Test modules to calculate Dutta multiplicities

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Dedicated to Professor David A. Buchsbaum for his 70th birthday

Abstract

Using Frobenius maps, the Dutta multiplicity $\chi_\infty(\mathbb{F}.)$ was first defined by Dutta for a complex $\mathbb{F}.$ over a local ring of positive characteristic. Nowadays, using localized Chern characters or Adams operations, it is defined for a complex not only over a local ring of positive characteristic but also over a ring that is a homomorphic image of an arbitrary regular local ring. In the paper, we shall give another characterization to Dutta multiplicity using some Galois extension.

Furthermore, we shall define the notion of test modules. If a test module for a local ring exists, then the positivity conjecture of Dutta multiplicities is true over the ring. In the paper, it is proved that, if the small Cohen-Macaulay modules conjecture is true, then test modules always exist for complete local domains whose field of fractions are of characteristic 0.

Key words: Dutta multiplicity, singular Riemann-Roch theory, test modules,

1 Introduction

In 1983, using Frobenius maps, S. P. Dutta [5] defined the limit multiplicity $\chi_\infty(\mathbb{F}.)$ (that we refer as the *Dutta multiplicity*) for a bounded finite free complex $\mathbb{F}.$ with homology of finite length over a local ring A of characteristic

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positive (Remark 2.2), and showed that it was a rational number. In 1989, P. C. Roberts [20] proved that, for a local ring A of positive characteristic, if the complex $\mathbb{F}.$ is not exact and its length equals the dimension of A, then its Dutta multiplicity is positive. The fact played an essential role in his proof of the new intersection theorem in the mixed-characteristic case.

On the other hand, using localized Chern characters (Fulton [8], Roberts [21]) or Adams operations (Gillet-Soulé [10], Soulé [23]), the notion of Dutta multiplicity was extended as an invariant for a bounded finite free complex with homology of finite length over a local ring that is a homomorphic image of a regular local ring ([12], [15] and [21]).

It is conjectured that the Dutta multiplicity $\chi_\infty(\mathbb{F}.)$ is positive if the complex $\mathbb{F}.$ is not exact and its length equals the dimension of the ring A without the assumption that A is of positive characteristic. We refer it as the positivity conjecture of Dutta multiplicities. In [15], it was solved affirmatively in the case where A contains a field, but it is still open in the mixed-characteristic case. The conjecture is deeply related to Serre's positivity conjecture as we shall see in Remark 3.4. If the conjecture is true, so is Serre's positivity conjecture of intersection multiplicities in the case where one of two modules is Cohen-Macaulay (Proposition 3.3 in [12] or Corollary 5.1 in [15]).

Even if a ring A is a Gorenstein ring, it is not known whether the positivity conjecture of Dutta multiplicities is true for complexes over A. However, the conjecture is true if A has a *test module*, that is, a finitely generated maximal Cohen-Macaulay module M such that the Todd class $\tau_{A/S}([M])$ is contained in the Chow group $\mathrm{A}_d(A)_\mathbb{Q}$ of A in degree $d = \dim A$.

In the paper, we study the theory of test modules.

In Section 2, we define the Dutta multiplicity for a complex and give some characterizations. The main aim in Section 2 is to prove the following:

Theorem 1.1 *Let (A, m) be a Nagata local domain and put $\dim A = d$. Suppose that A has a Noether normalization S (i.e., S is a regular local subring of A such that A is a finite S-module) such that $Q(A)$ is a separable extension of $Q(S)$, where $Q(\ )$ denotes the field of fractions.*

Let L be a finite Galois extension of $Q(S)$ containing $Q(A)$. Let B be the integral closure of A in L. (B is a finite A-module since A is a Nagata ring, but B may not be a local ring.)

Then, for a bounded complex $\mathbb{F}.$ of finitely generated free A-modules with homology of finite length, we have

$$\chi_\infty(\mathbb{F}.) = \frac{\sum_i (-1)^i \ell_A(H_i(\mathbb{F}. \otimes_A B))}{[L : Q(A)]},$$

where $[L : Q(A)]$ denotes the degree of the field extension $L/Q(A)$.

In Section 3, we state the positivity conjecture of Dutta multiplicities. By virtue of Corollary 2.6, we know that the conjecture is reduced to the case where the given ring A is a Galois extension of a regular local ring. Furthermore, when this is the case, the Dutta multiplicity coincides with the alternating sum of lengths of homology modules. In Remark 3.4 we shall see the relation between the positivity conjecture of Dutta multiplicities and Serre's conjecture in the following theorem that is proved in Section 3:

Theorem 1.2 *Let (T, n) be a regular local ring and (A, m) be a homomorphic image of T. Put $d = \dim A > 0$. Let*

$$\mathbb{F}. \ : \ 0 \to F_s \to \cdots \to F_0 \to 0$$

be a perfect A-complex with $\mathrm{Supp}(\mathbb{F}.) = \{m\}$. *Assume that there exists a T-free complex $\mathbb{G}.$ such that $\mathbb{F}. = \mathbb{G}. \otimes_T A$.*

(1) $\chi_\infty(\mathbb{F}.) = \sum_i (-1)^i \ell_A(H_i(\mathbb{F}.))$ is satisfied.

(2) Assume $s = d$. Then, we have the following:

- *(i) $H_i(\mathbb{G}.) = 0$ is satisfied for $i > 0$.*
- *(ii) Put $A = T/I$ and* $\mathrm{Supp}(H_0(\mathbb{G}.)) = \mathrm{Spec}\, T/J$. *Then, $I + J$ is an n-primary ideal.*
- *(iii) $\chi_\infty(\mathbb{F}.) = \sum_i (-1)^i \ell_T(\mathrm{Tor}_i^T(T/I, H_0(\mathbb{G}.)))$.*
- *(iv) We have $\dim T/I + \dim T/J = \dim T$.*
- *(v) We have $\chi_\infty(\mathbb{F}.) \geq 0$ always, and even $\chi_\infty(\mathbb{F}.) > 0$ if T contains a field or is unramified of mixed characteristic.*

Here, a mixed characteristic regular local ring is said to be *unramified* (resp. *ramified*) if the characteristic of the residue class field is not contained in (resp. is contained in) the square of the maximal ideal.

In Section 4, we shall define the notion of test modules. If a test module for a local ring A exists, then the positivity conjecture of Dutta multiplicities is true over A. By the next theorem, if the small Cohen-Macaulay modules conjecture (Conjecture 4.5) is true, so is the positivity of Dutta multiplicities.

Theorem 1.3 *Let A be a d-dimensional complete local domain such that there exists a Noether normalization S of A with $Q(A)/Q(S)$ separable.*

Let L be a finite Galois extension of $Q(S)$ containing $Q(A)$ and B be the integral closure of A in L. (B is a complete local normal domain.)

If B has a maximal Cohen-Macaulay module, then A has a test module.

We prove the theorem in Section 4.

In Section 5, we give some examples.

We refer the reader to Fulton [8], Matsumura [16] and Roberts [21] for unexplained terminologies.

For a Noetherian ring R, $\mathrm{K}_0(R)$ denotes the Grothendieck group of finitely generated R-modules. For an R-module M, $[M]$ denotes the element in $\mathrm{K}_0(R)$ corresponding to M. If a ring homomorphism $f : R \to S$ is finite, we have the induced homomorphism $f^* : \mathrm{K}_0(S) \to \mathrm{K}_0(R)$ of additive groups defined by $f^*([M]) = [{}_f M]$ for an S-module M, where ${}_f M = M$ is an R-module whose R-module structure is given through f.

