On Roberts rings

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Abstract

In 1985, P. C. Roberts [14] proved the vanishing theorem of intersection multiplicities for a local ring such that the Todd class of the ring is a homogeneous element in the Chow group. We refer such rings as Roberts rings. For rings of characteristic positive, we can characterize Roberts rings by the Frobenius maps. For rings with field of fractions of characteristic 0, we can characterize Roberts rings by some Galois extensions. We shall give basic properties and examples of Roberts rings in the paper.

1 Introduction

In 1985, P. C. Roberts [16] proved the vanishing theorem of intersection multiplicities for a local ring that satisfies $\tau_{A/S}([A]) = [\operatorname{Spec} A]_{\dim A}$, that we refer as a *Roberts ring* in the paper. Let k be a perfect field of characteristic $p > 0$, and put $A = k[[x_1, \ldots, x_n]]/(f_1, \ldots, f_s)$. Furthermore put ${}_fA = k[[x_1, \ldots, x_n]]/(f_1^p, \ldots, f_s^p)$. We define a ring homomorphism $f : A \to {}_fA$ by $f(y) = y^p$ for each $y \in A$. Then, the condition $\tau_{A/S}([A]) = [\operatorname{Spec} A]_{\dim A}$ is equivalent to that the cycle $[{}_fA]$ corresponding to the A-module ${}_fA$ coincides with $p^{\dim A}[A]$ in $\mathrm{K}_0(A)_{\mathbb{Q}}$, that is the Grothendieck group of finitely generated A-modules with rational coefficient. If A is a regular local ring, then ${}_fA$ is a free A-module of rank $p^{\dim A}$ and, therefore, A satisfies $[{}_fA] = p^{\dim A}[A]$ in $\mathrm{K}_0(A)_{\mathbb{Q}}$. If the equality $[{}_fA] = p^{\dim A}[A]$ in $\mathrm{K}_0(A)_{\mathbb{Q}}$ is satisfied, then Dutta multiplicity for a complex coincides with the alternating sum of lengths of homology modules (Application 3.2), and Szpiro's conjectures are true for A (Application 3.4).

Put

$$\tau_{A/S}([A]) = \tau_d + \tau_{d-1} + \cdots + \tau_0 \quad (\tau_i \in \mathrm{A}_i(A)_{\mathbb{Q}}),$$

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where $d = \dim A$. We sometimes call $\tau_{A/S}([A])$ the *Todd class* of A. Then, A is a Roberts ring if and only if $\tau_i = 0$ for $i < d$. Such τ_i's inherit homological properties on A. For example, for a normal local ring A, $\tau_{d-1} = 0$ if and only if the canonical class is a torsion in the divisor class group $\mathrm{Cl}(A)$ (Lemma 3.5 in [5]). If A is a complete intersection, then $\tau_i = 0$ for $i < d$, that is, A is a Roberts ring (see Roberts [16]). If A is Gorenstein, then $\tau_{d-j} = 0$ for each odd integer j (e.g., see the proof of Lemma 3.5 in [5]).

In the paper, we shall study Roberts rings.

In Section 2, we shall give the definition of Roberts rings. Roberts rings are characterized by using the Frobenius maps in the case of positive characteristic as in Remark 2.3. Furthermore, we show that Roberts rings are characterized in terms of Adams operations (Gillet-Soulé [3], Soulé [21]) in Proposition 2.2. By the following theorem that will be proved in Section 2, Roberts rings are characterized in terms of some Galois extensions:

Theorem 1.1 *Let A be a d-dimensional Nagata local domain. Assume that A has a Noether normalization S (i.e., S is a regular local subring of A such that A is a finitely generated S-module) such that the extension $Q(A)/Q(S)$ is separable, where $Q(\ )$ denotes the field of fractions.*

Then, the following conditions are equivalent:

(1) $\tau_{A/S}([A]) \in \mathrm{A}_d(A)_{\mathbb{Q}}$.

(2) For some finite-dimensional field extension L of $Q(A)$ such that L is Galois over $Q(S)$, the following is satisfied:

- *Let B be the integral closure of A in L. (B is a finite A-module since A is a Nagata ring. However, B may not be a local ring.) Then, $[B] = \mathrm{rank}_A B \cdot [A]$ in $\mathrm{K}_0(A)_{\mathbb{Q}}$ is satisfied.*

(3) For any finite-dimensional field extension L of $Q(A)$ such that L is Galois over $Q(S)$, the following is satisfied:

- *Let B be the integral closure of A in L. Then, $[B] = \mathrm{rank}_A B \cdot [A]$ in $\mathrm{K}_0(A)_{\mathbb{Q}}$ is satisfied.*

In Section 3, we give some applications of the theory of Roberts rings. The author are most interested in Application 3.2. That is to say, Dutta multiplicity of a complex coincides with the alternating sum of lengths of homology modules of the complex over a Roberts ring. Furthermore, the positivity conjecture of Dutta multiplicities is reduced to the case where the given ring is a Roberts ring.

Let A be a local ring of finite type over a regular ring S. Then the construction of the Riemann-Roch map $\tau_{A/S} : \mathrm{K}_0(A)_{\mathbb{Q}} \to \mathrm{A}_*(A)_{\mathbb{Q}}$ depends not only on A but also on S. However, in Section 4, we prove that $\tau_{A/S}$ is independent of the choice of S in many important cases as follows:

Proposition 1.2 *Let A be a local ring of finite type over a regular (not necessary local) ring S, that is, A is finitely generated over S as a ring. If one of the following two conditions is satisfied, then the Riemann-Roch map $\tau_{A/S}$ is independent of the choice of S:*

- *A is an excellent henselian local ring.*
- *A is essentially of finite type over a field or $\mathbb{Z}$.*

In Section 5, we shall state basic properties of Roberts rings, that is, we prove the following theorem:

Theorem 1.3 *Let (A, m) be a d-dimensional local ring that is a homomorphic image of a regular local ring S.*

(1) If A is a complete intersection, then A is a Roberts ring. There are examples of Gorenstein non-Roberts rings.

(2) If A is a normal Roberts ring, then the isomorphism class $\mathrm{cl}(K_A)$ containing the canonical module K_A of A is a torsion in the divisor class group $\mathrm{Cl}(A)$.

(3) If $\dim A \leq 1$, then A is a Roberts ring.

(4) If A is a Roberts ring, then A is equi-dimensional.

(5) Let $S \to T$ be a flat extension of regular local rings that may not be a local homomorphism. Let I be an ideal of S. If S/I is a Roberts ring, then so is T/IT.

In particular, if A is a Roberts ring, then so are its completion $\hat{A}$, its henselization ${}^{h}A$ and its localization A_P for any prime ideal P of A.

(6) Let T be a finite-dimensional regular (not necessary local) ring and $R = T/I$ be a homomorphic image of T. Then, the set

$$\{P \in \operatorname{Spec} T \mid P \supseteq I \text{ and } \tau_{R_P/T_P}([R_P]) \in \mathrm{A}_{\dim R_P}(R_P)_{\mathbb{Q}}\}$$

is open in $\operatorname{Spec} R$.

(7) Assume that A is an excellent henselian local ring. If its completion $\hat{A}$ is a Roberts ring, then so is A.

(8) Let x be a non-zero-divisor of A. If A is a Roberts ring, then so is A/xA.

(9) Let I be an ideal of A contained in the 0-th local cohomology group $H^0_m(A)$ with respect to the maximal ideal m of A. Then, A is a Roberts ring if and only if so is A/I.

(10) *Let $x_1, \ldots, x_s$ be a filter regular sequence of A. If A is a Roberts ring, then so is $A/(x_1, \ldots, x_s)$.*

(11) *Assume that there exists a regular local ring T containing A such that the inclusion $i : A \to T$ is finite. Then, A is a Roberts ring.*

In particular, an invariant subring of a regular local ring with respect to a finite group is a Roberts ring.

In Section 6, we give some examples.

We refer the reader to Fulton [2], Matsumura [10] and Roberts [18] for unexplained terminologies.

For a Noetherian ring R, $\mathrm{K}_0(R)$ denotes the Grothendieck group of finitely generated R-modules. For an R-module M, $[M]$ denotes the element in $\mathrm{K}_0(R)$ corresponding to M. If a ring homomorphism $f : R \to S$ is finite, we have the induced homomorphism $f^* : \mathrm{K}_0(S) \to \mathrm{K}_0(R)$ of additive groups defined by $f^*([M]) = [{}_fM]$ for an S-module M, where ${}_fM = M$ is an R-module whose R-module structure is given through f.

