm{G}_0(A)_{\mathbb{Q}} \to \mathrm{A}_*(A)_{\mathbb{Q}}$[1] . Put

$$\tau_A([A]) = \tau_d + \tau_{d-1} + \cdots + \tau_0, \quad (\tau_i \in \mathrm{A}_i(A)_{\mathbb{Q}}).$$

[1] Let A be a homomorphic image of a regular local ring S. Then, τ_A may depend not only on A but also on S by its construction (Fulton [6]). However, in many cases (e.g., the case where A is a complete local ring or A is essentially of finite type over a field or $\mathbb{Z}$ (Section 4 in [16])), it is proved that τ_A does not depend on the choice of S.

The author knows no example that τ_A actually depends on the choice of a regular local ring S.

These τ_i's enjoy interesting properties as follows:

PROPOSITION 3.1 *With notation as above, the following are satisfied:*

(1) If A is a Cohen-Macaulay ring, then

$$\tau_A([\omega_A]) = \tau_d - \tau_{d-1} + \tau_{d-2} - \cdots + (-1)^i \tau_{d-i} + \cdots$$

is satisfied (by the definition of τ_A and Example 18.1.2 in [6]), where ω_A denotes the canonical module of A.

(2) If A is a Gorenstein ring, then we have $\tau_{d-i} = 0$ for each odd i.

(3) If A is a complete intersection, then we have $\tau_i = 0$ for $i < d$ (by the definition of τ_A and Corollary 18.1.2 in [6]).

(4) τ_d is equal to $[\operatorname{Spec} A]_d$ (Theorem 18.3 (5) in [6]), where

$$[\operatorname{Spec} A]_d = \sum_{\substack{\mathfrak{p} \in \operatorname{Spec} A \\ \dim A/\mathfrak{p} = d}} \ell_{A_\mathfrak{p}}(A_\mathfrak{p})[\operatorname{Spec} A/\mathfrak{p}] \in \mathrm{A}_d(A)_\mathbb{Q}.$$

In particular, we have $\tau_d \neq 0$.

(5) Assume that A is normal. Then, by definition, $\mathrm{A}_{d-1}(A)$ coincides with the divisor class group $\mathrm{Cl}(A)$ of A. Let $\mathrm{cl}(\omega_A) \in \mathrm{Cl}(A)$ be the isomorphism class containing ω_A. Then, we have $\tau_{d-1} = \mathrm{cl}(\omega_A)/2$ in $\mathrm{A}_{d-1}(A)_\mathbb{Q} = \mathrm{Cl}(A)_\mathbb{Q}$ (Lemma 3.5 in [12]).

If A is a ring as in Section 2, then it is known that $\tau_A(\mathrm{L}_i\mathrm{G}_0\,(A)_\mathbb{Q}) = \mathrm{A}_i(A)_\mathbb{Q}$ (we can prove it using Theorem 18.3 (1) and Example 18.3.12 in [6]). Therefore, in the case, we obtain

$$\tau_A(q_i) = \tau_i$$

for each i.

By the equality as above, we can naturally extend the notion of Roberts rings and Dutta multiplicities to the general case as follows ([12], [16]):

DEFINITION 3.2 Let $\mathbb{F}.$ be a perfect A-complex with support in $\{m\}$. The rational number $\chi_{\mathbb{F}.}(\tau_A^{-1}(\tau_d))$ is called the *Dutta multiplicity* of the complex $\mathbb{F}.$ and denoted by $\chi_\infty(\mathbb{F}.)$.

We say that A is a *Roberts ring* if $\tau_i = 0$ for $i = 0, 1, \ldots, d-1$.

By Proposition 3.1 (3), a complete intersection is a Roberts ring. By Proposition 3.1 (5), we can easily find examples of normal Cohen-Macaulay non-Roberts rings. We shall give some examples of normal Gorenstein non-Roberts rings later (Remark 3.4 and Section 6 (R1)).

By definition, we know that the Dutta multiplicity $\chi_\infty(\mathbb{F}.)$ coincides with the Euler characteristic $\chi_{\mathbb{F}.}([A])$ if A is a Roberts ring. Here we give some sufficient conditions for $\chi_\infty(\mathbb{F}.) = \chi_{\mathbb{F}.}([A])$.