For a Noetherian ring R, $\mathrm{A}_*(R)$ is the Chow group of the scheme $\operatorname{Spec} R$. For a prime ideal $\mathfrak{p}$ of R of $\dim R/\mathfrak{p} = i$, $[\operatorname{Spec} R/\mathfrak{p}]$ denotes the cycle in $\mathrm{A}_i(R)$ corresponding to the closed subscheme $\operatorname{Spec} R/\mathfrak{p}$. If a ring homomorphism $f : R \to S$ is finite, we have the induced homomorphism $f^* : \mathrm{A}_*(S) \to \mathrm{A}_*(R)$ of additive groups defined as in 1.4 in Fulton [8]. (f^* is the push-forward of cycles for the proper morphism $\operatorname{Spec} S \to \operatorname{Spec} R$.)

For an additive group N, $N_{\mathbb{Q}}$ denotes $N \otimes_{\mathbb{Z}} \mathbb{Q}$.

2 Definition and some characterizations of Dutta multiplicity

In this section we give some characterizations to Dutta multiplicity. We first recall the original definition of Dutta multiplicity in the positive characteristic case due to Dutta [5].

Let A be a local ring, and let $\mathbb{F}.$ be a bounded complex of free A-modules. If $\mathbb{F}.$ has homology of finite length, we define the Euler characteristic $\chi_{\mathbb{F}.} : \mathrm{K}_0(A) \to \mathbb{Z}$ (or $\chi_{\mathbb{F}.} : \mathrm{K}_0(A)_{\mathbb{Q}} \to \mathbb{Q}$) to be

$$\chi_{\mathbb{F}.}([M]) = \sum_t (-1)^t \ell_A(H_t(\mathbb{F}. \otimes_A M)),$$

where $H_t(\mathbb{F}. \otimes_A M)$ is the t-th homology module of the complex $\mathbb{F}. \otimes_A M$ and $\ell_A(H_t(\mathbb{F}. \otimes_A M))$ denotes its length as an A-module. We sometimes denote $\chi_{\mathbb{F}.}([A])$ simply by $\chi(\mathbb{F}.)$.

Definition 2.1 Let A be a complete local ring of positive characteristic p with perfect coefficient field, and let $\mathbb{F}.$ be a bounded complex of finitely generated free A-modules with homology of finite length. Let d be the dimension of A. We denote by $f : A \to A$ the Frobenius map. Then the *Dutta multiplicity* of the complex $\mathbb{F}.$ is defined to be

$$\chi_\infty(\mathbb{F}.) = \lim_{e \to \infty} \frac{\chi_{\mathbb{F}.}([{}_{f^e} A])}{p^{de}}.$$

Dutta proved in [5] that the Dutta multiplicity as above is a rational number.

Remark 2.2 With notation as in Definition 2.1, let $F^e(\mathbb{F}.)$ be a bounded A-free complex such that entries of the boundaries of $F^e(\mathbb{F}.)$ are the p^e-th power of those of $\mathbb{F}.$. Then, $\chi_{\mathbb{F}.}([{}_{f^e}A]) = \chi(F^e(\mathbb{F}.))$ is satisfied. Therefore,

$$\chi_\infty(\mathbb{F}.) = \lim_{e\to\infty} \frac{\chi(F^e(\mathbb{F}.))}{p^{de}}$$

is satisfied.

Here, assume that $H_i(\mathbb{F}.) = 0$ for $i \neq 0$. By the equation as above, we know that $\chi_\infty(\mathbb{F}.)$ coincides with $\chi_\infty(H_0(\mathbb{F}.), A)$ in the sense of Dutta [5].

Let M and N be finitely generated A-modules such that the projective dimension of M is finite and $\dim M + \dim N \leq \dim A$. Assume that $\ell_A(M \otimes_A N)$ is finite. Put $r = \dim A - \dim M$. Let $\mathbb{G}.$ be the minimal A-free resolution of M. Then, we have

$$\chi_\infty(M,N) = \begin{cases} 0 & \text{if } r > \dim N \\ \displaystyle\sum_{\dim A/P = r} \ell_{A_P}(N_P)\chi_\infty(\mathbb{G}. \otimes_A A/P) & \text{if } r = \dim N \end{cases}$$

where the above sum is taken over all minimal prime ideals of N that satisfy $\dim A/P = r$.

Let X be a scheme of finite type over a regular scheme. A bounded complex of locally free $\mathcal{O}_X$-modules of finite rank is called a *perfect* $\mathcal{O}_X$-complex. For a perfect $\mathcal{O}_X$-complex $\mathbb{F}.$, we define the *support* of $\mathbb{F}.$ by

$$\mathrm{Supp}(\mathbb{F}.) = \sum_t \mathrm{Supp}(H_t(\mathbb{F}.)).$$

The support of $\mathbb{F}.$ is a closed set of X consisting of those points at which $\mathbb{F}.$ is not exact. Let $\mathbb{F}.$ be a perfect $\mathcal{O}_X$-complex with support in Y. We denote the *localized Chern character* with respect to the perfect complex $\mathbb{F}.$ by

$$\mathrm{ch}_Y^X(\mathbb{F}.) = \sum_{i\geq 0} \mathrm{ch}_i(\mathbb{F}.)$$

If no confusion is possible, we denote it simply by $\mathrm{ch}(\mathbb{F}.)$. We refer the reader to Fulton [8] or Roberts [21] for the definition and basic properties of localized Chern characters. We recall that localized Chern characters are defined as operators on the Chow group, and that if η is a cycle of dimension j in $\mathrm{A}_j(X)_\mathbb{Q}$, then $\mathrm{ch}_i(\mathbb{F}.) \cap \eta$ is an element of $\mathrm{A}_{j-i}(Y)_\mathbb{Q}$.

Let $\mathbb{F}.$ be a perfect $\mathcal{O}_X$-complex with support in Y. We define $\chi_{\mathbb{F}.} : \mathrm{K}_0(X) \to \mathrm{K}_0(Y)$ by $\chi_{\mathbb{F}.}([\mathcal{F}]) = \sum_i (-1)^i [H_i(\mathbb{F}. \otimes_{\mathcal{O}_X} \mathcal{F})]$ for a coherent $\mathcal{O}_X$-module $\mathcal{F}$ as in Example 18.3.12 in Fulton [8].

Using localized Chern characters, we can define the notion of Dutta multiplicities not only over a ring of positive characteristic but also over a homomorphic image of an arbitrary regular local ring (Remark 2.4) as follows:

Definition 2.3 Let (A, m) be a homomorphic image of a regular local ring. Put $d = \dim A$. Let $\mathbb{F}.$ be a perfect A-complex with support in $\{m\}$. Then, we define the *Dutta multiplicity* of a perfect A-complex $\mathbb{F}.$ by letting

$$\chi_\infty(\mathbb{F}.) = \mathrm{ch}(\mathbb{F}.) \cap [\mathrm{Spec}\, A]_d.$$

Here, $[\mathrm{Spec}\, A]_d$ denotes the element of the Chow group of A in dimension d defined by taking the sum of $\ell_{A_\mathfrak{p}}(A_\mathfrak{p})[\mathrm{Spec}\, A/\mathfrak{p}]$, where the sum is taken over all prime ideals of A with $\dim A/\mathfrak{p} = d$. By definition, $\chi_\infty(\mathbb{F}.) \in \mathrm{A}_0(A/m)_\mathbb{Q} = \mathbb{Q} \cdot [\mathrm{Spec}\, A/m]$. Hence, we may regard the Dutta multiplicity $\chi_\infty(\mathbb{F}.)$ as a rational number.