For a Noetherian ring R, $\mathrm{A}_*(R)$ is the Chow group of the scheme $\operatorname{Spec} R$. For a prime ideal $\mathfrak{p}$ of R of $\dim R/\mathfrak{p} = i$, $[\operatorname{Spec} R/\mathfrak{p}]$ denotes the cycle in $\mathrm{A}_i(R)$ corresponding to the closed subscheme $\operatorname{Spec} R/\mathfrak{p}$. If a ring homomorphism $f : R \to S$ is finite, we have the induced homomorphism $f^* : \mathrm{A}_*(S) \to \mathrm{A}_*(R)$ of additive groups defined as in 1.4 in Fulton [2]. (f^* is the push-forward of cycles for the proper morphism $\operatorname{Spec} S \to \operatorname{Spec} R$.)

For an additive group N, $N_{\mathbb{Q}}$ denotes $N \otimes_{\mathbb{Z}} \mathbb{Q}$.

2 Definition and some characterizations of Roberts rings

In this section we give definition and some characterizations of Roberts rings.

First we define the notion of Roberts rings.

Definition 2.1 Let A be a d-dimensional Noetherian local ring. We say that A is a *Roberts ring* if there exists a regular local ring S that satisfies the following two conditions;

1. A is of finite type over S, that is, A is finitely generated over S as a ring,
2. $\tau_{A/S}([A]) \in \mathrm{A}_d(A)_{\mathbb{Q}}$.

Here, $\tau_{A/S} : \mathrm{K}_0(A)_{\mathbb{Q}} \to \mathrm{A}_*(A)_{\mathbb{Q}}$ is an isomorphism of $\mathbb{Q}$-vector spaces defined in Chapter 18 in Fulton [2]. We refer to $\tau_{A/S}$ as a *Riemann-Roch map.* As in Chapter 20 of Fulton [2], it is assumed that all schemes are of finite type over a fixed regular scheme in the singular Riemann-Roch theory. In fact, the construction of the isomorphism $\tau_{A/S} : \mathrm{K}_0(A)_{\mathbb{Q}} \to \mathrm{A}_*(A)_{\mathbb{Q}}$ of $\mathbb{Q}$-vector spaces depends not only on A but also on the fixed regular ring S. However, $\tau_{A/S}$ is independent of the choice of S in many important cases as we shall see in Section 4.

Let A be a local ring, and $\mathbb{F}.$ be a bounded complex of free A-modules. If $\mathbb{F}.$ has homology of finite length, we define the Euler characteristic $\chi_{\mathbb{F}.} : \mathrm{K}_0(A) \to \mathbb{Z}$ (or $\chi_{\mathbb{F}.} : \mathrm{K}_0(A)_{\mathbb{Q}} \to \mathbb{Q}$) to be

$$\chi_{\mathbb{F}.}([M]) = \sum_t (-1)^t \ell_A(H_t(\mathbb{F}. \otimes_A M)),$$

where $H_t(\mathbb{F}. \otimes_A M)$ is the t-th homology module of the complex $\mathbb{F}. \otimes_A M$ and $\ell_A(H_t(\mathbb{F}. \otimes_A M))$ denotes its length.

Let X be a scheme of finite type over a regular scheme. A bounded complex of locally free $\mathcal{O}_X$-modules of finite rank is called a *perfect* complex. For a perfect $\mathcal{O}_X$-complex $\mathbb{F}.$, we define the *support* of $\mathbb{F}.$ by

$$\mathrm{Supp}(\mathbb{F}.) = \bigcup_t \mathrm{Supp}(H_t(\mathbb{F}.)).$$

The support of $\mathbb{F}.$ is a closed set of X consisting of those points at which $\mathbb{F}.$ is not exact. Let $\mathbb{F}.$ be a perfect $\mathcal{O}_X$-complex with support in Y. We denote the *localized Chern character* with respect to the perfect complex $\mathbb{F}.$ by

$$\mathrm{ch}_Y^X(\mathbb{F}.) = \bigoplus_{i \geq 0} \mathrm{ch}_i(\mathbb{F}.).$$

If no confusion is possible, we denote it simply by $\mathrm{ch}(\mathbb{F}.)$. We refer the reader to Fulton [2] or Roberts [18] for the definition and basic properties of localized Chern characters. We recall that localized Chern characters are defined as operators on the Chow group, and that if η is a cycle of dimension j in $\mathrm{A}_j(X)_{\mathbb{Q}}$, then $\mathrm{ch}_i(\mathbb{F}.) \cap \eta$ is an element of $\mathrm{A}_{j-i}(Y)_{\mathbb{Q}}$.

Let $\mathbb{F}.$ be a perfect $\mathcal{O}_X$-complex with support in Y. We define $\chi_{\mathbb{F}.} : \mathrm{K}_0(X) \to \mathrm{K}_0(Y)$ by $\chi_{\mathbb{F}.}([\mathcal{F}]) = \sum_i (-1)^i [H_i(\mathbb{F}. \otimes_{\mathcal{O}_X} \mathcal{F})]$ for a coherent $\mathcal{O}_X$-module $\mathcal{F}$ as in Example 18.3.12 in Fulton [2].

We can describe Roberts rings in terms of Adams operations on compleces (Gillet-Soulé [3], Sooulé [21]) as follows:

Proposition 2.2 *Let A be a d-dimensional Noetherian local ring that is a homomorphic image of a regular local ring S. Put $k = \dim S - \dim A$. Then the following conditions are equivalent:*

1. $\tau_{A/S}([A]) \in \mathrm{A}_d(A)_{\mathbb{Q}}$.
2. *Letting $\mathbb{F}.$ be a finite S-free resolution of A, $\psi^t([\mathbb{F}.]) = t^k[\mathbb{F}.]$ is satisfied in $\mathrm{K}_0^{\mathrm{Spec}\,A}(\mathrm{Spec}\,S)_{\mathbb{Q}}$ for some $t \geq 2$.*

3. *With notation as above,* $\psi^t([\mathbb{F}.]) = t^k[\mathbb{F}.]$ *is satisfied in* $\mathrm{K}_0^{\mathrm{Spec}\,A}(\mathrm{Spec}\,S)_{\mathbb{Q}}$ *for any* $t \geq 1$.

Here, $\mathrm{K}_0^{\mathrm{Spec}\,A}(\mathrm{Spec}\,S)_{\mathbb{Q}}$ *is the Grothendieck group of perfect* S*-compleces with support in* $\mathrm{Spec}\,A$ *and* ψ^t *denotes the* t*-th Adams operation (see Gillet-Soulé [3], Soulé [21]).*

Proof. Let $\mathbb{F}.$ be a finite S-free resolution of A. Then, by the definition of $\tau_{A/S}$ (Chapter 18 in Fulton [2]), we have

$$\tau_{A/S}([A]) = \mathrm{ch}(\mathbb{F}.) \cap [\mathrm{Spec}\,S] = \sum_{i \geq k} \mathrm{ch}_i(\mathbb{F}.) \cap [\mathrm{Spec}\,S],$$

where $\mathrm{ch}_i(\mathbb{F}.) \cap [\mathrm{Spec}\,S] \in \mathrm{A}_{d+k-i}(A)_{\mathbb{Q}}$. Therefore, $\tau_{A/S}([A]) \in \mathrm{A}_d(A)_{\mathbb{Q}}$ if and only if $\mathrm{ch}_i(\mathbb{F}.) \cap [\mathrm{Spec}\,S] = 0$ for $i > k$.

Let t be an integer bigger than 1. By Theorem 3.1 in [9],

$$\mathrm{ch}_i(\psi^t(\mathbb{F}.)) = t^i \mathrm{ch}_i(\mathbb{F}.)$$

is satisfied for any i and t. Then, we have

$$\mathrm{ch}(\psi^t(\mathbb{F}.)) \cap [\mathrm{Spec}\,S] = \sum_{i \geq k} t^i \mathrm{ch}_i(\mathbb{F}.) \cap [\mathrm{Spec}\,S].$$

Therefore, we know that A is a Roberts ring if and only if

$$\mathrm{ch}(\psi^t(\mathbb{F}.)) \cap [\mathrm{Spec}\,S] = t^k \mathrm{ch}(\mathbb{F}.) \cap [\mathrm{Spec}\,S]$$

is satisfied. Then, by Example 18.3.12 in Fulton [2], the map $\chi_{\psi^t(\mathbb{F}.)} : \mathrm{K}_0(S)_{\mathbb{Q}} \to \mathrm{K}_0(A)_{\mathbb{Q}}$ coincides with $t^k \chi_{\mathbb{F}.} = \chi_{\mathbb{F}.\oplus t^k}$ in the case. Remark that, by Lemma 1.9 in Gillet-Soulé [3], the map $\mathrm{K}_0^{\mathrm{Spec}\,A}(\mathrm{Spec}\,S)_{\mathbb{Q}} \to \mathrm{K}_0(A)_{\mathbb{Q}}$ induced by $[\mathbb{G}.] \mapsto \chi_{\mathbb{G}.}([S])$ is an isomorphism. Therefore, $\tau_{A/S}([A]) \in \mathrm{A}_d(A)_{\mathbb{Q}}$ if and only if $\psi^t([\mathbb{F}.]) = t^k[\mathbb{F}.]$ in $\mathrm{K}_0^{\mathrm{Spec}\,A}(\mathrm{Spec}\,S)_{\mathbb{Q}}$.