PROPOSITION 3.3 *If one of the following conditions is satisfied, then we have $\chi_\infty(\mathbb{F}.) = \chi_{\mathbb{F}.}([A])$:*

(1) *A is a Roberts ring.*

(2) $\mathbb{F}.$ *is* liftable *to a regular local ring, i.e., there exists a regular local ring S and an S-free complex $\mathbb{G}.$ such that A is a homomorphic image of S and $\mathbb{F}. = \mathbb{G}. \otimes_S A$ is satisfied (Theorem 1.2 in [15]).*

(3) *A is equi-dimensional with $d = \dim A \le 2$ (Proposition 3.4 in [12]).*

(4) *A is Gorenstein (or $\mathbb{Q}$-Gorenstein) with $d = \dim A \le 3$ (Proposition 3.4 in [12]).*

If $\mathbb{K}.$ is a Koszul complex with respect to a system of parameters for A, then $\mathbb{K}.$ is liftable to a regular local ring. Hence the Dutta multiplicity of $\mathbb{K}.$ coincides with the usual multiplicity of A with respect to the parameter ideal. As in [12], the Dutta multiplicity of a complex enjoys properties like the usual multiplicity.

REMARK 3.4 Using an example of a negative intersection multiplicity due to Dutta-Hochster-MacLaughlin [5] [10], there exists a 3-dimensional normal Cohen-Macaulay ring (A, m) and a perfect A-complex $\mathbb{F}.$ with support $\{m\}$ such that $\chi_\infty(\mathbb{F}.) \neq \chi_{\mathbb{F}.}([A])$ (p295 in [26]). Miller-Singh [19] constructed 5-dimensional normal Gorenstein local ring (A, m) and $\mathbb{F}.$ such that $\chi_\infty(\mathbb{F}.) \neq \chi_{\mathbb{F}.}([A])$.

By Proposition 3.1 (4), we have $\chi_\infty(\mathbb{F}.) = \chi_{\mathbb{F}.}(\tau_A^{-1}([\mathrm{Spec}\, A]_d))$. By the local Riemann-Roch formula (Example 18.3.12 in [6]), $\chi_{\mathbb{F}.} = \mathrm{ch}(\mathbb{F}.)\tau_A$ is satisfied, where $\mathrm{ch}(\mathbb{F}.) : \mathrm{A}_*(A)_\mathbb{Q} \to \mathbb{Q}$ is the localized Chern character defined by MacPherson's graph construction [6]. Therefore, we have

$$\chi_\infty(\mathbb{F}.) = \mathrm{ch}(\mathbb{F}.) \cap [\mathrm{Spec}\, A]_d = \mathrm{ch}_d(\mathbb{F}.) \cap [\mathrm{Spec}\, A]_d = \mathrm{ch}_d(\mathbb{F}.) \cap \tau_A([A]),$$

where $\mathrm{ch}_i(\mathbb{F}.) : \mathrm{A}_i(A)_\mathbb{Q} \to \mathbb{Q}$ is a map for each i such that $\mathrm{ch}(\mathbb{F}.) = \sum_{i \ge 0} \mathrm{ch}_i(\mathbb{F}.)$.

We can also characterize Roberts rings and Dutta multiplicities in terms of Adams operations ψ^e (defined by Gillet-Soulé [8]) on complexes as below (Lemma 4.4 in [17] and Proposition 2.2 in [16]). They are immediate consequences of the equality

$$\mathrm{ch}_i(\psi^e(\mathbb{F}.)) = e^i\, \mathrm{ch}_i(\mathbb{F}.)$$

proved in Theorem 3.1 in [17].

THEOREM 3.5 *Let (A, m) be a homomorphic image of a regular local ring and put $\dim A = d$. Let $\mathbb{F}.$ be a perfect A-complex with support in $\{m\}$. Let ψ^e denote the e-th Adams operation.*

(1) *For any $k \ge 2$, we have*

$$\chi_\infty(\mathbb{F}.) = \lim_{e \to \infty} \frac{\chi(\psi^e(\mathbb{F}.))}{e^d} = \lim_{e \to \infty} \frac{\chi((\psi^k)^e(\mathbb{F}.))}{k^{de}}.$$

(2) *There exist rational numbers $a_1, \ldots, a_{d+1}$ (depending only on d) such that*

$$\chi_\infty(\mathbb{F}.) = \sum_{e=1}^{d+1} a_e \cdot \chi(\psi^e(\mathbb{F}.)).$$

THEOREM 3.6 *Let A be a d-dimensional Noetherian local ring that is a homomorphic image of a regular local ring S. Put $k = \dim S - \dim A$. Then the following conditions are equivalent:*

(1) $\tau_A([A]) \in \mathrm{A}_d(A)_{\mathbb{Q}}$, *i.e., A is a Roberts ring.*

(2) Letting $\mathbb{G}.$ be a finite S-free resolution of A, $\psi^t([\mathbb{G}.]) = t^k[\mathbb{G}.]$ is satisfied in $\mathrm{K}_0^{\mathrm{Spec}\, A}(\mathrm{Spec}\, S)_{\mathbb{Q}}$ *for some $t \geq 2$.*