We refer the reader to [12] for basic properties on the Dutta multiplicity defined by the localized Chern character.

Remark 2.4 Let (A, m) be a complete equal-characteristic Noetherian local ring such that the coefficient field is perfect of positive characteristic. Then, by Theorem 18.3 (5) and Example 18.3.12 in [8], the Dutta multiplicity $\chi_\infty(\mathbb{F}.)$ as in Definition 2.3 coincides with the original one defined in Definition 2.1 (Szpiro [24], Roberts [20], [21]).

We also make one comment on the relation of Dutta multiplicity to the usual Euler characteristic. Let A be a d-dimensional local ring that is of finite type over a regular local ring S.

By the local Riemann-Roch theorem (Example 18.3.12 in Fulton [8]), there is an element of $\mathrm{A}_*(A)_\mathbb{Q}$ denoted

$$\tau_{A/S}([A]) = \tau_d + \tau_{d-1} + \cdots + \tau_0,$$

where $\tau_i \in \mathrm{A}_i(A)_\mathbb{Q}$ for each i, $\tau_d = [\mathrm{Spec}\, A]_d$ (Theorem 18.3 (5) in [8]), and

$$\chi(\mathbb{F}.) = \sum_{i=0}^{d} \mathrm{ch}_i(\mathbb{F}.) \cap \tau_i.$$

We refer to $\tau_{A/S}$ as the *Riemann-Roch map.* (It is assumed that all schemes are of finite type over a fixed regular scheme in the singular Riemann-Roch theory as in Chapter 20 of Fulton [8]. In fact, the construction of the isomorphism $\tau_{A/S} : \mathrm{K}_0(A)_\mathbb{Q} \to \mathrm{A}_*(A)_\mathbb{Q}$ of $\mathbb{Q}$-vector spaces depends not only on A but also on the fixed regular ring S. However, $\tau_{A/S}$ is independent of the choice of S in many important cases as we shall see in Section 4 in [14].) Comparing this expression for the Euler characteristic to Dutta multiplicity, we see that the Dutta multiplicity of $\mathbb{F}.$ is equal to the term in this sum for which $i = d$. If $\tau_i = 0$ for $i < d$ (see [14] for the situation which sometimes occurs), the Euler characteristic and the Dutta multiplicity of $\mathbb{F}.$ are equal.

Remark 2.5 Recently, using Adams operations ψ^k on the Grothendieck group of complexes (Gillet-Soulé [10] or Soulé [23]), a new method to describe Dutta multiplicities has been found (see Lemma 4.4 in [15]), that is,

Let (A, m) be a homomorphic image of a regular local ring and put $\dim A = d$. *Let $\mathbb{F}.$ be a perfect A-complex with support in $\{m\}$.*

1. *For any $k \geq 2$, we have*
$$\chi_\infty(\mathbb{F}.) = \lim_{e\to\infty} \frac{\chi(\psi^e(\mathbb{F}.))}{e^d} = \lim_{e\to\infty} \frac{\chi((\psi^k)^e(\mathbb{F}.))}{k^{de}}.$$
2. *There exist rational numbers $a_1, \ldots, a_{d+1}$ (depending only on d) such that*
$$\chi_\infty(\mathbb{F}.) = \sum_{e=1}^{d+1} a_e \cdot \chi(\psi^e(\mathbb{F}.)).$$

The main aim in the section is to give a new characterization to Dutta multiplicities as in Theorem 1.1.

Now, we start to prove Theorem 1.1.

Proof. Let G be the Galois group of the field extension $L/Q(S)$. For each $g \in G$, the S-automorphism $g : B \to B$ induces $g^* : \mathrm{K}_0(B) \to \mathrm{K}_0(B)$ and $g^* : \mathrm{A}_*(B) \to \mathrm{A}_*(B)$. Therefore, the Galois group G acts on $\mathrm{A}_*(B)_{\mathbb{Q}}$. By Example 1.7.6 in Fulton [8], the composite map of
$$(\mathrm{A}_t(B)_{\mathbb{Q}})^G \hookrightarrow \mathrm{A}_t(B)_{\mathbb{Q}} \xrightarrow{i^*} \mathrm{A}_t(S)_{\mathbb{Q}}$$
is an isomorphism for each t, where $i : S \to B$ is the inclusion. Since S is a regular local ring, it is easy to see that $\mathrm{K}_0(S) \simeq \mathbb{Z}$. Since $\tau_{S/S} : \mathrm{K}_0(S)_{\mathbb{Q}} \to \mathrm{A}_*(S)_{\mathbb{Q}}$ is an isomorphism, we have $\mathrm{A}_d(S)_{\mathbb{Q}} \simeq \mathbb{Q}$ and $\mathrm{A}_t(S)_{\mathbb{Q}} \simeq 0$ if $t < d$. Therefore we have $(\mathrm{A}_d(B)_{\mathbb{Q}})^G \simeq \mathbb{Q}$ and $(\mathrm{A}_t(B)_{\mathbb{Q}})^G \simeq 0$ if $t < d$.

On the other hand, since $g : B \to B$ is an S-automorphism, the diagram
$$\begin{array}{ccc} \mathrm{K}_0(B)_{\mathbb{Q}} & \xrightarrow{\tau_{B/S}} & \mathrm{A}_*(B)_{\mathbb{Q}} \\ g^* \downarrow & & g^* \downarrow \\ \mathrm{K}_0(B)_{\mathbb{Q}} & \xrightarrow{\tau_{B/S}} & \mathrm{A}_*(B)_{\mathbb{Q}} \end{array}$$
is commutative. Since $g^*([B]) = [{}_gB] = [B]$, $\tau_{B/S}([B])$ is a G-invariant. Therefore $\tau_{B/S}([B])$ is contained in $(\mathrm{A}_*(B)_{\mathbb{Q}})^G = (\mathrm{A}_d(B)_{\mathbb{Q}})^G = \mathrm{A}_d(B)_{\mathbb{Q}}$. By Theorem 18.3 (5) in Fulton [8], we have $\tau_{B/S}([B]) = [\mathrm{Spec}\, B]$.