It is easy to see that $\psi^t([\mathbb{F}.]) = t^k[\mathbb{F}.]$ is satisfied for any $t \geq 1$ if $\tau_{A/S}([A]) \in \mathrm{A}_d(A)_{\mathbb{Q}}$. **q.e.d.**

Remark 2.3 Arguments here are found in Section 2 in [7].

Let A be a d-dimensional complete equal-characteristic local ring with residue class field perfect of characteristic $p > 0$. Then, it is easy to see that the Frobenius map $f : A \to A$ is finite. The map f induces $f^* : \mathrm{K}_0(A)_{\mathbb{Q}} \to \mathrm{K}_0(A)_{\mathbb{Q}}$. Put

$$L_i\,\mathrm{K}_0(A)_{\mathbb{Q}} = \{c \in \mathrm{K}_0(A)_{\mathbb{Q}} \mid f^*(c) = p^i c\}$$

for $i = 0, 1, \ldots, d$, i.e., $L_i\,\mathrm{K}_0(A)_{\mathbb{Q}}$ is the eigenspace of f^* with eigenvalue p^i. Then we can easily prove

$$\mathrm{K}_0(A)_{\mathbb{Q}} = \bigoplus_{i=0}^{d} L_i\,\mathrm{K}_0(A)_{\mathbb{Q}}.$$

Hence we have a unique representation

$$[A] = q_d + q_{d-1} + \cdots + q_0, \quad (q_i \in L_i \mathrm{K}_0(A)_{\mathbb{Q}}).$$

On the other hand, $\tau_{A/S}(L_i \mathrm{K}_0(A)_{\mathbb{Q}}) = \mathrm{A}_i(A)_{\mathbb{Q}}$ is satisfied for any i and, therefore, we have $\tau_{A/S}(q_i) \in \mathrm{A}_i(A)_{\mathbb{Q}}$ for each i, where S is a formal power series ring over a field such that A is a homomorphic image of S. Thus, we know that the following statements are equivalent:

(1) A is a Roberts ring.

(2) $\tau_{A/S}([A]) \in \mathrm{A}_d(A)_{\mathbb{Q}}$.

(3) $q_i = 0$ for $i = 0, 1, \ldots, d-1$.

(4) $[{}_f A] = p^d [A]$ in $\mathrm{K}_0(A)_{\mathbb{Q}}$.

(5) For any $e \geq 1$, $[{}_{f^e} A] = p^{de} [A]$ in $\mathrm{K}_0(A)_{\mathbb{Q}}$.

As in Proposition 1.2 that will be proved in Section 4, the map $\tau_{A/S}([A])$ is independent of the choice of S if A is a complete local ring. Therefore, (1) is equivalent to (2).

In the case where A is a complete local domain with field of fractions $Q(A)$ of characteristic 0, we can characterize Roberts rings by some Galois extension as in Theorem 1.1.

Proof of Theorem 1.1. Let L be a finite Galois extension of $Q(S)$ containing $Q(A)$, and G be the Galois group of the field extension $L/Q(S)$. Let B denote the integral closure of A in L.

For each $g \in G$, the S-automorphism $g : B \to B$ induces $g^* : \mathrm{K}_0(B) \to \mathrm{K}_0(B)$ and $g^* : \mathrm{A}_*(B) \to \mathrm{A}_*(B)$. Therefore, the Galois group G acts on $\mathrm{A}_*(B)_{\mathbb{Q}}$. By Example 1.7.6 in Fulton [2], the composite map of

$$(\mathrm{A}_t(B)_{\mathbb{Q}})^G \hookrightarrow \mathrm{A}_t(B)_{\mathbb{Q}} \xrightarrow{i^*} \mathrm{A}_t(S)_{\mathbb{Q}}$$

is an isomorphism for each t, where $i : S \to B$ is the inclusion. Since S is a regular local ring, it is easy to see that $\mathrm{K}_0(S) \simeq \mathbb{Z}$. Since $\tau_{S/S} : \mathrm{K}_0(S)_{\mathbb{Q}} \to \mathrm{A}_*(S)_{\mathbb{Q}}$ is an isomorphism, we have $\mathrm{A}_d(S)_{\mathbb{Q}} \simeq \mathbb{Q}$ and $\mathrm{A}_t(S)_{\mathbb{Q}} \simeq 0$ if $t < d$. Therefore we know $(\mathrm{A}_d(B)_{\mathbb{Q}})^G \simeq \mathbb{Q}$ and $(\mathrm{A}_t(B)_{\mathbb{Q}})^G \simeq 0$ if $t < d$.

On the other hand, since $g : B \to B$ is an S-automorphism, the diagram

$$\begin{array}{ccc} \mathrm{K}_0(B)_{\mathbb{Q}} & \xrightarrow{\tau_{B/S}} & \mathrm{A}_*(B)_{\mathbb{Q}} \\ g^* \downarrow & & g^* \downarrow \\ \mathrm{K}_0(B)_{\mathbb{Q}} & \xrightarrow{\tau_{B/S}} & \mathrm{A}_*(B)_{\mathbb{Q}} \end{array}$$

is commutative by Theorem 18.3 (1) in Fulton [2]. Since $g^*([B]) = [{}_g B] = [B]$, $\tau_{B/S}([B])$ is a G-invariant. Therefore $\tau_{B/S}([B])$ is contained in $(\mathrm{A}_*(B)_{\mathbb{Q}})^G =$

$(\mathrm{A}_d(B)_{\mathbb{Q}})^G = \mathrm{A}_d(B)_{\mathbb{Q}}$. By Theorem 18.3 (5) in Fulton [2], we have $\tau_{B/S}([B]) = [\mathrm{Spec}\, B]$.

Let $j : A \to B$ be the inclusion. Then, by Theorem 18.3 (1) in Fulton [2], j induces the following commutative diagram:

$$\begin{array}{ccc} \mathrm{K}_0(B)_{\mathbb{Q}} & \xrightarrow{\tau_{B/S}} & \mathrm{A}_*(B)_{\mathbb{Q}} \\ {\scriptstyle j^*}\downarrow & & {\scriptstyle j^*}\downarrow \\ \mathrm{K}_0(A)_{\mathbb{Q}} & \xrightarrow{\tau_{A/S}} & \mathrm{A}_*(A)_{\mathbb{Q}} \end{array}$$

By definition, we have $j^*([\mathrm{Spec}\, B]) = \mathrm{rank}_A B \cdot [\mathrm{Spec}\, A] \in \mathrm{A}_d(A)_{\mathbb{Q}}$. By the commutativity of the diagram above, $\tau_{A/S}([B]) = \mathrm{rank}_A B \cdot [\mathrm{Spec}\, A]$ is satisfied.

By Theorem 18.3 (5) in Fulton [2], the condition (1) is satisfied if and only if $\tau_{A/S}([A]) = [\mathrm{Spec}\, A]$ holds. Since $\tau_{A/S}$ is an isomorphism, the condition (1) holds if and only if $[B] = \mathrm{rank}_A B \cdot [A]$ in $\mathrm{K}_0(A)_{\mathbb{Q}}$. **q.e.d.**

It is very difficult to know whether a local ring A has a Noether normalization or not (even if A is essentially of finite type over a field [4]).

The following corollary is an immediate consequence of Theorem 1.1.

Corollary 2.4 *Let A be a Noetherian local normal domain such that A has a Noether normalization S with $Q(A)/Q(S)$ Galois. Then, A is a Roberts ring.*

Corollary 2.4 implies that, if a Noetherian local normal ring A is not a Roberts ring, then A never have a Noether normalization S with $Q(A)/Q(S)$ Galois.

The fact in the next remark was suggested by P. Roberts.