(3) With notation as above, $\psi^t([\mathbb{G}.]) = t^k[\mathbb{G}.]$ is satisfied in $\mathrm{K}_0^{\mathrm{Spec}\, A}(\mathrm{Spec}\, S)_{\mathbb{Q}}$ *for any $t \geq 1$.*

Here, $\mathrm{K}_0^{\mathrm{Spec}\, A}(\mathrm{Spec}\, S)_{\mathbb{Q}}$ *is the Grothendieck group of perfect S-complexes with support in* $\mathrm{Spec}\, A$ *(see Gillet-Soulé [8], Soulé [28]).*

4 THE CASE OF DOMAINS

In the section, we assume that (A, m) is a d-dimensional local domain that has a subring S satisfying the following two conditions; (1) S is a Nagata regular local ring, (2) A is a localization of a module-finite ring extension of S.

It is easy to see that, if A is a complete local domain or a local ring of an algebraic variety, then there exists a subring S satisfying the above two conditions.

Let L be a finite algebraic field extension of $Q(S)$ containing $Q(A)$, where $Q(\ )$ denotes the field of fractions. Let B_L be the integral closure of A in L. Since A is a Nagata domain, B_L is a finitely generated A-module with field of fractions L.

Then, we can characterize Roberts rings and Dutta multiplicities in terms of B_L as follows[2] :

THEOREM 4.1 *The following conditions are equivalent:*

(1) $\tau_A([A]) \in \mathrm{A}_d(A)_{\mathbb{Q}}$, *that is, A is a Roberts ring.*

(2) For some finite algebraic field extension L of $Q(A)$ such that L is normal over $Q(S)$, $[B_L] = \mathrm{rank}_A\, B_L \cdot [A]$ in $\mathrm{G}_0(A)_{\mathbb{Q}}$ is satisfied.

(3) For any finite algebraic field extension L of $Q(A)$ such that L is normal over $Q(S)$, $[B_L] = \mathrm{rank}_A\, B_L \cdot [A]$ in $\mathrm{G}_0(A)_{\mathbb{Q}}$ is satisfied.

THEOREM 4.2 *Let L be a finite-dimensional normal extension of $Q(S)$ containing $Q(A)$. For a perfect A-complex $\mathbb{F}.$ with support in $\{m\}$, we have*

$$\chi_\infty(\mathbb{F}.) = \frac{\chi_{\mathbb{F}.}([B_L])}{[L : Q(A)]} = \frac{\sum_i (-1)^i \ell_A(H_i(\mathbb{F}. \otimes_A B_L))}{[L : Q(A)]},$$

where $[L : Q(A)]$ denotes the dimension of the $Q(A)$-vector space L.

Then, we immediately obtain the following corollary:

[2] They are generalizations of Theorem 1.1 in [15] and Theorem 1.1 in [16]. Proofs are much the same, however.

COROLLARY 4.3 *Let A be a Noetherian local normal domain such that A has a* Noether normalization[3] *S with $Q(A)/Q(S)$ Galois. Then, A is a Roberts ring.*

Corollary 4.3 says that, if a Noetherian local normal domain A is not a Roberts ring, then A never have a Noether normalization S with $Q(A)/Q(S)$ Galois.

REMARK 4.4 Combining Theorem 4.1 and Definition 2.1, we have the following:
Suppose that (A, m) is a d-dimensional complete local domain with perfect coefficient field of characteristic $p > 0$. Take a Noether normalization S of A. Then, the following are equivalent:

(1) A is a Roberts ring.

(2) Let $f : A \to A$ be the Frobenius map. (By our assumption, f is finite.) Then,

$$[{}_fA] = \operatorname{rank}_A {}_fA \cdot [A] \quad \text{in } \mathrm{G}_0(A)_{\mathbb{Q}}$$

is satisfied.

(3) For some (any) finite-dimensional normal extension L of $Q(S)$ containing $Q(A)$,

$$[B_L] = \operatorname{rank}_A B_L \cdot [A] \quad \text{in } \mathrm{G}_0(A)_{\mathbb{Q}}$$

is satisfied, where B_L is the integral closure of A in L. (B_L is a finitely generated A-module since A is complete.)