Let $j : A \to B$ and $\bar{j} : A/m \to B/mB$ be the inclusions. Since the localized Chern character $\mathrm{ch}(\mathbb{F}.)$ is a bivariant class (Definition 17.1 and Theorem 18.1 in Fulton [8]), the following diagram is commutative:
$$\begin{array}{ccc} \mathrm{A}_d(B)_{\mathbb{Q}} & \xrightarrow{\mathrm{ch}(\mathbb{F}.\otimes_A B)} & \mathrm{A}_0(B/mB)_{\mathbb{Q}} \\ j^* \downarrow & & \bar{j}^* \downarrow \\ \mathrm{A}_d(A)_{\mathbb{Q}} & \xrightarrow{\mathrm{ch}(\mathbb{F}.)} & \mathrm{A}_0(A/m)_{\mathbb{Q}} \end{array}$$

By definition of j^*, we have $j^*([\operatorname{Spec} B]) = [L : Q(A)] \cdot [\operatorname{Spec} A]$. Therefore, we have

$$\begin{aligned}[L : Q(A)] \cdot \chi_\infty(\mathbb{F}.) &= \bar{j}^* \left(\operatorname{ch}(\mathbb{F}. \otimes_A B) \cap [\operatorname{Spec} B]\right) \\ &= \bar{j}^* \left(\operatorname{ch}(\mathbb{F}. \otimes_A B) \cap \tau_{B/S}([B])\right).\end{aligned}$$

By Example 18.3.12 in Fulton [8], we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{K}_0(B)_{\mathbb{Q}} & \xrightarrow{\tau_{B/S}} & \mathrm{A}_*(B)_{\mathbb{Q}} \\ \chi_{\mathbb{F}.\otimes B} \downarrow & & \operatorname{ch}(\mathbb{F}.\otimes_A B) \downarrow \\ \mathrm{K}_0(B/mB)_{\mathbb{Q}} & \xrightarrow{\tau_{(B/mB)/S}} & \mathrm{A}_*(B/mB)_{\mathbb{Q}} \end{array}$$

Since the diagram

$$\begin{array}{ccc} \mathrm{K}_0(B/mB)_{\mathbb{Q}} & \xrightarrow{\tau_{(B/mB)/S}} & \mathrm{A}_*(B/mB)_{\mathbb{Q}} \\ \bar{j}^* \downarrow & & \bar{j}^* \downarrow \\ \mathrm{K}_0(A/m)_{\mathbb{Q}} & \xrightarrow{\tau_{(A/m)/S}} & \mathrm{A}_*(A/m)_{\mathbb{Q}} \end{array}$$

is commutative and $\tau_{(A/m)/S}([A/m]) = [\operatorname{Spec} A/m]$ by Theorem 18.3 (5) in Fulton [8], we have

$$\begin{aligned} & \bar{j}^* \left(\operatorname{ch}(\mathbb{F}. \otimes_A B) \cap \tau_{B/S}([B])\right) \\ = \; & \bar{j}^* \tau_{(B/mB)/S} \left(\sum_i (-1)^i [H_i(\mathbb{F}. \otimes_A B)]\right) \\ = \; & \tau_{(A/m)/S} \bar{j}^* \left(\sum_i (-1)^i [H_i(\mathbb{F}. \otimes_A B)]\right) \\ = \; & \tau_{(A/m)/S} \left(\sum_i (-1)^i [H_i(\mathbb{F}. \otimes_A B)]\right) \\ = \; & \sum_i (-1)^i \ell_A(H_i(\mathbb{F}. \otimes_A B)). \end{aligned}$$

q.e.d.

Next, we give an immediate corollary of Theorem 1.1

Corollary 2.6 *Let (A, m) be a normal local domain and put $\dim A = d$. Suppose that A has a Noether normalization S such that $Q(A)$ is a finite Galois extension of $Q(S)$.*

Then, for a perfect A-complex $\mathbb{F}.$ with support in $\{m\}$, we have

$$\chi_\infty(\mathbb{F}.) = \sum_i (-1)^i \ell_A(H_i(\mathbb{F}.)).$$

When we prove the above corollary, we have only to put $A = B$ in Theorem 1.1.

Remark 2.7 There exists a perfect A-complex $\mathbb{F}.$ with support in $\{m\}$ such that

$$\chi_\infty(\mathbb{F}.) \neq \sum_i (-1)^i \ell_A(H_i(\mathbb{F}.)).$$

In fact, using an example of a negative intersection multiplicity due to Dutta-Hochster-MacLaughlin [6], there exists a 3-dimensional normal Cohen-Macaulay ring (A, m) and a perfect A-complex $\mathbb{F}.$ with support $\{m\}$ such that its Dutta multiplicity does not coincides with the alternating sum of lengths of homologies (p295 in Roberts [21]). Miller-Singh [17] constructed 5-dimensional normal Gorenstein local ring (A, m) and a perfect A-complex $\mathbb{F}.$ with support $\{m\}$ having the same property as above.

3 Positivity of Dutta multiplicity

In the section, we discuss the following positivity conjecture of Dutta multiplicities.

Conjecture 3.1 Let (A, m) be a homomorphic image of a regular local ring. Put $d = \dim A$. Let

$$\mathbb{F}. \ : \ 0 \to F_d \to \cdots \to F_0 \to 0$$

be a perfect A-complex with $\mathrm{Supp}(\mathbb{F}.) = \{m\}$. (In particular, the support is not empty, and $\mathbb{F}.$ is not exact.) Then $\chi_\infty(\mathbb{F}.) > 0$.

The above conjecture was affirmatively solved by Roberts [20] if (A, m) is a complete equal-characteristic Noetherian local ring such that the residue class field is perfect of positive characteristic. Recently, it has been solved in [15] if A contains a field.

Conjecture 3.1 is open even if A is a Gorenstein ring. As in Remark 3.6 and 3.7 in [12], Conjecture 3.1 is true if A is a Cohen-Macaulay ring of dimension less than or equal to 3.

Assume that A is of mixed characteristic. By the argument at the beginning of Section 4 in [15], we can reduce Conjecture 3.1 to the case where A is a complete local ring with A/m algebraically closed. Furthermore, by Remark 2.4 in [12], we may assume that A is an integral domain. Therefore, we may assume that A is a complete local domain of mixed characteristic with residue class field algebraically closed without loss of generality. Then, A has a Noether normalization S with $Q(A)/Q(S)$ separable. We may assume that the residue class field of S coincides with that of A. Here, take a finite Galois extension L over $Q(S)$ containing $Q(A)$. Let B be the integral closure of A in L. (Remark that B is a complete local normal domain.) Then, for a perfect A-complex $\mathbb{F}.$ with support in $\{m\}$, we have

$$\begin{aligned}\chi_\infty(\mathbb{F}.) &= \frac{\sum_i (-1)^i \ell_A(H_i(\mathbb{F}. \otimes_A B))}{[L : Q(A)]} \\ &= \frac{\sum_i (-1)^i \ell_B(H_i(\mathbb{F}. \otimes_A B))}{[L : Q(A)]} \\ &= \frac{\chi(\mathbb{F}. \otimes_A B)}{[L : Q(A)]}\end{aligned}$$

by Theorem 1.1 and Corollary 2.6. Therefore, we have only to show $\chi(\mathbb{F}. \otimes_A B) > 0$.

Hence, Conjecture 3.1 is equivalent to the following conjecture:

Conjecture 3.2 Let (A, m) be a d-dimensional complete local normal domain and assume that A has a Noether normalization S with $Q(A)/Q(S)$ Galois. Let

$$\mathbb{F}. \ : \ 0 \to F_d \to \cdots \to F_0 \to 0$$

be a perfect A-complex with $\mathrm{Supp}(\mathbb{F}.) = \{m\}$. Then $\chi(\mathbb{F}.) = \sum_i (-1)^i \ell_A(H_i(\mathbb{F}.)) > 0$.

It is easy to see that both conjectures as above are true if $d \leq 2$.

Conjecture 3.2 is true if A contains a field, because so is Conjecture 3.1 [15] in the case.

Conjecture 3.1 is deeply related to the following Serre's positivity conjecture [22]:

Conjecture 3.3 Let T be a regular local ring and M, N be finitely generated T-modules such that $0 < \ell_T(M \otimes_T N) < \infty$ and $\dim M + \dim N = \dim T$. Then, $\sum_i (-1)^i \ell_T(\mathrm{Tor}_i^T(M, N)) > 0$.