Remark 2.5 Let S be a formal power series ring with d variables over a perfect field k of characteristic $p > 0$. Let L be a finite-dimensional normal extension of $Q(S)$, and A be the integral closure of S in L. *Then, A is a Roberts ring.* Here, we give an outline of a proof. Let M be the intermediate field of the extension $L/Q(S)$ such that $M/Q(S)$ (resp. L/M) is purely inseparable (resp. Galois). Put $S = k[[x_1, \ldots, x_d]]$. Let B be the integral closure of S in M. Then it is easy to see that there exists a positive integer e such that

$$S \subseteq B \subseteq S^{p^{-e}} = k[[x_1^{p^{-e}}, \ldots, x_d^{p^{-e}}]].$$

Let i be the second inclusion as above. The map i induces $i^* : \mathrm{A}_*(S^{p^{-e}})_{\mathbb{Q}} \to \mathrm{A}_*(B)_{\mathbb{Q}}$. By the lying-over theorem, i^* is a surjection. Since $S^{p^{-e}}$ is a regular local ring, $\mathrm{A}_*(S^{p^{-e}})_{\mathbb{Q}}$ is a $\mathbb{Q}$-vector space of dimension 1 as in the proof of Theorem 1.1. Since $\mathrm{A}_d(B)_{\mathbb{Q}} = \mathbb{Q} \cdot [\mathrm{Spec}\, B] \neq 0$, we conclude that $\mathrm{A}_d(B)_{\mathbb{Q}} = \mathbb{Q} \cdot [\mathrm{Spec}\, B] \simeq \mathbb{Q}$ and $\mathrm{A}_t(B)_{\mathbb{Q}} = 0$ if $t < d$. Using the fact, we can prove $\tau_{A/S}([A]) \in \mathrm{A}_d(A)_{\mathbb{Q}}$ in the same way as in the proof of Theorem 1.1.

As we shall see in Example 6.1, there exists a Noetherian local normal ring A that is not a Roberts ring such that A has a subring S satisfying the following three conditions; (1) S is a normal Roberts ring, (2) the inclusion $S \to A$ is finite, and (3) the extension $Q(A)/Q(S)$ is normal.

3 Motivation and Application

In the section, we give three applications of the theory of Roberts rings. The main motivation is in Application 3.2 below.

Application 3.1 In 1985, P. C. Roberts ([14], [16]) proved the vanishing theorem of intersection multiplicities for Roberts rings as follows:

> *Let A be a Roberts ring. Suppose that finitely generated A-modules M and N satisfy following three conditions:*
>
> 1. $\operatorname{pd}_A M < \infty$, $\operatorname{pd}_A N < \infty$.
> 2. $\ell_A(M \otimes_A N) < \infty$.
> 3. $\dim M + \dim N < \dim A$.
>
> *Then,* $\sum_i (-1)^i \ell_A(\operatorname{Tor}_i^A(M, N)) = 0$ *is satisfied.*

It is open whether the above statement is still true without the assumption that A is a Roberts ring.

Application 3.2 Let (A, m) be a d-dimensional Noetherian local ring that is a homomorphic image of a regular local ring, and $\mathbb{F}.$ be a perfect A-complex with support in $\{m\}$. Then the characteristic free Dutta multiplicity ([5], [6], [7], [8], [18]) is defined by

$$\chi_\infty(\mathbb{F}.) = \operatorname{ch}(\mathbb{F}.) \cap [\operatorname{Spec} A]_d,$$

where $[\operatorname{Spec} A]_d$ denotes the element of the Chow group of A in dimension d defined by taking the sum of $\ell_{A_\mathfrak{p}}(A_\mathfrak{p})[\operatorname{Spec} A/\mathfrak{p}]$, where the sum is taken over all prime ideals of A with $\dim A/\mathfrak{p} = d$.

If A is a Roberts ring, then we have

$$\begin{aligned} \chi_\infty(\mathbb{F}.) &= \operatorname{ch}(\mathbb{F}.) \cap [\operatorname{Spec} A]_d \\ &= \operatorname{ch}(\mathbb{F}.) \cap \tau_{A/S}([A]) \\ &= \chi(\mathbb{F}.) \\ &= \sum_{i \geq 0} (-1)^i \ell_A(H_i(\mathbb{F}.)) \end{aligned}$$

by Example 18.3.12 in Fulton [2].

It is conjectured that, if $\mathbb{F}.$ is not exact and of length d, then $\chi_\infty(\mathbb{F}.)$ is positive (Conjecture 3.1 in [8]). We refer the conjecture as the *positivity conjecture of Dutta multiplicities*. In [8], it is proved that the conjecture is equivalent to the following conjecture:

Conjecture 3.3 *Let (A, m) be a d-dimensional complete normal domain and assume that A has a Noether normalization S with $Q(A)/Q(S)$ Galois. Let*

$$\mathbb{F}. \ : \ 0 \to F_d \to \cdots \to F_0 \to 0$$

be a perfect A-complex with $\operatorname{Supp}(\mathbb{F}.) = \{m\}$. *Then* $\sum_i (-1)^i \ell_A(H_i(\mathbb{F}.)) > 0$.

That is to say, by virtue of Corollary 2.4, we have only to discuss the positivity conjecture of Dutta multiplicities in the case where A is a Roberts ring. Furthermore, as we have already seen, if A is a Roberts ring,

$$\chi_\infty(\mathbb{F}.) = \chi(\mathbb{F}.) = \sum_{i \geq 0} (-1)^i \ell_A(H_i(\mathbb{F}.))$$

is satisfied. It is the reason why the author is interested in Roberts rings.

As in [8], the small Cohen-Macaulay modules conjecture implies the positivity of Dutta multiplicities. Furthermore, if the positivity of Dutta multiplicities is true, so is Serre's positivity conjecture of intersection multiplicities in the case where one of two modules is Cohen-Macaulay.

Application 3.4 Szpiro gave the following two conjectures (Conjecture C2 in [22]):

(1) *Let (A, m) be a d-dimensional Noetherian local ring and $\mathbb{F}.$ be a perfect A-complex with support in $\{m\}$. Put $\mathbb{F}.^\vee = \mathrm{Hom}_A(\mathbb{F}., A)$, where $F_i^\vee$ is regarded as the $(-i)$-th part of the complex $\mathbb{F}.^\vee$. Then,*

$$\chi(\mathbb{F}.) = (-1)^d \chi(\mathbb{F}.^\vee)$$

is satisfied.

(2) *Let (A, m) be a Noetherian local ring of characteristic $p > 0$. For a perfect A-complex $\mathbb{F}.$ with support in $\{m\}$, $F^e(\mathbb{F}.)$ denotes the complex defined by matrices whose entries are the p^e-th power of those of the complex $\mathbb{F}.$. Then,*

$$\chi(F^e(\mathbb{F}.)) = p^{de} \chi(\mathbb{F}.)$$

is satisfied for any e.

Nowadays, unfortunately, it is known that both two conjectures as above are not true (see 13.3 in Roberts [18]).

Here, we prove that *both two conjectures are true for Roberts rings.*

First of all, remark that both compleces $\mathbb{F}.^\vee$ and $F^e(\mathbb{F}.)$ as above are perfect A-compleces with support in $\{m\}$.

Let A be a Roberts ring. Assume that A is of finite type over a regular local ring S and $\tau_{A/S}([A]) \in \mathrm{A}_d(A)_{\mathbb{Q}}$ is satisfied. By Example 18.1.2, 18.3.12 and Theorem 18.3 (3) in Fulton [2], we have

$$\begin{aligned} \chi(\mathbb{F}.^\vee) &= \mathrm{ch}(\mathbb{F}.^\vee) \cap \tau_{A/S}([A]) \\ &= \mathrm{ch}_d(\mathbb{F}.^\vee) \cap [\mathrm{Spec}\, A]_d \\ &= (-1)^d \mathrm{ch}_d(\mathbb{F}.) \cap [\mathrm{Spec}\, A]_d \\ &= (-1)^d \mathrm{ch}(\mathbb{F}.) \cap \tau_{A/S}([A]) \\ &= (-1)^d \chi(\mathbb{F}.). \end{aligned}$$

The first statement has been proved.

Here, we start to prove the second one. We may assume that (A, m) is a complete local ring with residue class field algebraically closed. (Our assumption that A is a Roberts ring is preserved by Theorem 1.3 (5).) Let $f : A \to A$ be the Frobenius map. Then, it is easy to see

$$\chi(F^e(\mathbb{F}.)) = \chi_{\mathbb{F}.}([{}_{f^e}A]).$$

Since A is a Roberts ring, we have $[{}_{f^e}A] = p^{de}[A]$ in $\mathrm{K}_0(A)_{\mathbb{Q}}$ as in Remark 2.3. Then, we immediately obtain $\chi(F^e(\mathbb{F}.)) = p^{de}\chi(\mathbb{F}.)$.