5 THE ORIGIN OF ROBERTS RINGS AND THEIR APPLICATIONS

In 1985, P. Roberts [21] proved the following: Assume that A is a Roberts ring. Let M and N be finitely generated A-modules satisfying the following three conditions; (1) $\mathrm{pd}_A M < \infty$, $\mathrm{pd}_A N < \infty$, (2) $\ell_A(M \otimes_A N) < \infty$, (3) $\dim M + \dim N < \dim A$, where pd_A denotes the *projective dimension* as an A-module. Then, $\sum_i (-1)^i \ell_A(\mathrm{Tor}_i^A(M, N)) = 0$ is satisfied. (It was conjectured by Serre [27] when A is a regular local ring, and it was independently solved by P. Roberts [21] and Gillet-Soulé [7].)

The above was the beginning of the theory of Roberts rings. P. Roberts proved the above vanishing theorem by proving a corresponding equality on Dutta multiplicities (Theorem 5.1 (3)) as below.

Before stating Theorem 5.1, we give a short remark.
Let

$$\mathbb{F}. \; : \; 0 \to F_s \to \cdots \to F_0 \to 0$$

be a perfect A-complex. Then, the A-dual

$$\mathbb{F}.^\vee \; : \; 0 \to F_0^\vee \to \cdots \to F_s^\vee \to 0$$

of $\mathbb{F}.$ is defined. Here, we regard $F_i^\vee$ as the part in degree $-i$ of the complex $\mathbb{F}.^\vee$. It is easy to see that the support of $\mathbb{F}.$ coincides with that of $\mathbb{F}.^\vee$.

[3] We say that regular subring S of A is a *Noether normalization* if the inclusion $S \hookrightarrow A$ is finite.

Let $\mathbb{F}.$ and $\mathbb{G}.$ be perfect A-complexes. Then, $\mathbb{F}. \otimes_A \mathbb{G}.$ is a perfect A-complex with support $\operatorname{Supp} \mathbb{F}. \cap \operatorname{Supp} \mathbb{G}.$.

THEOREM 5.1 *Let A be a d-dimensional local ring that is a homomorphic image of a regular local ring. Let $\mathbb{F}.$ and $\mathbb{G}.$ be perfect A-complexes.*

(1) Assume that the support of $\mathbb{F}.$ is in $\{m\}$. Then,

$$\chi_\infty(\mathbb{F}.^\vee) = (-1)^d \chi_\infty(\mathbb{F}.)$$

is satisfied (Example 18.1.2 in [6]).

(2) Assume that the support of $\mathbb{F}.$ is in $\{m\}$ and A contains a field of characteristic $p > 0$. Let $F^e(\mathbb{F}.)$ be the perfect A-complex whose entries are p^e-th power of those of $\mathbb{F}.$. (It is easy to see $\operatorname{Supp}(F^e(\mathbb{F}.)) = \operatorname{Supp}(\mathbb{F}.)$.) Then,

$$\chi_\infty(F^e(\mathbb{F}.)) = p^{de} \chi_\infty(\mathbb{F}.)$$

is satisfied for each $e > 0$. (Using Proposition 2.6 in [12], we may assume that A is a complete local ring with perfect coefficient field of characteristic $p > 0$. Then, by the equation (3), it will be proved.)

(3) Assume that $\operatorname{Supp} \mathbb{F}. \cap \operatorname{Supp} \mathbb{G}.$ is in $\{m\}$. If $\dim \operatorname{Supp} \mathbb{F}. + \dim \operatorname{Supp} \mathbb{G}. < d$, then

$$\chi_\infty(\mathbb{F}. \otimes_A \mathbb{G}.) = 0$$

is satisfied (P. Roberts [21]).

(4) Assume that $\operatorname{Supp} \mathbb{F}. \cap \operatorname{Supp} \mathbb{G}.$ is in $\{m\}$. If $\dim \operatorname{Supp} \mathbb{F}. + \dim \operatorname{Supp} \mathbb{G}. \leq d$, then

$$\chi_\infty(\mathbb{F}.^\vee \otimes_A \mathbb{G}.) = (-1)^{d-\dim \operatorname{Supp} \mathbb{F}.} \chi_\infty(\mathbb{F}. \otimes_A \mathbb{G}.)$$

is satisfied (C-Y. Chan [3]).

(5) Assume that (A, m) contains a field. Let $\mathbb{F}.$ be a perfect A-complex of length d with $\operatorname{Supp}(\mathbb{F}.) = \{m\}$. (In particular, the support is not empty, and $\mathbb{F}.$ is not exact.) Then

$$\chi_\infty(\mathbb{F}.) > 0$$

is satisfied [17].

If one replace the Dutta multiplicity $\chi_\infty(\ )$ by the Euler characteristic $\chi(\ )$ in Theorem 5.1, then all of (1), (2), (3), (4), (5) are false (see 13.3 in Roberts [26]). However, if A is a Roberts ring, then the Dutta multiplicity of a complex coincides with the Euler characteristic. That is to say, if A is a Roberts ring, then Theorem 5.1 is still true for the Euler characteristic.