If T contains a field or is unramified of mixed characteristic, Conjecture 3.3 is true by Serre's theorem [22]. If T is ramified of mixed characteristic, Gabber [2] proved $\sum_i (-1)^i \ell_T(\mathrm{Tor}_i^T(M, N)) \geq 0$ under the same condition as in Conjecture 3.3.

Remark 3.4 If Conjecture 3.1 is true, then so is Conjecture 3.3 in the case where either M or N is a Cohen-Macaulay module (e.g., Proposition 3.3 in [12] or Corollary 5.1 in [15]). Here, we say that M is a Cohen-Macaulay module if $\dim M = \mathrm{depth}\, M$. We shall give an outline of a proof. Assume that M is a Cohen-Macaulay module and put $d = \dim N$. Let $\mathbb{G}.$ be the minimal T-free resolution of M. Let P be a minimal prime ideal of N such that $d = \dim T/P$. Put $A = T/P$. Then, the length of the complex $\mathbb{F}. = \mathbb{G}. \otimes_T A$ is equal to d.

Therefore, if Conjecture 3.1 is true, $\chi_\infty(\mathbb{F}.) > 0$ is satisfied. On the other hand, by Theorem 1.2 (2) (iii), we have

$$\sum_i (-1)^i \ell_T(\mathrm{Tor}_i^T(M, A)) = \chi_\infty(\mathbb{F}.) > 0.$$

By the vanishing theorem of intersection multiplicities (Roberts [18], Gillet-Soulé [9]), we have

$$\sum_i (-1)^i \ell_T(\mathrm{Tor}_i^T(M, N)) > 0$$

immediately.

Conversely, if Conjecture 3.3 is true, then so is Conjecture 3.1 if the complex $\mathbb{F}.$ is liftable to a regular local ring as in Theorem 1.2.

Proof of Theorem 1.2. Put $A = T/I$ and let J and K be ideals of T such that $\mathrm{Supp}(H_0(\mathbb{G}.)) = \mathrm{Spec}\, T/J$ and $\mathrm{Supp}(\mathbb{G}.) = \mathrm{Spec}\, T/K$.

It is easy to see that $I + K$ is an n-primary ideal because

$$\{m\} = \mathrm{Supp}(\mathbb{F}.) = \mathrm{Supp}(\mathbb{G}. \otimes_T T/I) = \mathrm{Supp}(\mathbb{G}.) \cap \mathrm{Spec}\, T/I.$$

Then, by Serre's Theorem (p141 of [22]), we have $d + \mathrm{Supp}(\mathbb{G}.) = \dim T/I + \dim T/K \leq \dim T$.

On the other hand, by Theorem 3 in Roberts [19], $\mathrm{ch}_i(\mathbb{F}.)$ vanishes if $i < \dim T - \dim \mathrm{Supp}(\mathbb{G}.)$. Therefore, if $i < d$, then $\mathrm{ch}_i(\mathbb{F}.) = 0$.

Put $\tau_{A/T}([A]) = \tau_d + \tau_{d-1} + \cdots + \tau_0$, where $\tau_i \in \mathrm{A}_i(A)_{\mathbb{Q}}$. By Example 18.3.12 in Fulton [8], we have

$$\begin{aligned}
\left(\sum_i (-1)^i \ell_A(H_i(\mathbb{F}.))\right) \cdot [\mathrm{Spec}\, A/m] &= \tau_{(A/m)/T}(\chi_{\mathbb{F}.}([A])) \\
&= \mathrm{ch}(\mathbb{F}.) \cap \tau_{A/T}([A]) \\
&= \sum_{i=0}^{d} \mathrm{ch}_i(\mathbb{F}.) \cap \tau_i \\
&= \mathrm{ch}_d(\mathbb{F}.) \cap [\mathrm{Spec}\, A]_d \\
&= \chi_\infty(\mathbb{F}.).
\end{aligned}$$

The assertion (1) has been proved.

Put

$$\mathbb{G}. \ : \ 0 \longrightarrow G_d \xrightarrow{\partial_d} G_{d-1} \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_1} G_0 \longrightarrow 0$$

and $r_i = \mathrm{rank}_T\, G_i$ for $i = 0, 1, \ldots, d$. Since $\mathbb{F}. = \mathbb{G}. \otimes_T A$, $\mathbb{F}.$ is of the form

$$\mathbb{F}. \ : \ 0 \longrightarrow F_d \xrightarrow{\overline{\partial_d}} F_{d-1} \xrightarrow{\overline{\partial_{d-1}}} \cdots \xrightarrow{\overline{\partial_1}} F_0 \longrightarrow 0,$$

where $\overline{\partial_i} = \partial_i \otimes_T 1$. Put $t_i = r_i - r_{i+1} + \cdots + (-1)^{d-i} r_d$ for $i = 1, \ldots, d$. We denote by $I_{t_i}(\overline{\partial_i})$ the ideal of A generated by all t_i by t_i minors of an r_{i-1} by r_i matrix representing the A-linear map $\overline{\partial_i} : F_i \to F_{i-1}$. Since $\operatorname{Supp}(\mathbb{F}.) = \{m\}$, $\mathbb{F}. \otimes_A A_{\mathfrak{p}}$ is split exact for any prime ideal $\mathfrak{p}$ except for m. Therefore, $I_{t_i}(\overline{\partial_i})$ must be an m-primary ideal for $i = 1, \ldots, d$. Since $I_{t_i}(\overline{\partial_i}) = (I_{t_i}(\partial_i) + I)/I$, $I_{t_i}(\partial_i) + I$ is an n-primary ideal for $i = 1, \ldots, d$. Then, by Serre's theorem [22], we have $\operatorname{ht} I_{t_i}(\partial_i) + \operatorname{ht} I \geq \dim T$. Therefore, we have

$$\operatorname{grade}(I_{t_i}(\partial_i), T) = \operatorname{ht} I_{t_i}(\partial_i) \geq \dim T - \operatorname{ht} I = \dim A = d.$$

Using Buchsbaum-Eisenbud's theorem [3], we know that $\mathbb{G}.$ is acyclic, that is, $H_i(\mathbb{G}.) = 0$ if $i \neq 0$. The assertion (2) (i) has been proved.

Since $\operatorname{Spec} T/K = \operatorname{Supp}(\mathbb{G}.) = \operatorname{Supp}(H_0(\mathbb{G}.)) = \operatorname{Spec} T/J$, we have $\sqrt{J} = \sqrt{K}$. Therefore, $I + J$ is an n-primary ideal.

Since $\mathbb{G}.$ is a T-free resolution of $H_0(\mathbb{G}.)$, the projective dimension $\operatorname{pd}_T H_0(\mathbb{G}.)$ is at most d. Therefore, we have

$$\dim T \geq \dim H_0(\mathbb{G}.) + d \geq \dim H_0(\mathbb{G}.) + \operatorname{pd}_T H_0(\mathbb{G}.) \geq \operatorname{depth} H_0(\mathbb{G}) + \operatorname{pd}_T H_0(\mathbb{G}.).$$

Then, by the Auslander-Buchsbaum formula [16], we have $\dim T = \dim H_0(\mathbb{G}.) + d$. We have proved (iv).

Since $\mathbb{G}.$ is a T-free resolution of $H_0(\mathbb{G}.)$ by (i), we have

$$\sum_i (-1)^i \ell_T(\operatorname{Tor}_i^T(T/I, H_0(\mathbb{G}.))) = \sum_i (-1)^i \ell_A(H_i(\mathbb{F}.)) = \chi_\infty(\mathbb{F}.)$$

by (1).