4 Does the Riemann-Roch map $\tau_{A/S}$ depend on the choice of S?

Let A be a local ring of finite type over a regular (not necessary local) ring S. Then, the Riemann-Roch map $\tau_{A/S}$ is defined as in Chapter 18, 20 in Fulton [2]. The construction of $\tau_{A/S}$ depends not only on A but also on the regular base ring S. In the section, we shall prove that $\tau_{A/S}$ is independent of the choice of S in many important cases as in Proposition 1.2. It was pointed out (without a proof) by Roberts (p270 in [18]) in the case where A is complete or essentially of finite type over a field. The author does not know any example that the map $\tau_{A/S}$ actually depends on the choice of S.

Now, we start to prove Proposition 1.2.

Proof. Since $\operatorname{Spec} A$ is connected, A is of finite type over a connected component S' of S. Here, note that $\operatorname{Spec} S'$ is an open subscheme of $\operatorname{Spec} S$. Then, we have $\tau_{A/S} = \tau_{A/S'}$ by the definition of the Riemann-Roch map in Fulton [2]. (By the same argument as above, even if the regular base scheme is not affine, we can replace it with $\operatorname{Spec} S$, where S is a regular domain.)

Assume that A is of finite type over a regular domain S. Let S" be a polynomial ring over S with some variables such that there is a surjective S-algebra homomorphism $g : S" \to A$. Since $\Omega_{S"/S}$ is an S"-free module, we have $\tau_{S"/S}([S"]) = [\operatorname{Spec} S"]$ by the definition of the Riemann-Roch map. Let $\mathbb{F}.$ be a finite S"-free resolution of an A-module M. Then, we have

$$\tau_{A/S}([M]) = \mathrm{ch}(\mathbb{F}.) \cap [\operatorname{Spec} S"] = \tau_{A/S"}([M])$$

by Example 18.3.12 in Fulton [2]. Therefore, we may assume that A is a homomorphic image of a regular domain S.

Here, we need to define flat pull-backs of Chow groups for flat morphisms that are not necessary of finite type, and show the compatibility of localized Chern characters with such flat-pull backs as follows. (In [2], all schemes are assumed to be of finite type over a fixed regular scheme.)

Lemma 4.1 *Let $h : S \to T$ be a flat homomorphism of regular (not necessary local) domains. Let I be an ideal of S. Put $A = S/I$ and $B = T/IT$. Let $h' : A \to B$ be the flat ring homomorphism induced by h.*

(a) As in 1.7 in Fulton [2], h' induces the flat pull-back of Chow groups ${h'}_* : \mathrm{A}_*(A) \to \mathrm{A}_*(B)$ defined by

$$h'_*([\operatorname{Spec} A/P]) = [\operatorname{Spec} B/PB] = \sum_{Q \in \operatorname{Min}_B(B/PB)} \ell_{B_Q}(B_Q/PB_Q)[\operatorname{Spec} B/Q],$$

where P is a prime ideal of A, and $\operatorname{Min}_B(B/PB)$ denotes the set of minimal prime ideals of a B-module B/PB.

(b) Let J be an ideal of A. Let $\mathbb{F}.$ be a perfect A-complex with support in $\operatorname{Spec} A/J$. (Then $\mathbb{F}. \otimes_A B$ is a perfect B-complex with support in $\operatorname{Spec} B/JB$.) Let $h" : A/J \to B/JB$ be the flat ring homomorphism induced by h. Then, for any $i \geq 0$, the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{A}_*(A)_{\mathbb{Q}} & \xrightarrow{\mathrm{ch}_i(\mathbb{F}.)} & \mathrm{A}_*(A/J)_{\mathbb{Q}} \\ {h'}_* \downarrow & & h"_* \downarrow \\ \mathrm{A}_*(B)_{\mathbb{Q}} & \xrightarrow{\mathrm{ch}_i(\mathbb{F}.\otimes_A B)} & \mathrm{A}_*(B/JB)_{\mathbb{Q}}. \end{array}$$

(c) We define ${h'}_* : \mathrm{K}_0(A)_{\mathbb{Q}} \to \mathrm{K}_0(B)_{\mathbb{Q}}$ by ${h'}_*([M]) = [M \otimes_A B]$ for each finitely generated A-module M. Then, the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{K}_0(A)_{\mathbb{Q}} & \xrightarrow{\tau_{A/S}} & \mathrm{A}_*(A)_{\mathbb{Q}} \\ {h'}_* \downarrow & & {h'}_* \downarrow \\ \mathrm{K}_0(B)_{\mathbb{Q}} & \xrightarrow{\tau_{B/T}} & \mathrm{A}_*(B)_{\mathbb{Q}}. \end{array}$$

We omit a proof of the above lemma, because proofs in Fulton [2] (Theorem 1.7, Definition 18.1 and the definition of τ) are also valid in the case we are considering.

Now, we go back to the proof of Proposition 1.2. Let A be a homomorphic image of a regular domain S. We denote by P the prime ideal of S that is the inverse image of the maximal ideal of A. Applying Lemma 4.1 (c) to the flat homomorphism $S \to S_P$, we have $\tau_{A/S} = \tau_{A/S_P}$.

Therefore, we know that $\tau_{A/S} = \tau_{A/T}$ is satisfied for some regular local ring T such that A is a homomorphic image of T. Suppose that A is also a homomorphic image of another regular local ring R. We have only to prove $\tau_{A/T} = \tau_{A/R}$.

First, assume that A is a complete local domain. Then, by Lemma 4.1 (c), we have $\tau_{A/T} = \tau_{A/\hat{T}}$ and $\tau_{A/R} = \tau_{A/\hat{R}}$. Therefore, we may assume that both T and R are complete regular local rings. We denote by f_1 (resp. f_2) the surjective ring homomorphism $T \to A$ (resp. $R \to A$). Here, put

$$B = \{(t, r) \in T \times R \mid f_1(t) = f_2(r)\}.$$

Then the diagram

$$\begin{array}{ccc} B & \longrightarrow & R \\ \downarrow & & f_2 \downarrow \\ T & \xrightarrow{f_1} & A \end{array}$$

is commutative. Note that all ring homomorphisms in the diagram are surjective. Furthermore, it is easy to see that B is a complete Noetherian local ring. Therefore, B is a homomorphic image of a regular local ring C, and we have the following commutative diagram:

$$\begin{array}{ccc} C & \longrightarrow & R \\ \downarrow & & f_2 \downarrow \\ T & \xrightarrow{f_1} & A \end{array}$$

Note that all ring homomorphisms in the diagram as above are surjective. Then, we shall obtain $\tau_{A/T} = \tau_{A/C} = \tau_{A/R}$ by the following claim:

Claim 4.2 *Let C and T be regular local rings, and A be a Noetherian local ring. Suppose that there exist surjective ring homomorphisms $C \to T$ and $T \to A$. Then, $\tau_{A/C} = \tau_{A/T}$ is satisfied.*

Proof. Let $\mathbb{F}.$ (resp. $\mathbb{G}.$) be a finite C-free resolution (resp. a finite T-free resolution) of an A-module M. Then, by the definition of the Riemann-Roch map (Chapter 18 in [2]), we have

$$\begin{aligned} \tau_{A/C}([M]) &= \mathrm{ch}(\mathbb{F}.) \cap [\operatorname{Spec} C] \\ \tau_{A/T}([M]) &= \mathrm{ch}(\mathbb{G}.) \cap [\operatorname{Spec} T]. \end{aligned}$$

Since T is a regular local ring, we have $\mathrm{A}_*(T)_{\mathbb{Q}} = \mathbb{Q}\cdot[\operatorname{Spec} T]$. Then, $\tau_{T/C}([T]) = [\operatorname{Spec} T]$ is satisfied by Theorem 18.3 (5) in Fulton [2]. Then, by Example 18.3.12 in [2],

$$\begin{aligned} \tau_{A/C}([M]) &= \tau_{A/C}(\chi_{\mathbb{G}.}([T])) \\ &= \mathrm{ch}(\mathbb{G}.) \cap \tau_{T/C}([T]) \\ &= \mathrm{ch}(\mathbb{G}.) \cap [\operatorname{Spec} T] \end{aligned}$$

is satisfied. **q.e.d.**

Let T and R be regular local rings and suppose that A is a homomorphic image of both of them. Let $h : A \to \hat{A}$ be the natural map, where $\hat{A}$ is the completion of A. Since the diagrams

$$\begin{array}{ccc} \mathrm{K}_0(A)_{\mathbb{Q}} & \xrightarrow{\tau_{A/T}} & \mathrm{A}_*(A)_{\mathbb{Q}} \\ h_* \downarrow & & h_* \downarrow \\ \mathrm{K}_0(\hat{A})_{\mathbb{Q}} & \xrightarrow{\tau_{\hat{A}/\hat{T}}} & \mathrm{A}_*(\hat{A})_{\mathbb{Q}} \end{array} \qquad \begin{array}{ccc} \mathrm{K}_0(A)_{\mathbb{Q}} & \xrightarrow{\tau_{A/R}} & \mathrm{A}_*(A)_{\mathbb{Q}} \\ h_* \downarrow & & h_* \downarrow \\ \mathrm{K}_0(\hat{A})_{\mathbb{Q}} & \xrightarrow{\tau_{\hat{A}/\hat{R}}} & \mathrm{A}_*(\hat{A})_{\mathbb{Q}} \end{array}$$

are commutative, we have

$$h_*\tau_{A/T}([M]) = \tau_{\hat{A}/\hat{T}}h_*([M]) = \tau_{\hat{A}/\hat{R}}h_*([M]) = h_*\tau_{A/R}([M])$$

for any finitely generated A-module M because of $\tau_{\hat{A}/\hat{T}} = \tau_{\hat{A}/\hat{R}}$. Therefore, if $h_* : \mathrm{A}_*(A)_{\mathbb{Q}} \to \mathrm{A}_*(\hat{A})_{\mathbb{Q}}$ is injective, then we have $\tau_{A/T} = \tau_{A/R}$.

