The Positivity of Intersection Multiplicities and Symbolic Powers of Prime Ideals

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Abstract

Serre's nonnegativity conjecture for intersection multiplicities has recently been proven by O. Gabber. In this paper we investigate Serre's positivity conjecture using the methods which he developed. We show in particular that the positivity conjecture has implications for properties of symbolic powers of prime ideals in regular local rings.

Introduction

Around forty years ago, Serre [16] introduced a homological definition of intersection multiplicity for modules over a regular local ring and showed that it satisfied many of the properties which should hold for intersection multiplicities. We denote this multiplicity $\chi_R(M, N)$, where M and N are finitely generated modules over the regular local ring R such that $M \otimes_R N$ is a module of finite length (we recall the definition of $\chi_R(M, N)$ in section 1). Serre showed that the condition that $M \otimes_R N$ has finite length implies that $\dim(M) + \dim(N) \leq \dim(R)$, and he made the following conjectures:

1. (Vanishing) If $\dim(M) + \dim(N) < \dim(R)$, then $\chi_R(M, N) = 0$.
2. (Nonnegativity) It is always true that $\chi_R(M, N) \geq 0$.
3. (Positivity) If $\dim(M) + \dim(N) = \dim(R)$, then $\chi_R(M, N) > 0$.

Serre proved these results for equicharacteristic local rings and unramified rings of mixed characteristic, leaving open the case in which R is a ramified regular local ring of mixed characteristic. The vanishing conjecture was proven about ten years ago by Roberts [13] and Gillet and Soulé [5] using K-theoretic

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methods. Recently, Gabber (see Berthelot [2], Hochster [6], or Roberts [15]) proved the nonnegativity conjecture using a recent result on resolution of singularities of de Jong [7]. In this paper we discuss some implications of these new results and investigate the conditions which are necessary for the positivity conjecture to hold.

In the first section, we describe Gabber's construction and give a criterion for positivity which follows from this construction in a fairly straightforward manner. We identify a certain subring and a certain ideal of a bigraded ring which is used in Gabber's construction, and we show that positivity is equivalent to the condition that the intersection of this subring with this ideal is zero. In the second section we prove a reduction which shows that we may restrict attention to a smaller subring which is easier to investigate. Finally, in section 3, we discuss the existence of nonzero elements in this intersection in detail and show that the positivity conjecture implies a condition on the intersections of symbolic powers of prime ideals in regular local rings.

1 A criterion for positivity

In this section we set up notation, describe the construction of Gabber that we use, and give a criterion (Theorem 1.2) for the positivity conjecture to hold.

Let $(R, \mathfrak{m})$ be a d-dimensional regular local ring with residue class field $k = R/\mathfrak{m}$. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R that satisfy

1. $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$
2. $\mathrm{ht}_R\, \mathfrak{p} + \mathrm{ht}_R\, \mathfrak{q} = d$.

Serre defined the intersection multiplicity of $R/\mathfrak{p}$ and $R/\mathfrak{q}$ to be

$$\chi_R(R/\mathfrak{p}, R/\mathfrak{q}) = \sum_{i=0}^{d} (-1)^i \mathrm{length}(\mathrm{Tor}_i^R(R/\mathfrak{p}, R/\mathfrak{q})).$$

Serre's positivity conjecture states that in this situation, it follows that $\chi_R(R/\mathfrak{p}, R/\mathfrak{q})$ is positive.

In the next few paragraphs we describe the construction used in Gabber's proof of the nonnegativity conjecture, after which we state the consequence of his theorem that we will use. The whole proof is based on a theorem of de Jong [7] on the existence of regular alterations, which can be considered a slightly weaker version of resolution of singularities. It follows from de Jong's theorem that there exists a graded prime ideal I in the graded polynomial ring $A = R[X_0, X_1, \ldots, X_n]$ for some n such that the following three conditions hold:

1. $I \cap R = \mathfrak{q}$.
2. $\mathrm{Proj}(A/I)$ is a regular scheme.

3. the morphism $\mathrm{Proj}(A/I) \to \mathrm{Spec}\, R/\mathfrak{p}$ is generically finite; that is, the extension of function fields defined by this morphism is finite.

We remark that the existence of the ideal I with the above properties requires that the regular local ring R is essentially of finite type over a field or a complete discrete valuation ring, but that the multiplicity conjectures can be reduced to this case. We refer to Berthelot [2] and Hochster [6] for details of this reduction. Throughout this section we assume that these properties hold and that this construction can be carried out. We let X denote $\mathrm{Proj}(A/I)$.