By (ii), (iii) and (iv), we obtain (v) immediately from Serre's positivity theorem (p138 of [22]) and Gabber's non-negativity theorem [2]. **q.e.d.**

Remark 3.5 Complexes defined in Roberts [21] or Miller-Singh [17] do not satisfy the equality in Theorem 1.2 (1) as in Remark 2.7. Therefore, these complexes are never liftable to regular local rings.

4 Test modules

First, we define the notion of test modules.

Definition 4.1 Let A be a homomorphic image of a regular local ring S. Put $d = \dim A$. Then, a finitely generated A-module M is called a *test module* for A if the following two conditions are satisfied:

(1) M is a (non-zero) maximal Cohen-Macaulay A-module, that is, $\operatorname{depth} M = \dim M = d$.

(2) $\tau_{A/S}([M]) \in \mathrm{A}_d(A)_{\mathbb{Q}}$.

As we remarked in Section 2, the construction of $\tau_{A/S}$ depends not only on A but also on S. Therefore, the concept of test modules may depend on the choice of S. However, in many important cases (Section 4 in [14]), $\tau_{A/S}$ is independent of the choice of S.

Remark 4.2 With notation as in Definition 4.1, assume that A is an equal-characteristic complete local ring with residue class field perfect of characteristic $p > 0$. Let $f : A \to A$ be the Frobenuis endomorphism. Then, the second condition in Definition 4.1 is equivalent to $[{}_fM] = p^d[M]$ in $\mathrm{K}_0(A)_{\mathbb{Q}}$.

The following is the reason why we consider test modules.

Proposition 4.3 *Let A be a d-dimensional integral domain that is a homomorphic image of a regular local ring S. Assume that A has a test module M. Then, for a perfect A-complex*

$$\mathbb{F}. \ : \ 0 \to F_d \to \cdots \to F_0 \to 0$$

with support in $\{m\}$, we have

$$\chi_\infty(\mathbb{F}.) = \frac{\ell_A(H_0(\mathbb{F}. \otimes_A M))}{\mathrm{rank}_A M}.$$

In particular, if $\mathbb{F}.$ is not exact, we have $\chi_\infty(\mathbb{F}.) > 0$.

Proof. Put $r = \mathrm{rank}_A M$. Then, $[M] - r[A] \in \mathrm{K}_0(A)_{\mathbb{Q}}$ is a $\mathbb{Q}$-linear combination of cycles of dimension less than d. Then, by Theorem 18.3 (5) in Fulton [8], $\tau_{A/S}([M] - r[A]) \in \oplus_{i=0}^{d-1} \mathrm{A}_i(A)_{\mathbb{Q}}$. Therefore, we have $\tau_{A/S}([M]) \equiv r \cdot \tau_{A/S}([A]) \equiv r \cdot [\mathrm{Spec}\, A] \bmod \left(\oplus_{i=0}^{d-1} \mathrm{A}_i(A)_{\mathbb{Q}}\right)$. Since $\tau_{A/S}([M]) \in \mathrm{A}_d(A)_{\mathbb{Q}}$ by the definition of test modules, we have $\tau_{A/S}([M]) = r \cdot [\mathrm{Spec}\, A]$. Then, by Example 18.3.12 in Fulton [8], we have

$$\begin{aligned} \chi_{\mathbb{F}.}([M]) &= \mathrm{ch}(\mathbb{F}.) \cap \tau_{A/S}([M]) \\ &= r \cdot (\mathrm{ch}(\mathbb{F}.) \cap [\mathrm{Spec}\, A]) \\ &= r \cdot \chi_\infty(\mathbb{F}.). \end{aligned}$$

On the other hand, $H_i(\mathbb{F}. \otimes_A M) = 0$ is satisfied for $i > 0$ by the depth sensitivity due to Buchsbaum-Rim [4]. Therefore, $\chi_{\mathbb{F}.}([M])$ coincides with $\ell_A(H_0(\mathbb{F}. \otimes_A M))$. We have proved the first assertion.

If $\mathbb{F}.$ is not exact, then we have $H_0(\mathbb{F}. \otimes_A M) \neq 0$. Therefore, the second assertion immediately follows from it. **q.e.d.**

Remark 4.4 With notation as in Definition 4.1, if A is a Cohen-Macaulay ring with $\tau_{A/S}([A]) \in \mathrm{A}_d(A)_{\mathbb{Q}}$, then A itself is a test module for A. Therefore, Conjecture 3.1 is true for A in the case. If A is a complete intersection, then $\tau_{A/S}([A]) \in \mathrm{A}_d(A)_{\mathbb{Q}}$ is satisfied by Corollary 18.1.2 in Fulton [8].

Noetherian local rings with $\tau_{A/S}([A]) \in \mathrm{A}_d(A)_{\mathbb{Q}}$ will be studied in [14].

The author does not know whether there exists a test module for A even if A is a Gorenstein ring of equal characteristic. However, if the small Cohen-Macaulay modules conjecture (Conjecture 4.5 below) due to Hochster [11] is true, any complete local domain A whose field of fractions $Q(A)$ is of characteristic 0 has a test module as in Theorem 1.3 that is proved below.

Here, recall the small Cohen-Macaulay modules conjecture.

Conjecture 4.5 If R is a complete local ring, then R has a maximal Cohen-Macaulay module.

By Theorem 1.3, we know that <u>Conjecture 4.5 implies Conjecture 3.1</u>.

Proof of Theorem 1.3. Let G be the Galois group of the field extension $L/Q(S)$. Suppose that M is a maximal Cohen-Macaulay B-module.

Put
$$N = \bigoplus_{g \in G} {}_gM,$$
where ${}_gM = M$ is a B-module whose B-module structure is given through $g : B \to B$. We claim that N is a test module for A.

For each $g \in G$, the S-automorphism $g : B \to B$ induces $g^* : \mathrm{K}_0(B)_{\mathbb{Q}} \to \mathrm{K}_0(B)_{\mathbb{Q}}$ and $g^* : \mathrm{A}_*(B)_{\mathbb{Q}} \to \mathrm{A}_*(B)_{\mathbb{Q}}$. Therefore, G acts on $\mathrm{K}_0(B)_{\mathbb{Q}}$ and $\mathrm{A}_*(B)_{\mathbb{Q}}$. For any $g \in G$, ${}_gN$ is isomorphic to N as a B-module. Therefore, we have $g^*([N]) = [{}_gN] = [N]$ in $\mathrm{K}_0(B)_{\mathbb{Q}}$ for any $g \in G$. Hence, $[N] \in \left(\mathrm{K}_0(B)_{\mathbb{Q}}\right)^G$.

Since $g : B \to B$ is an S-automorphism, the following diagram is commutative:
$$\begin{array}{ccc}
\mathrm{K}_0(B)_{\mathbb{Q}} & \xrightarrow{\tau_{B/S}} & \mathrm{A}_*(B)_{\mathbb{Q}} \\
g^* \downarrow & & g^* \downarrow \\
\mathrm{K}_0(B)_{\mathbb{Q}} & \xrightarrow{\tau_{B/S}} & \mathrm{A}_*(B)_{\mathbb{Q}}
\end{array}$$
Therefore, $\tau_{B/S}\left((\mathrm{K}_0(B)_{\mathbb{Q}})^G\right) = \left((\mathrm{A}_*(B)_{\mathbb{Q}})^G\right)$ is satisfied. As we have seen in the proof of Theorem 1.1, $\left(\mathrm{A}_*(B)_{\mathbb{Q}}\right)^G$ coincides with $\mathrm{A}_d(B)_{\mathbb{Q}}$. Therefore, we have $\tau_{B/S}([N]) \in \mathrm{A}_d(B)_{\mathbb{Q}}$.