We shall obtain $\tau_{A/T} = \tau_{A/R}$ in the case where A is an excellent henselian local ring by the following claim:

Claim 4.3 *Let A be an excellent henselian local ring that is a homomorphic image of a regular local ring. Then, the map $h_* : \mathrm{A}_*(A)_{\mathbb{Q}} \to \mathrm{A}_*(\hat{A})_{\mathbb{Q}}$ is injective.*

Proof. Note that if $h_* : \mathrm{K}_0(A) \to \mathrm{K}_0(\hat{A})$ is injective, then so is $h_* : \mathrm{A}_*(A)_{\mathbb{Q}} \to \mathrm{A}_*(\hat{A})_{\mathbb{Q}}$. We shall show that $h_* : \mathrm{K}_0(A) \to \mathrm{K}_0(\hat{A})$ is injective.

Let I be an ideal of a regular local ring T, and suppose $A = T/I$. Let T' be the I-adic completion of T. Then, T' is also a regular local ring that is IT'-adic complete and $T'/IT' = A$ is satisfied.

Therefore, we may assume that T is I-adic complete. Since $A = T/I$ is excellent henselian, so is T (Theorem 3 in Rotthaus [19]).

Then, applying Popescu-Ogoma's approximation theorem ([13], [12]) to T, we can prove the claim in much the same way as in the proof of Lemma 3.10 in [6]. **q.e.d.**

We have proved Proposition 1.2 in the case where A is excellent henselian.

Before proving Proposition 1.2 in the other case, we prove the following claim that is a little stronger than Claim 4.2.

Claim 4.4 *Let C and T be regular local rings, and A be a Noetherian local ring. Suppose that there exist ring homomorphisms $h : C \to T$ and $f : T \to A$ such that both f and fh are surjections. Then, $\tau_{A/C} = \tau_{A/T}$ is satisfied.*

Proof. Put $g = fh$. We denote by m_C, m_T and m_A the maximal ideals of C, T and A, respectively. Dividing T by a part of a regular system of parameters for T contained in the kernel of f (Claim 4.2), we may assume that the dimension of T is equal to the embedding dimension of A, that is, $\dim T = \dim_{A/m_A} m_A/m_A^2$.

Let I (resp. J) be the kernel of f (resp. g), respectively.

Let T' be the I-adic completion of T. Then, T' is a regular local ring which is flat over T, and $A = T'/IT'$ is satisfied. Then, by Lemma 4.1 (c), we have $\tau_{A/T} = \tau_{A/T'}$. Here, note that T' is IT'-adic complete.

Therefore we may assume that T is I-adic complete.

Since $h(J) \subseteq I$, h is factored as $C \to C' \to T$, where C' is the J-adic completion of C. Since $\tau_{A/C} = \tau_{A/C'}$ is satisfied as in the case of T, we may assume that C is J-adic complete.

Since h is a local homomorphism, the diagram

$$\begin{tikzcd} & C \arrow[dl, "h"'] \arrow[d, "g"] \\ T \arrow[r, "f"] & A \end{tikzcd} \tag{1}$$

induces the commutative diagram of A/m_A-vector spaces as follows:

$$\begin{tikzcd} & m_C/m_C^2 \arrow[dl, "h'"'] \arrow[d, "g'"] \\ m_T/m_T^2 \arrow[r, "f'"] & m_A/m_A^2 \end{tikzcd}$$

Since g' is surjective and f' is an isomorphism, h' is surjective. Therefore, we have

$$m_T = h(m_C) + m_T^2 \subseteq m_C T + m_T^2.$$

Then, $m_T = m_C T$ is satisfied by Nakayama's lemma.

On the other hand, by tensoring rings in Diagram (1) with C/J, we obtain the following diagram:

$$\begin{tikzcd} & C/J \arrow[dl, "\bar{h}"'] \arrow[d, "\bar{g}"] \\ T/JT \arrow[r, "\bar{f}"] & A \end{tikzcd}$$

Remark that $\bar{g}$ is an isomorphism. Put $T" = T/JT$. We have $\mathrm{Ker}(\bar{f}) = I/JT = IT"$. As a C/J-module, we have

$$T" = \bar{h}(C/J) \oplus \mathrm{Ker}(\bar{f}) = A \oplus IT".$$

The maximal ideal of $T"$ coincides with $m_A \oplus IT"$. Since $m_T = m_C T$,

$$m_T/JT = m_C T" = m_A(A \oplus IT") = m_A \oplus m_A IT"$$

is satisfied. Therefore, we obtain $IT" = m_A IT"$. Hence, $IT" = m_{T"} IT"$ is satisfied. Then, by Nakayama's lemma, we have $IT" = 0$. Therefore, $I = JT$ holds and $\bar{f}$ is an isomorphism. Since C is J-adic complete, we know that T is generated by 1 as a C-module, that is, h is a surjection. Then, $\tau_{A/T} = \tau_{A/C}$ follows from Claim 4.2. **q.e.d.**

We go back to the proof of Proposition 1.2.

First, assume that A is essentially of finite type over a field K of characteristic 0. Let S be a local ring of a polynomial ring over K with some variables such that A is a homomorphic image of S. Let T be another regular local ring such that A is also a homomorphic image of T. We shall prove $\tau_{A/S} = \tau_{A/T}$.

Remark that T contains the field of rationals $\mathbb{Q}$. Let I be the kernel of the map $T \to A$. As in the proof of Claim 4.4, we may assume that T is I-adic complete. Consider the following commutative diagram:

$$\begin{array}{ccc} K & \longrightarrow & A \\ \uparrow & & \uparrow \\ \mathbb{Q} & \longrightarrow & T \end{array}$$

Here, the map $K \to A$ in the diagram is the composition of $K \to S \to A$. Since the extension $\mathbb{Q} \to K$ is separable, K is smooth over $\mathbb{Q}$ (see Matsumura [10]). Then, since T is I-adic complete, we obtain a ring homomorphism $K \to T$ which makes the above diagram commutative. Then, we obtain the following commutative diagram:

$$\begin{array}{ccc} S & \to & A \\ & \searrow & \uparrow \\ & & T \end{array}$$

Then, by Claim 4.4, we have $\tau_{A/S} = \tau_{A/T}$.

Next, assume that A is essentially of finite type over a field K of characteristic $p > 0$. (W, pW) denotes the Witt ring of K, that is, W is the complete discrete valuation ring with $W/pW = K$. Let S be a local ring of a polynomial ring over W with some variables such that A is a homomorphic image of S. Let T be another regular local ring such that A is also a homomorphic image of T. We shall prove $\tau_{A/S} = \tau_{A/T}$. Let I be the kernel of the map $T \to A$. We may assume that T is I-adic complete.

Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ with respect to (p). Consider the following commutative diagram:

$$\begin{array}{ccc} W & \longrightarrow & A \\ \uparrow & & \uparrow \\ \mathbb{Z}_{(p)} & \longrightarrow & T \end{array}$$

Here, the map $W \to A$ in the diagram is the composition of $W \to S \to A$. Remark that the morphism $\mathbb{Z}_{(p)} \to W$ is formally smooth in pW-adic topology ([10]) since it is flat and the extension $K/\mathbb{F}_p$ of their residue class fields is separable. Since A is of characteristic p and T is I-adic complete, we obtain a ring homomorphism $W \to T$ which makes the above diagram commutative. Then, we obtain the following commutative diagram:

$$\begin{array}{ccc} S & \to & A \\ & \searrow & \uparrow \\ & & T \end{array}$$

Then, by Claim 4.4, we have $\tau_{A/S} = \tau_{A/T}$.