By Theorem 5.1 (1) and (2), we have the following:

(S1) *Let (A, m) be a d-dimensional Roberts ring and $\mathbb{F}.$ be a perfect A-complex with support in $\{m\}$. Then,*

$$\chi(\mathbb{F}.) = (-1)^d \chi(\mathbb{F}.^\vee)$$

is satisfied.

(S2) *Let (A, m) be a d-dimensional Roberts ring of characteristic $p > 0$. For a perfect A-complex $\mathbb{F}.$ with support in $\{m\}$,*

$$\chi(F^e(\mathbb{F}.)) = p^{de}\chi(\mathbb{F}.)$$

is satisfied for any e.

They were conjectured by Szpiro (Conjecture C2 in [29]) without the assumption that A is a Roberts ring. (There exists an example (13.3 in Roberts [26]) of a non-Roberts ring A and a perfect A-complex $\mathbb{F}.$ with support $\{m\}$ such that both (S1) and (S2) are not satisfied.)

Let M and N be finitely generated A-modules satisfying the following two conditions; (1) $\mathrm{pd}_A M < \infty$, $\mathrm{pd}_A N < \infty$, (2) $\ell_A(M \otimes_A N) < \infty$. Let $\mathbb{F}.$ (resp. $\mathbb{G}.$) be the minimal A-free resolutions of M (resp. N). Then, it is easy to see

$$\begin{aligned}
\sum_i (-1)^i \ell_A(\mathrm{Tor}_i^A(M, N)) &= \chi(\mathbb{F}. \otimes_A \mathbb{G}) \\
\sum_i (-1)^i \ell_A(\mathrm{Ext}_A^i(M, N)) &= \chi(\mathbb{F}.^\vee \otimes_A \mathbb{G}.).
\end{aligned}$$

By Theorem 5.1 (3), $\sum_i (-1)^i \ell_A(\mathrm{Tor}_i^A(M, N))$ vanishes if A is a Roberts ring with $\dim M + \dim N < \dim A$. That is a vanishing theorem of P. Roberts [24].

Assume $\dim M + \dim N \leq \dim A$. By Theorem 5.1 (4), we have

$$\sum_i (-1)^i \ell_A(\mathrm{Tor}_i^A(M, N)) = (-1)^{d - \dim M} \sum_i (-1)^i \ell_A(\mathrm{Ext}_A^i(M, N))$$

if A is a Roberts ring. That is a theorem due to C-Y. Chan [3].

Theorem 5.1 (5) was proved by P. Roberts [24] when (A, m) is a complete equal-characteristic Noetherian local ring such that the residue class field is perfect of positive characteristic. Using the fact, he proved the new intersection theorem in the mixed-characteristic case. Recently, it has been solved in [17] when A contains a field. Here, it is natural to ask:

CONJECTURE 5.2 Let (A, m) be a homomorphic image of a regular local ring. Put $d = \dim A$. Let $\mathbb{F}.$ be a perfect A-complex of length d with $\mathrm{Supp}(\mathbb{F}.) = \{m\}$. Then $\chi_\infty(\mathbb{F}.) > 0$.

Conjecture 5.2 is open even if A is a Gorenstein ring. As in Remark 3.6 and 3.7 in [12], Conjecture 5.2 is true if A is a Cohen-Macaulay ring of dimension less than or equal to 3. If A is a Cohen-Macaulay Roberts ring in Conjecture 5.2, then we have

$$\chi_\infty(\mathbb{F}.) = \chi(\mathbb{F}.) = \ell_A(H_0(\mathbb{F}.)) > 0$$

by the depth sensitivity [1]. Conjecture 5.2 is true if A has a test module [15].

REMARK 5.3 As in Theorem 1.2 in [15], the conjecture as above is deeply related to the Serre's positivity conjecture as below (Conjecture 5.4).

In fact, if the complex in Conjecture 5.2 is liftable (see Proposition 3.3 (2)) to a regular local ring, then we can prove that the Dutta multiplicity of $\mathbb{F}.$ is non-negative by Gabber's non-negativity theorem [2]. Conversely, if Conjecture 5.2 is ture, so is Conjecture 5.4 in the case where either M or N is a (not necessary maximal) Cohen-Macaulay module.

CONJECTURE 5.4 Let (S, n) be a regular local ring. Let M and N be a finitely generated S-modules such that $\dim M + \dim N = \dim S$ and $0 < \ell_S(M \otimes_S N) < \infty$. Then, $\sum_i (-1)^i \ell_S(\mathrm{Tor}_i^S(M, N)) > 0$ is satisfied.