Let $j : A \to B$ be the inclusion. Then the diagram
$$\begin{array}{ccc}
\mathrm{K}_0(B)_{\mathbb{Q}} & \xrightarrow{\tau_{B/S}} & \mathrm{A}_*(B)_{\mathbb{Q}} \\
j^* \downarrow & & j^* \downarrow \\
\mathrm{K}_0(A)_{\mathbb{Q}} & \xrightarrow{\tau_{A/S}} & \mathrm{A}_*(A)_{\mathbb{Q}}
\end{array}$$
is commutative. Therefore, we have $\tau_{A/S}([N]) \in \mathrm{A}_d(A)_{\mathbb{Q}}$.

It is easy to see that N is a maximal Cohen-Macaulay A-module. **q.e.d.**

5 Some examples

In the section, we shall give some examples of test modules for some Cohen-Macaulay rings.

Example 5.1 Let k be an algebraically closed field of characteristic 0. Let $T = k[x_1, \ldots, x_n]$ be the graded polynomial ring over k with $\deg(x_1) = \cdots = \deg(x_n) = 1$ and I be a homogeneous ideal of T with $I \subseteq (x_1, \ldots, x_n)^2$. Put $A = (T/I)_{(x_1,\ldots,x_n)}$.

Suppose that A is a Cohen-Macaulay domain of minimal multiplicity. (If (A, m) is a Cohen-Macaulay ring, then it is easy to see

$$e(m, A) \geq \dim_{A/m} m/m^2 - \dim A + 1,$$

where $e(m, A)$ is the multiplicity of A with respect to m. We say that (A, m) is of *minimal multiplicity* if $e(m, A) = \dim_{A/m} m/m^2 - \dim A + 1$ is satisfied.)

We claim that A has a test module.

Put $B = T/I$. Then, by the classification due to Bertini (p166 in Yoshino [25]), we know that B is isomorphic to one of the following three types of rings;

(i) $k[x_1, \ldots, x_n]/(q)$, where q is a quadratic polynomial,

(ii) $k[x_1, x_2, x_3]^{(2)} = k[x_1^2, x_2^2, x_3^2, x_1x_2, x_2x_3, x_3x_1]$, and

(iii) scrolls.

If B is isomorphic to (i), then A is a complete intersection. In the case, by Remark 4.4, A itself is a test module for A.

Assume that B is isomorphic to the second Veronesean subring $k[x_1, x_2, x_3]^{(2)}$ of the polynomial ring $T = k[x_1, x_2, x_3]$. Then $k[x_1, x_2, x_3]_{(x_1,x_2,x_3)}$ is a test module for A (Theorem 1.3 (11) in [14]) since the regular local ring $k[x_1, x_2, x_3]_{(x_1,x_2,x_3)}$ is finite over A. (In the case, A itself is a test module for A. See Example 6.3 in [14].)

If B is isomorphic to a scroll, then one can prove that $A \oplus K_A$ is a test module for A.

Example 5.2 Let k be an algebraically closed field of characteristic 0. Let $T = k[x_1, \ldots, x_n]$ be the graded polynomial ring over k with $\deg(x_1) = \cdots = \deg(x_n) = 1$ and I be a homogeneous ideal of T. Put $A = \widehat{(T/I)_{(x_1,\ldots,x_n)}}$, that is the completion of $(T/I)_{(x_1,\ldots,x_n)}$.

Assume that A is a Cohen-Macaulay ring with only finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay A-modules.

We claim that A has a test module.

In fact, by Eisenbud-Herzog's theorem [7], we know that $B = T/I$ is isomorphic to one of the following eight types of rings;

(i) $k[x_1, \ldots, x_n]$,

(ii) $k[x]/(x^m)$ for some $m \geq 1$,

(iii) $k[x_1, \ldots, x_n]/(x_1^2 + \cdots + x_n^2)$,

(iv) $k[x_1, x_2]/(x_1^2 x_2 + x_2^3)$,

(v) $k[x_1, x_2, x_3]/(x_1x_2, x_2x_3, x_3x_1)$,

(vi) the scroll of type (m) for some m,

(vii) the scroll of type $(2, 1)$, and

(viii) $k[x_1, x_2, x_3]^{(2)} = k[x_1^2, x_2^2, x_3^2, x_1x_2, x_2x_3, x_3x_1]$.

If B is isomorphic to (i), (ii), (iii) or (iv), then A is a complete intersection and, therefore, A itself is a test module (Remark 4.4).

If B is isomorphic to (vi), (vii) or (viii), then A is of minimal multiplicity. Then, the existence of test modules will be proved in the same way as in Example 5.1.

Finally, assume that B is isomorphic to the ring (v). Then, $\dim A$ is equal to 1. It is easy to see $\mathrm{A}_0(A)_{\mathbb{Q}} = 0$. Therefore, we have $\tau_{A/S}([A]) \in \mathrm{A}_1(A)_{\mathbb{Q}}$, where $S = \widehat{T_{(x_1,\ldots,x_n)}}$. Therefore, A itself is a test module.

Example 5.3 Let k be a field and A be the completion of the $(2,1)$-scroll, i.e.,

$$A = k[[s_0, s_1, s_2, t_0, t_1]] \Big/ I_2 \begin{pmatrix} s_0 & s_1 & t_0 \\ s_1 & s_2 & t_1 \end{pmatrix}.$$

We shall describe all test modules for A. We refer the reader to Yoshino [25] for the theory of maximal Cohen-Macaulay modules. Let $S = k[x_1, x_2, x_3, x_4]$ be the graded polynomial ring over k with $\deg(x_1) = 2$, $\deg(x_2) = 1$, $\deg(x_3) = -1$ and $\deg(x_4) = -1$. We denote by S_n the homogeneous component of S in degree n. Then, we have

$$\begin{aligned} S_0 &= k[x_1x_3^2, x_1x_3x_4, x_1x_4^2, x_2x_3, x_2x_4] \\ &\simeq k[s_0, s_1, s_2, t_0, t_1] \Big/ I_2 \begin{pmatrix} s_0 & s_1 & t_0 \\ s_1 & s_2 & t_1 \end{pmatrix} \\ S_{-2} &= x_3^2 S_0 + x_3x_4 S_0 + x_4^2 S_0 \\ S_{-1} &= x_3 S_0 + x_4 S_0 \\ S_1 &= x_1x_3 S_0 + x_1x_4 S_0 + x_2 S_0 \\ M &= \mathrm{Syz}^1_{S_0} S_1. \end{aligned}$$

Note that $A = \widehat{S_0}$. Then, by Proposition (16.12) in Yoshino [25], the set of isomorphism classes of indecomposable maximal Cohen-Macaulay A-modules consists of

$$A, \quad K_A = S_{-1} \otimes_{S_0} A, \quad S_{-2} \otimes_{S_0} A, \quad S_1 \otimes_{S_0} A, \quad M \otimes_{S_0} A.$$