Next, assume that A is essentially of finite type over $\mathbb{Z}$. Let S be a local ring of a polynomial ring over $\mathbb{Z}$ with some variables such that A is a homomorphic image of S. Let T be another regular local ring such that A is also a homomorphic image of T. Consider the following commutative diagram:

$$\begin{array}{ccc} S & \longrightarrow & A \\ \uparrow & & \uparrow \\ \mathbb{Z} & \longrightarrow & T \end{array}$$

Then, it is easy to see that there exists a ring homomorphism $S \to T$ which makes the diagram commutative. Then, by Claim 4.4, we have $\tau_{A/S} = \tau_{A/T}$.

We have completed the proof of Proposition 1.2.

5 Basic properties on Roberts rings

The aim of the section is to prove Theorem 1.3. Before proving it, we make some comments.

Remark 5.1 Using (2) or (4) in Theorem 1.3, it is easy to see that rings of dimension 2 are not necessary Roberts rings.

The author does not know whether (7) is true without the assumption that A is an excellent henselian local ring.

The converse of (8) is not true. It is easy to find a counterexample using (1) and (3).

If we assume that T is not regular but a Roberts ring, then (11) is not true. See Example 6.1.

Now, we start to prove Theorem 1.3.

Proof of (1). If A is a complete intersection, then we have $\tau_{A/S}(A) \in \mathrm{A}_d(A)_{\mathbb{Q}}$ by Corollary 18.1.2 in Fulton [2]. (It was pointed out by Roberts [16].) There are examples of Gorenstein non-Roberts rings as in Example 6.2.

Proof of (2). Put

$$\tau_{A/S}(A) = \tau_d + \tau_{d-1} + \cdots + \tau_0,$$

where $\tau_i \in \mathrm{A}_i(A)_{\mathbb{Q}}$. Then, by Lemma 3.5 in [5], we have $\tau_{d-1} = \mathrm{cl}(K_A)/2$ in $\mathrm{A}_{d-1}(A)_{\mathbb{Q}} = \mathrm{Cl}(A)_{\mathbb{Q}}$, where $\mathrm{cl}(K_A)$ denotes the isomorphism class containing the canonical module K_A.

Proof of (3). If $\dim A = 0$, then we have $\mathrm{A}_*(A)_{\mathbb{Q}} = \mathrm{A}_0(A)_{\mathbb{Q}}$. Therefore, $\tau_{A/S}([A]) \in \mathrm{A}_0(A)_{\mathbb{Q}}$ is satisfied.

Suppose that the dimension of A is positive. Then it is easy to see that $[\mathrm{Spec}\, A/m] = 0$ in $\mathrm{A}_*(A)_{\mathbb{Q}}$. Hence, $\mathrm{A}_0(A)_{\mathbb{Q}} = 0$. Therefore, if $\dim A = 1$, then $\tau_{A/S}([A]) \in \mathrm{A}_1(A)_{\mathbb{Q}}$ is satisfied.

Proof of (4). Let P be a minimal prime ideal of A. Then, the coefficient of $[\mathrm{Spec}\, A/P]$ in $\tau_{A/S}([A])$ is equal to $\ell_{A_P}(A_P)$ by Theorem 18.3 (5) in [2]. If there exists a minimal prime ideal P such that $\dim A/P < d = \dim A$, then $\tau_{A/S}([A]) \notin \mathrm{A}_d(A)_{\mathbb{Q}}$.

Proof of (5). It immediately follows from Lemma 4.1 (c).

Proof of (6). We may assume that $\operatorname{Spec} T$ is connected. Therefore, assume that T is a finite-dimensional regular domain.

We denote by $\mathrm{Rob}_T(R)$ the set defined in (6). In order to prove (6), we have only to verify two conditions in (22.B) Lemma 2 in Matsumura [10], i.e.,

- $\mathrm{Rob}_T(R)$ is stable under generalization, and
- if $Q \in \mathrm{Rob}_T(R)$, then $\mathrm{Rob}_T(R)$ contains a non-empty open set of the irreducible closed set $V(Q)$.

The first one immediately follows from (5).

Suppose that Q is contained in $\mathrm{Rob}_T(R)$. We shall prove that $V(Q) \cap \mathrm{Rob}_T(R)$ contains a non-empty open subset of $V(Q)$. Since $R_Q = T_Q/IT_Q$ is equi-dimensional by (4), there exists $a \in T \setminus Q$ such that all minimal prime ideals of $IT[a^{-1}]$ have the same height. Replacing T with $T[a^{-1}]$, we may assume that all minimal prime ideals of I have the same height. Put

$$\tau_{R/T}([R]) = \tau_s + \tau_{s-1} + \cdots + \tau_0, \ \ (\tau_i \in \mathrm{A}_i(R)_{\mathbb{Q}}),$$

where $s = \dim R$. Since $\tau_{R_Q/T_Q}([R_Q]) \in \mathrm{A}_{\dim R_Q}(R_Q)_{\mathbb{Q}}$, we have $\tau_i = 0$ in $\mathrm{A}_*(R_Q)_{\mathbb{Q}}$ for $i < s$. Then, it is easy to see that there exists an element $b_i \in T \setminus Q$ such that $\tau_i = 0$ in $\mathrm{A}_*(R \otimes_T T[b_i^{-1}])_{\mathbb{Q}}$ for $i < s$. Put $b = b_0 \cdots b_{s-1}$. Then, we have $\tau_i = 0$ in $\mathrm{A}_*(R \otimes_T T[b^{-1}])_{\mathbb{Q}}$ for $i < s$. By Lemma 4.1 (c), we have $\tau_{R_P/T_P}([R_P]) \in \mathrm{A}_{\dim R_P}(R_P)_{\mathbb{Q}}$ if P is a prime ideal of T such that $Q \subseteq P \not\ni b$.

Proof of (7). It is an immediate consequence of Lemma 4.1 (c) and Claim 4.3.

Proof of (8). We put

$$\mathbb{F}. : 0 \to A \xrightarrow{x} A \to 0.$$

Then, by Example 18.3.12 in Fulton [2], we have $\tau_{A'/S}([A']) = \mathrm{ch}(\mathbb{F}.) \cap \tau_{A/S}([A])$, where $A' = A/xA$. Since A is a Roberts ring, we have $\tau_{A/S}([A]) = [\operatorname{Spec} A]$. By Lemma 1.7.2 and Corollary 18.1.2 in [2], we have

$$\mathrm{ch}(\mathbb{F}.) \cap [\operatorname{Spec} A] = [\operatorname{Spec} A'].$$

Proof of (9). Put $A' = A/I$. Then, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{K}_0(A')_{\mathbb{Q}} & \xrightarrow{\tau_{A'/S}} & \mathrm{A}_*(A')_{\mathbb{Q}} \\ p^* \downarrow & & p^* \downarrow \\ \mathrm{K}_0(A)_{\mathbb{Q}} & \xrightarrow{\tau_{A/S}} & \mathrm{A}_*(A)_{\mathbb{Q}} \end{array}$$

Here, $p : A \to A/I$ is the projection. Since I is an ideal consists of nilpotent elements, the vertical maps in the diagram are isomorphisms (see Example 1.3.1 and 15.1.7 in [2]).

If $\dim A = 0$, then both A and A' are Roberts rings by (3). Assume that $d = \dim A > 0$. Then, we have $[A] = [A']$ in $\mathrm{K}_0(A)_{\mathbb{Q}}$ since $[I] = \ell_A(I) \cdot [A/m] = 0$ in $\mathrm{K}_0(A)_{\mathbb{Q}}$. Therefore, $\tau_{A/S}([A]) \in \mathrm{A}_d(A)_{\mathbb{Q}}$ if and only if $\tau_{A'/S}([A']) \in \mathrm{A}_d(A')_{\mathbb{Q}}$.

Proof of (10). It follows from (8) and (9).

Proof of (11). By the lying-over theorem, $i^* : \mathrm{A}_t(T)_{\mathbb{Q}} \to \mathrm{A}_t(A)$ is a surjection for each t. Since T is a regular local ring, we know $\mathrm{A}_d(T)_{\mathbb{Q}} \simeq \mathbb{Q}$ and $\mathrm{A}_t(T)_{\mathbb{Q}} \simeq 0$ if $t < d$. Therefore, we have $\mathrm{A}_d(A)_{\mathbb{Q}} \simeq \mathbb{Q}$ and $\mathrm{A}_t(A)_{\mathbb{Q}} \simeq 0$ if $t < d$. Hence, we have $\tau_{A/S}([A]) \in \mathrm{A}_d(A)_{\mathbb{Q}}$.