It is well-known that the following conjecture (that is called *Small Cohen-Macaulay Modules Conjecture [9]*) implies the Serre's positivity conjecture.

CONJECTURE 5.5 If A is a complete local ring, then A has a maximal Cohen-Macaulay module M, i.e., M is a finitely generated A-module with $\operatorname{depth} M = \dim A$.

By Theorem 1.3 in [15], Small Cohen-Macaulay Modules Conjecture also implies Conjecture 5.2 using the theory of test modules.

By Theorem 4.2, we know that Conjecture 5.2 is equivalent to the following one:

CONJECTURE 5.6 Let (A, m) be a d-dimensional complete local normal domain and assume that A has a Noether normalization S with $Q(A)/Q(S)$ Galois. Let

$$\mathbb{F}. \ : \ 0 \to F_d \to \cdots \to F_0 \to 0$$

be a perfect A-complex with $\mathrm{Supp}(\mathbb{F}.) = \{m\}$. Then $\chi(\mathbb{F}.) = \sum_i (-1)^i \ell_A(H_i(\mathbb{F}.)) > 0$.

6 BASIC PROPERTIES ON ROBERTS RINGS

In the section, we shall give basic properties on Roberts rings. For proofs, we refer the reader to [16]. However we shall give a proof to (R11), because it was not proved anywhere else.

Throughout the section, (A, m) is a homomorphic image of a regular local ring.

(R1) If A is a complete intersection, then A is a Roberts ring. However, a Gorenstein ring is not necessary a Roberts ring. Here, we give a concrete example.

Let t, m, n be integers such that $1 \leq t \leq m \leq n$. Let R be the polynomial ring $k[x_{ij} \mid 1 \leq i \leq m;\ 1 \leq j \leq n]$ over a field k divided by the ideal $I_t(x_{ij})$ generated by all t by t minors of the m by n matrix (x_{ij}). Using a criterion (Theorem 1.3 in [14]) below, we conclude that, for $t = 2$, $A = R_{(R_1)}$ is a Roberts ring if and only if R is a complete intersection as in Section 3 in [14]. Since any localization of a Roberts ring is a Roberts ring again (see (R4)), we know that, for any t, $A = R_{(R_1)}$ is a Roberts ring if and only if A is a complete intersection. Therefore, R is a Roberts ring if and only if $t = 1$ or $t = m = n$. Hence, if $1 < t < m = n$, then A is a Gorenstein non-Roberts ring.

THEOREM 6.1 *Let $R = \oplus_{n \geq 0} R_n = R_0[R_1]$ be a Noetherian graded ring over a field R_0. Assume that $X = Proj(R)$ is smooth over R_0. Then, the local ring $A = R_{(R_1)}$ is a Roberts ring if and only if*

$$\mathrm{td}(\Omega_X^\vee) \equiv 1 \mod c_1(\mathcal{O}_X(1)) \cdot \mathrm{CH}^*(X)_{\mathbb{Q}}.$$

In particular, if X is an abelian variety, then $A = R_{(R_1)}$ is a Roberts ring.

Here, $\mathrm{CH}^*(X)$ denotes the Chow ring of the smooth projective variety X, $c_1(\ )$ is the first Chern class, $\Omega_X^\vee$ is the tangent sheaf of X, and $\mathrm{td}(\Omega_X^\vee)$ is its Todd class (see [6]).

(R2) If A is a normal Roberts ring, then it is $\mathbb{Q}$-Gorenstein, i.e., the isomorphism class $\mathrm{cl}(\omega_A)$ containing the canonical module ω_A of A is a torsion in the divisor class group $\mathrm{Cl}(A)$.

Assume that A is a 2-dimensional normal domain. Then A is a Roberts ring if and only if A is $\mathbb{Q}$-Gorenstein.

(R3) If $\dim A \leq 1$, then A is a Roberts ring.

If A is a Roberts ring, then A is equi-dimensional, i.e., $\dim A = \dim A/P$ is satisfied for any minimal prime ideal P of A.

(R4) Let $S \to T$ be a flat extension of regular local rings that may not be a local homomorphism. Let I be an ideal of S. If S/I is a Roberts ring, then so is T/IT.

In particular, if A is a Roberts ring, then so are its completion $\hat{A}$, its henselization ${}^hA$ and its localization A_P for any prime ideal P of A.

(R5) Assume that A is an excellent henselian local ring. If its completion $\hat{A}$ is a Roberts ring, then so is A.