Put $B = S_{-2} \otimes_{S_0} A$, $C = S_1 \otimes_{S_0} A$ and $D = M \otimes_{S_0} A$. We have the Auslander-Reiten sequences as follows:

$$
\begin{array}{l}
0 \to D \to B \oplus A^{\oplus 2} \to K_A \to 0 \\
0 \to C \to D \to B \to 0 \\
0 \to B \to K_A \oplus A \to C \to 0 \\
0 \to A \to K_A^{\oplus 2} \oplus C \to D \to 0
\end{array}
$$

Put $\alpha = [A]$, $\beta = [B]$, $\gamma = [C]$, $\delta = [D]$ and $\epsilon = [K_A]$. It is easy to see that $\mathrm{K}_0(A)$ is generated by α, β, γ, δ and ϵ. Then, by Auslander-Reiten's theorem [1] (Theorem (13.7) in Yoshino [25]), relations on the modules in the Grothendieck group are generated by the Auslander-Reiten sequences, that is,

$$
\begin{aligned}
\mathrm{K}_0(A) &= \frac{\mathbb{Z}\alpha \oplus \mathbb{Z}\beta \oplus \mathbb{Z}\gamma \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\epsilon}{(\delta + \epsilon - 2\alpha - \beta,\ \beta + \gamma - \delta,\ \beta + \gamma - \alpha - \epsilon,\ \alpha + \delta - \gamma - 2\epsilon)} \\
&= \mathbb{Z}\alpha \oplus \mathbb{Z}\epsilon,
\end{aligned}
$$

where $\beta = 2\epsilon - \alpha$, $\gamma = 2\alpha - \epsilon$ and $\delta = \alpha + \epsilon$ in $\mathrm{K}_0(A)$.

Put $T = k[[s_0, s_1, s_2, t_0, t_1]]$. Since $\dim A$ is equal to 3, we have the Riemann-Roch map

$$\mathrm{K}_0(A)_{\mathbb{Q}} \xrightarrow{\tau_{A/T}} \mathrm{A}_*(A)_{\mathbb{Q}} = \oplus_{i=0}^{3} \mathrm{A}_i(A)_{\mathbb{Q}}.$$

Put $\tau_{A/T}([A]) = \tau_3 + \tau_2 + \tau_1$. (Since $\dim A$ is positive, $\mathrm{A}_0(A)_{\mathbb{Q}}$ vanishes.) Then, we have $\tau_{A/T}([K_A]) = \tau_3 - \tau_2 + \tau_1$ (e.g., Lemma 3.5 in [12]). Since $\{\alpha, \epsilon\}$ is a basis of $\mathrm{K}_0(A)_{\mathbb{Q}}$ as a $\mathbb{Q}$-vector space, $\{\tau_{A/T}(\alpha), \tau_{A/T}(\epsilon)\}$ is a basis of $\mathrm{A}_*(A)_{\mathbb{Q}}$ as a $\mathbb{Q}$-vector space. Then, it is easy to see $\mathrm{A}_*(A)_{\mathbb{Q}} = \mathbb{Q}\tau_3 \oplus \mathbb{Q}\tau_2$. Therefore we have $\mathrm{A}_3(A)_{\mathbb{Q}} = \mathbb{Q}\tau_3 \simeq \mathbb{Q}$, $\mathrm{A}_2(A)_{\mathbb{Q}} = \mathbb{Q}\tau_2 \simeq \mathbb{Q}$, $\mathrm{A}_1(A)_{\mathbb{Q}} = 0$. Hence,

$$
\begin{aligned}
\tau_{A/T}([A]) &= \tau_3 + \tau_2 \\
\tau_{A/T}([K_A]) &= \tau_3 - \tau_2 \\
\tau_{A/T}([B]) &= 2\tau_{A/T}([K_A]) - \tau_{A/T}([A]) \\
&= \tau_3 - 3\tau_2 \\
\tau_{A/T}([C]) &= 2\tau_{A/T}([A]) - \tau_{A/T}([K_A]) \\
&= \tau_3 + 3\tau_2 \\
\tau_{A/T}([D]) &= \tau_{A/T}([A]) + \tau_{A/T}([K_A]) \\
&= 2\tau_3
\end{aligned}
$$

are satisfied. Therefore, a maximal Cohen-Macaulay A-module

$$A^{\oplus n_1} \oplus K_A^{\oplus n_2} \oplus B^{\oplus n_3} \oplus C^{\oplus n_4} \oplus D^{\oplus n_5}$$

is a test module for A if and only if $n_1, \ldots, n_5$ are non-negative integers satisfying $n_1 - n_2 - 3n_3 + 3n_4 = 0$.

Remark 5.4 With notation as in Example 5.3, let b_2 and b_3 be rational numbers. Then, it is easy to see that there exist non-negative rational numbers a_1, $\ldots$, a_5 satisfying

$$b_3\tau_3+b_2\tau_2 = a_1\tau_{A/T}([A])+a_2\tau_{A/T}([K_A])+a_3\tau_{A/T}([B])+a_4\tau_{A/T}([C])+a_5\tau_{A/T}([D])$$

if and only if $3b_3 \geq |b_2|$.

Let

$$\mathbb{F}. : 0 \to F_3 \to F_2 \to F_1 \to F_0 \to 0$$

be a perfect A-complex with support in $\{m\}$. Let b_2 and b_3 be rational numbers such that $0 \neq 3b_3 \geq |b_2|$. Then, there exist a maximal Cohen-Macaulay A-module N and a positive rational number c such that $\tau_{A/T}([N]) = c(b_3\tau_3+b_2\tau_2)$. Then, we have

$$\begin{aligned} b_3 \cdot \chi_\infty(\mathbb{F}.) + b_2 \cdot \mathrm{ch}_2(\mathbb{F}.) \cap \tau_2 &= \mathrm{ch}(\mathbb{F}.) \cap (b_3\tau_3 + b_2\tau_2) \\ &= \frac{1}{c} \cdot \mathrm{ch}(\mathbb{F}.) \cap \tau_{A/T}([N]) \\ &= \frac{1}{c} \cdot \sum_{i=0}^{3} (-1)^i \ell_A(H_i(\mathbb{F}. \otimes_A N)) \\ &= \frac{1}{c} \cdot \ell_A(H_0(\mathbb{F}. \otimes_A N)) > 0 \end{aligned}$$

by Example 18.3.12 in Fulton [8]. Therefore,

$$\chi_\infty(\mathbb{F}.) > 3\,|\mathrm{ch}_2(\mathbb{F}.) \cap \tau_2|$$

is satisfied.

If there exists a perfect S-complex $\mathbb{G}.$ such that $\mathbb{F}. = \mathbb{G}. \otimes_T A$, we have $\mathrm{ch}_2(\mathbb{F}.) \cap \tau_2 = 0$ by Theorem 3 in Roberts [19].

Put $B = k[[x,y,z,w]]/(xw-yz)$. We define a ring homomorphism $\phi : B \to A$ by $\phi(x) = s_0$, $\phi(y) = t_0^2$, $\phi(z) = s_2$, $\phi(w) = t_1^2$. Let $\mathbb{H}.$ be a perfect B-complex of length 3 defined in Dutta-Hochster-MacLaughlin [6], and put $\mathbb{F}. = \mathbb{H}. \otimes_B A$. Then, we have $\chi_\infty(\mathbb{F}.) = 60$ and $\mathrm{ch}_2(\mathbb{F}.) \cap \tau_2 = 1$.

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