The first step in Gabber's proof of nonnegativity reduces the computation of intersection multiplicities to a computation on associated graded rings. Let $gr_I(A)$ be the associated graded ring of I; since I is a graded ideal of the graded ring A, $gr_I(A)$ is a bigraded ring. One grading, which we sometimes refer to as the first grading, is induced by the grading on A, while the second grading is determined by the powers of I. To the ring $gr_I(A)$ we associate a scheme Y as follows. The component of $gr_I(A)$ of degree 0 in the second grading is A/I, which defines the projective scheme $X = \mathrm{Proj}(A/I)$. For each element x of degree one in A/I there is an open affine subset of $\mathrm{Proj}(A/I)$ with associated ring consisting of the elements of degree zero in the localization $(A/I)_x$. Each such x defines an element of $gr_I(A)$ of degree zero in the second grading. For each such x, we define an open affine subset of Y by taking $\mathrm{Spec}(gr_I(A)_{(x)})$, where $gr_I(A)_{(x)}$ is the ring of elements of degree zero in the first grading in the localization $gr_I(A)_x$. We denote Y by $\mathrm{Proj}(gr_I(A))$ and use similar notation for analogous schemes defined by other bigraded rings. Note that since I is locally generated by a regular sequence, the associated graded ring $gr_I(A)$ is locally isomorphic to the symmetric algebra of I/I^2 over A/I. Thus if we define a scheme $\mathrm{Proj}(\mathrm{Sym}_{A/I}(I/I^2))$ associated to the bigraded ring $\mathrm{Sym}_{A/I}(I/I^2)$ using a similar definition, the map induced on schemes by the natural surjection from $\mathrm{Sym}_{A/I}(I/I^2)$ to $gr_I(A)$ is an isomorphism.

Throughout this paper, we are concerned both with graded rings and with the schemes that they define. To maintain consistency, when we speak of the dimension of a graded ring or module, we mean the dimension of the associated scheme or sheaf, which is generally one less than the dimension as a ring or module.

Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R satisfying the conditions stated at the beginning of this section. Let $\tilde{I}$ be the image of the ideal I in $A/\mathfrak{p}A$. We have a surjective map from $\mathrm{Sym}_{A/I}(I/I^2)$ to $gr_{\tilde{I}}(A/\mathfrak{p}A)$. Let K denote the kernel of this map, and let $\mathcal{K}$ denote the associated sheaf of ideals in $\mathcal{O}_Y$, where $\mathcal{O}_Y$ denotes the structure sheaf of Y. Then $\mathcal{K}$ defines a closed subscheme of Y which is the projective scheme defined by $gr_{\tilde{I}}(A/\mathfrak{p}A)$. The dimension of this subscheme is equal to the dimension of $\mathrm{Proj}(A/\mathfrak{p}A)$, which is $n+\dim(R/\mathfrak{p})$. Let $\mathcal{I}$ denote the sheaf of ideals in $\mathcal{O}_Y$ defined by the ideal $I/I^2 \oplus I^2/I^3 \oplus \cdots$ of $gr_I(A)$.

We wish to define an Euler characteristic $\chi_Y(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{K})$. We first recall the construction of $\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{K})$. The sheaf $\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{K})$ can be computed using a locally free resolution in either variable and tensoring with the other, or by taking the associated sheaves of the local computations of Tor. The locally free resolutions are finite since the graded ring $gr_I(A)$ is locally a polynomial ring over a regular ring, which follows from the fact that $\mathrm{Proj}(A/I)$ is a regular scheme. Each $\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{K})$ is a coherent sheaf over $\mathrm{Proj}(A/I)$, so we can then take its sheaf cohomology and obtain a finitely generated R-module. Furthermore, since $\mathfrak{p} + \mathfrak{q}$ is primary to $\mathfrak{m}$, the support of $\mathcal{O}_Y/\mathcal{I} \otimes \mathcal{O}_Y/\mathcal{K}$ lies over the closed point of R. Hence the same is true for $\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{K})$ for each i, and the sheaf cohomology modules they define are thus R-modules of finite length. For any coherent sheaf $\mathcal{F}$ of $\mathrm{Proj}(A/I)$ with support lying over the maximal ideal of R, we define the Euler characteristic to be the alternating sum of lengths of sheaf cohomology:

$$\chi(\mathcal{F}) = \sum_i (-1)^i \mathrm{length}(H^i(X, \mathcal{F})).$$

We then define

$$\chi_Y(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{K}) = \sum_i (-1)^i \chi(\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{K}))$$

$$= \quad \sum_i \sum_j (-1)^{i+j} \mathrm{length}(H^j(X, \mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{K}))).$$

We will use these definitions and this notation in similar situations below for other modules and for ideals and modules in other graded rings.