It is natual to ask the following:

QUESTION 6.2 Suppose that A is a homomorphic image of a regular local ring. When the completion $\hat{A}$ is a Roberts ring, is A so?

It is proved that the above question is equivalent to the following one:

QUESTION 6.3 Suppose that A is a homomorphic image of a regular local ring. Is the natural map $\mathrm{G}_0(A)_{\mathbb{Q}} \to \mathrm{G}_0(\hat{A})_{\mathbb{Q}}$ injective?

The author does not know whether they are true or not. If they are true, then we know that the Riemann-Roch map $\tau_A : \mathrm{G}_0(A)_{\mathbb{Q}} \to \mathrm{A}_*(A)_{\mathbb{Q}}$ does not depend on the choice of a regular local ring S where A is a homomorphic image of S.

(R6) Let T be a finite-dimensional regular (not necessary local) ring and $R = T/I$ be a homomorphic image of T. Then, the set

$$\{P \in \mathrm{Spec}\, T \mid P \supseteq I \text{ and } \tau_{R_P/T_P}([R_P]) \in \mathrm{A}_{\dim R_P}(R_P)_{\mathbb{Q}}\}$$

is an open set of $\operatorname{Spec} R$, where $\tau_{R_P/T_P} : \mathrm{G}_0(R_P)_{\mathbb{Q}} \to \mathrm{A}_*(R_P)_{\mathbb{Q}}$ denotes the Riemann-Roch map determined by R_P and T_P.

(R7) Let x be a non-zero-divisor of A. If A is a Roberts ring, then so is A/xA. (The converse is not true in general.)

(R8) Let I be an ideal of A contained in the 0-th local cohomology group $H^0_m(A)$ with respect to the maximal ideal m of A. Then, A is a Roberts ring if and only if so is A/I.

(R9) Let $x_1, \ldots, x_s$ be a filter regular sequence of A. If A is a Roberts ring, then so is $A/(x_1, \ldots, x_s)$.

(R10) Assume that there exists a regular local ring T containing A such that the inclusion $i : A \to T$ is finite. Then, A is a Roberts ring.

In particular, an invariant subring of a regular local ring with respect to a finite group is a Roberts ring if the regular local ring is finite over the invariant subring.

Furthermore, let H be a simplicial submonoid of $\mathbb{N}_0^d$. (Here, $\mathbb{N}_0$ denotes the set of non-negative integers. We say that a submonoid H of $\mathbb{N}_0^d$ is *simplicial* if there is a positive integer n such that $n \cdot \mathbb{N}_0^d \subseteq H$.) Let $k[\mathbb{N}_0^d]$ (resp. $k[H]$) be the semi-group ring of $\mathbb{N}_0^d$ (resp. H) over a field k. Then, since the inclusion $k[H]_{(H\setminus\{0\})} \hookrightarrow k[\mathbb{N}_0^d]_{(\mathbb{N}_0^d\setminus\{0\})}$ is finite, we know that $k[H]_{(H\setminus\{0\})}$ is a Roberts ring.

(R11) Let $F = \{F_k\}_{k\in\mathbb{Z}}$ be a filtration of ideals of A, i.e., F_k's are ideals of A such that (1) $F_k \supseteq F_{k+1}$ for $k \in \mathbb{Z}$, (2) $F_0 = A \supset m \supseteq F_1$, (3) $F_iF_j \subseteq F_{i+j}$ for $i, j \in \mathbb{Z}$. Put $G = \oplus_{k\geq 0} F_k/F_{k+1}$, $R' = A[\{F_kt^k \mid k \in \mathbb{Z}\}] \subset A[t, t^{-1}]$. Assume that R' is Noetherian. Then we have the following:

(1) There exists a graded homomorphism $\rho : \mathrm{A}_*(A)_{\mathbb{Q}} \to \mathrm{A}_*(G)$ such that $\rho(\tau_A([A])) = \tau_G([G])$.

(2) If A is a Roberts ring, then so is any local ring of G.

(3) Assume that both A and G are normal domains. If A is $\mathbb{Q}$-Gorenstein, then so is G.

Proof. Since $G = R'/t^{-1}R'$, we know that G is Noetherian. It is well-known that $\dim G = d$ and $\dim R' = d + 1$, where $d = \dim A$.

In the same way as Chapter 5 in [6], we shall construct the map ρ as below.