6 Examples

We shall give some examples of Roberts (or non-Roberts) rings in the section.

Example 6.1 We say that a ring A is a Galois (resp. purely inseparable) extension over a ring B if the following conditions are satisfied;

- both A and B are normal domains,
- B is a subring of A such that the inclusion $B \to A$ is finite, and
- the extension $Q(A)/Q(B)$ of their fields of fractions is Galois (resp. purely inseparable).

The ring C_n defined in p294 in Roberts [18] is a Galois extension over $A = k[x, y, z, w]/(xw - yz)$. Let A' be the localization $A_{(x,y,z,w)}$. Then, $C_n \otimes_A A'$ is also a Galois extension over A'. Then, A' is a Roberts ring by Theorem 1.3 (1), but $C_n \otimes_A A'$ is not so since $C_n \otimes_A A'$ does not satisfy Szpiro's conjecture (see Application 3.4) as in p295 in [18].

The ring R defined by Miller-Singh [11] is a purely inseparable extension over a hypersurface A. The ring A is a Roberts ring, but R is not so since R does not satisfy Szpiro's conjecture.

The ring A in Remark 5.4 in [8] is not a Roberts ring since A does not satisfy Szpiro's conjecture. If the characteristic of k is not 2, then A is a Galois extension over a hypersurface B. If the characteristic of k is 2, then A is a purely inseparable extension over a hypersurface B.

We have already seen that an invariant subring of a regular local ring with respect to a finite group is a Roberts ring (Theorem 1.3 (11)). It is not true if we remove the assumption that T is regular as follows: Let A be a complete local non-Roberts domain containing $\mathbb{Q}$. For example, put

$$A = \mathbb{Q}[[x_1, \ldots, x_6]] \Big/ I_2 \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix},$$

where $I_2(\ )$ denotes the ideal generated by all 2 by 2 minors of the given matrix. Remark that A is a 4-dimensional normal non-Roberts ring since the canonical class is not a torsion in the divisor class group (Theorem 1.3 (2)). Let

$B = \mathbb{Q}[[y_1, \ldots, y_4]]$ be a Noether normalization of A. Let L be a finite Galois extension of $Q(B)$ that contains $Q(A)$. Let C be the integral closure of A in L. Then, by Corollary 2.4, C is a Roberts ring. By the construction, C is a Galois extension of a non-Roberts ring A.

By the next example, we know when determinantal rings are Roberts rings.

Example 6.2 By a theorem in [7] as follows, we have a criterion for certain rings to be Roberts rings:

> *Let $R = \oplus_{n \geq 0} R_n = R_0[R_1]$ be a Noetherian graded ring over a field R_0. Assume that $X = Proj(R)$ is smooth over R_0. Then, the local ring $A = R_{(R_1)}$ is a Roberts ring if and only if*
>
> $$\mathrm{td}(\Omega_X^\vee) \equiv 1 \mod c_1(\mathcal{O}_X(1)) \cdot \mathrm{CH}^*(X)_{\mathbb{Q}}.$$
>
> *In particular, if X is an abelian variety, then $A = R_{(R_1)}$ is a Roberts ring.*

Here, $\mathrm{CH}^*(X)$ denotes the Chow ring of the smooth projective variety X, $c_1(\ )$ is the first Chern class, $\Omega_X^\vee$ is the tangent sheaf of X, and $\mathrm{td}(\Omega_X^\vee)$ is its Todd class (see [2]).

Let t, m, n be integers such that $1 \leq t \leq m \leq n$. Let R be the polynomial ring $k[x_{ij} \mid 1 \leq i \leq m;\ 1 \leq j \leq n]$ over a field k divided by the ideal $I_t(x_{ij})$ generated by all t by t minors of the m by n matrix (x_{ij}).

Using the criterion as above, we conclude that, for $t = 2$, $A = R_{(R_1)}$ is a Roberts ring if and only if R is a complete intersection as in Section 3 in [7].

Since any localization of a Roberts ring is a Roberts ring again (Theorem 1.3 (5)), we know that, for any t, $A = R_{(R_1)}$ is a Roberts ring if and only if A is a complete intersection. (S. Goto taught me the reduction.)

Therefore, R is a Roberts ring if and only if $t = 1$ or $m = n = t$.

Hence, if $1 < t < m = n$, then A is a Gorenstein non-Roberts ring.

The ring R defined in Miller-Singh [11] is also a Gorenstein non-Roberts ring.

Example 6.3 The localization of a simplicial semi-group ring at the homogeneous maximal ideal is a Roberts ring as follows.

Let $\mathbb{N}_0$ be the set of non-negative integers. Let d be a positive integer. Let H be a submonoid of the additive monoid $\mathbb{N}_0^d$. Here assume that there exists an integer $t > 0$ such that $t \cdot \mathbb{N}_0^d \subseteq H$. Let k be a field. We define the simplicial semi-group ring $k[H]$ with respect to H to be the subring of $k[t_1, \ldots, t_d]$ generated by

$$\{t_1^{a_1} t_2^{a_2} \cdots t_d^{a_d} \mid (a_1, a_2, \ldots, a_d) \in H\}$$

over k.

Let A be the localization of $k[H]$ at the homogeneous maximal ideal. Put $B = k[t_1, \ldots, t_d]_{(t_1, \ldots, t_d)}$. Then, the inclusion $A \to B$ is finite. Therefore, by Theorem 1.3 (11), A is a Roberts ring.

Let X be the d by d generic symmetric matrix, that is, $X = (x_{ij})$ where $x_{ij} = x_{ji}$. Then, the ring

$$R = k[x_{ij} \mid 1 \leq i \leq j \leq d]/I_2(X)$$

is isomorphic to the second Veronesean subring

$$k[t_i t_j \mid 1 \leq i \leq j \leq d]$$

of the polynomial ring $k[t_1, \ldots, t_d]$. Since it is a simplicial semi-group ring, the localization of R at the homogeneous maximal ideal is a Roberts ring.

Example 6.4 Put $R = k[x_1, \ldots, x_d]_{(x_1, \ldots, x_d)}$, where $x_1, \ldots, x_d$ are variables over a field k. Let I be an ideal of R and put $t = \mathrm{ht}_R\, I$. Assume that R/I is a $(d-t)$-dimensional Cohen-Macaulay ring. Let (T, m_T) be a regular local ring and $f : R \to T$ be a local ring homomorphism. Here, we shall prove that, if R/I is a Roberts ring and $\mathrm{ht}_T\, f(I)T \geq t$, then $T/f(I)T$ is also a Roberts ring. When this is the case, $\mathrm{ht}_T\, f(I)T = t$ is satisfied.

Put $T' = T[x_1, \ldots, x_d]_{(m_T, x_1, \ldots, x_d)}$. We define $g : R \to T'$ by $g(x_i) = x_i$ for each i. Then, we have

$$T = T'/(x_1 - f(x_1), \ldots, x_d - f(x_d))T'.$$

By Theorem 1.3 (5), we know that $T'/g(I)T'$ is a Roberts ring since g is flat. Note that, $T'/g(I)T'$ is Cohen-Macaulay and $\mathrm{ht}_{T'}\, g(I)T' = t$. It is easy to see that

$$T'/\{g(I)T' + (x_1 - f(x_1), \ldots, x_d - f(x_d))T'\} = T/f(I)T.$$

Then, calculating dimensions of both rings, we know that $x_1 - f(x_1)$, $\ldots$, $x_d - f(x_d)$ is a $T'/g(I)T'$-regular sequence and $\mathrm{ht}_T\, f(I)T = t$. Then, by Theorem 1.3 (8), we know that $T/f(I)T$ is a Roberts ring. We have completed the proof.

Let (T, m_T) be a regular local ring containing a field k and $\{a_{ij} \mid 1 \leq i \leq j \leq d\}$ be a subset of m_T. Putting $a_{ji} = a_{ij}$ for $i < j$, (a_{ij}) is a d by d symmetric matrix. We have already seen in Example 6.3 that

$$(k[x_{ij} \mid 1 \leq i \leq j \leq d]/I_2(x_{ij}))_{(x_{ij} \mid 1 \leq i \leq j \leq d)}$$

is a Roberts ring with $\mathrm{ht}\, I_2(x_{ij}) = d(d-1)/2$, where we put $x_{ji} = x_{ij}$ for $i < j$, and (x_{ij}) is the generic d by d symmetric matrix. Then, if $\mathrm{ht}_T\, I_2(a_{ij}) \geq d(d-1)/2$, then we know that $\mathrm{ht}\, I_2(a_{ij}) = d(d-1)/2$ and $T/I_2(a_{ij})$ is a Roberts ring.

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