Let $g : R' \to G$ be the projection and $\phi : R' \to R'[(t^{-1})^{-1}] = A[t, t^{-1}]$ the localization. Then, we have the following exact sequence

$$\mathrm{A}_*(G)_{\mathbb{Q}} \xrightarrow{g^*} \mathrm{A}_*(R')_{\mathbb{Q}} \xrightarrow{\phi_*} \mathrm{A}_*(A[t,t^{-1}]) \longrightarrow 0 \tag{4}$$

since $\operatorname{Spec} R' \setminus \operatorname{Spec} G = \operatorname{Spec} A[t, t^{-1}]$ (Proposition 1.8 in [6]). Since $\operatorname{Spec} G \xrightarrow{i} \operatorname{Spec} R'$ is an effective Cartier divisor, we have the map $i^! : \mathrm{A}_*(R')_{\mathbb{Q}} \to \mathrm{A}_*(G)_{\mathbb{Q}}$ (2.3 and Proposition 6.1 in [6]). By the definition of $i^!$, for a prime ideal P of R', we have $i^!([\operatorname{Spec} R'/P]) = [\operatorname{Spec} R'/P + (t^{-1})]$ (resp. 0) if $t^{-1} \notin P$ (resp.

$t^{-1} \in P$), where $[\operatorname{Spec} R'/P + (t^{-1})]$ is an element in the Chow group of R' defined in 1.5 in [6]. Since $i^! g^* = 0$ by the definition of $i^!$, we obtain the map $\psi : \mathrm{A}_*(A[t, t^{-1}])_{\mathbb{Q}} \to \mathrm{A}_*(G)_{\mathbb{Q}}$ such that $\psi\phi_* = i^!$ by the exactness of the sequence (4).

On the other hand, the flat map $\xi : A \to A[t, t^{-1}]$ induces the isomorphism $\xi_* : \mathrm{A}_*(A)_{\mathbb{Q}} \to \mathrm{A}_*(A[t, t^{-1}])_{\mathbb{Q}}$.

Put $\rho = \psi\xi_* : \mathrm{A}_*(A)_{\mathbb{Q}} \to \mathrm{A}_*(G)_{\mathbb{Q}}$. By definition, ρ is a graded homomorphism, i.e., $\rho(\mathrm{A}_i(A)_{\mathbb{Q}}) \subseteq \mathrm{A}_i(G)_{\mathbb{Q}}$ for each i.

In order to prove $\rho(\tau_A([A])) = \tau_G([G])$, it is suffucient to show the following three equalities:

(i) $\xi_*(\tau_A([A])) = \tau_{A[t,t^{-1}]}([A[t, t^{-1}]])$

(ii) $\phi_*(\tau_{R'}([R'])) = \tau_{A[t,t^{-1}]}([A[t, t^{-1}]])$

(iii) $i^!(\tau_{R'}([R'])) = \tau_G([G])$

Here, ξ and ϕ are smooth morphism. Therefore, by Theorem 18.3 (4), we obtain (i) and (ii) since $\Omega_{A[t,t^{-1}]/A} \simeq A[t, t^{-1}]$ and $\Omega_{A[t,t^{-1}]/R'} \simeq 0$. We have (iii) by Corollary 18.1.12 and Example 18.3.12 in [6]. We have completed the proof of (1).

Put

$$
\begin{aligned}
\tau_A([A]) &= \tau_d + \tau_{d-1} + \cdots + \tau_0 \\
\tau_G([G]) &= \tau'_d + \tau'_{d-1} + \cdots + \tau'_0,
\end{aligned}
$$

where $\tau_i \in \mathrm{A}_i(A)_{\mathbb{Q}}$ and $\tau'_i \in \mathrm{A}_i(G)_{\mathbb{Q}}$ for each i. Since ρ is graded and $\rho(\tau_A([A])) = \tau_G([G])$, we have $\rho(\tau_i) = \tau'_i$ for each i. In particular, if $\tau_i = 0$, then so is τ'_i.

Assume that A is a Roberts ring. Then we have $\tau_G([G]) = \tau'_d \in \mathrm{A}_d(G)_{\mathbb{Q}}$. We know that any local ring of G is a Roberts ring by Lemma 4.1 in [16]. The proof of (2) is completed.

Next assume that both A and G are normal domains. Then, by Proposition 3.1 (5), we have $\tau_{d-1} = \mathrm{cl}(\omega_A)/2 \in \mathrm{A}_{d-1}(A)_{\mathbb{Q}} = \mathrm{Cl}(A)_{\mathbb{Q}}$ and $\tau'_{d-1} = \mathrm{cl}(\omega_G)/2 \in \mathrm{A}_{d-1}(G)_{\mathbb{Q}} = \mathrm{Cl}(G)_{\mathbb{Q}}$. Therefore, if $\mathrm{cl}(\omega_A)$ is a torsion in $\mathrm{Cl}(A)$, then so is $\mathrm{cl}(\omega_G)$ in $\mathrm{Cl}(G)$.

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