The first main step in this construction is to reduce the computation of $\chi_R(R/\mathfrak{p}, R/\mathfrak{q})$ to that of the Euler characteristic $\chi_Y(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{K})$ which we just defined (we refer to the references [2], [6], and [15] cited above for details of how this is done). The second step of the construction is to replace the graded ring $gr_I(A)$ by a polynomial ring over a quotient of A/I. Let $B = (A/I) \otimes_R k$, where $k = R/\mathfrak{m}$. We define a map $\alpha : I/I^2 \to \Omega_A \otimes_A B$ by sending x to $dx \otimes 1$, where Ω_A is the module of differentials on A. Since I annihilates B and d is a derivation, α is a graded A-module homomorphism and induces a ring homomorphism $\beta : \mathrm{Sym}_{A/I}(I/I^2) \to \mathrm{Sym}_B(\Omega_A \otimes_A B)$. The main point is that the assumption that $\mathrm{Proj}(A/I)$ is regular implies that $\alpha \otimes_A 1 : I \otimes_A B \to \Omega_A \otimes_A B$ is locally a split injection so that the image in $\mathrm{Sym}_B(\Omega_A \otimes_A B)$ can be used compute Euler characteristics. We give an algebraic version of this construction.

Let $s_1, \ldots, s_d$ be a minimal set of generators for $\mathfrak{m}$. At this point we must assume that R is equicharacteristic or ramified, so that $\Omega_A \otimes_A B$ is a free B-module with basis $ds_1, \ldots, ds_d, dX_0, \ldots, dX_n$ (see [2], [6], or [15]). For simplicity of notation, we put $S_i = ds_i$, $T_j = dX_j$ for $i = 1, \ldots, d$ and $j = 0, \ldots, n$. Let $\{U_{jk} \mid j,\ k = 0, \ldots, n\}$ be an additional set of $(n+1)^2$ variables. Then we define a ring homomorphism

$$\varphi \ : \ B[S_1, \ldots, S_d, T_0, \ldots, T_n] \longrightarrow B[S_1, \ldots, S_d, U_{00}, U_{01}, \ldots, U_{nn}]$$

by letting $\varphi(T_j) = X_0U_{j0} + X_1U_{j1} + \cdots + X_nU_{jn}$ for each j. We then have maps

$$\begin{aligned} &\mathrm{Sym}_{A/I}(I/I^2) \xrightarrow{\beta} \mathrm{Sym}_B(\Omega_A \otimes_A B) \\ &= B[S_1, \ldots, S_d, T_0, \ldots, T_n] \xrightarrow{\varphi} B[S_1, \ldots, S_d, U_{00}, \ldots, U_{nn}]. \end{aligned}$$

We denote $B[S_1, \ldots, S_d, T_0, \ldots, T_n]$ by F and $B[S_1, \ldots, S_d, U_{00}, \ldots, U_{nn}]$ by G. Both F and G have natural structures of bigraded rings. In this case the first grading is induced by that of A; in the first grading we let S_i and U_{jk} have degree zero, and we let T_j have degree one. All of the variables have degree 1 in the second grading. With these assumptions, the above maps are maps of bigraded rings. We thus have schemes $\mathrm{Proj}(F)$ and $\mathrm{Proj}(G)$ defined as above. Put $Z = \mathrm{Proj}(F)$ and $W = \mathrm{Proj}(G)$. We denote I_F and I_G the ideals generated by S_i, T_j and S_i, U_{jk} respectively in F and G, and $\mathcal{I}_F$ and $\mathcal{I}_G$ the associated ideal sheaves to I_F and I_G, respectively. We let K_F and K_G denote the ideals generated by the images of K in F and G respectively, and $\mathcal{K}_F$ and $\mathcal{K}_G$ the associated ideal sheaves to K_F and K_G, respectively. Since the maps from $\mathrm{Sym}_{A/I}(I/I^2) \otimes_R k$ to F and G are locally inclusions of polynomial rings obtained by adjoining variables, the dimension of F/K_F is $n + d + 1$ and the dimension of G/K_G is $d + (n + 1)^2$. We have Euler characteristics $\chi_Z(\mathcal{O}_Z/\mathcal{I}_F, \mathcal{O}_Z/\mathcal{K}_F)$ and $\chi_W(\mathcal{O}_W/\mathcal{I}_G, \mathcal{O}_W/\mathcal{K}_G)$ defined by the same process which we outlined above. These Euler characteristics are simpler to compute than those on Y, since the ideals I_F and I_G are generated by variables in a polynomial ring, so that the Tors in the first step can be computed using Koszul complexes. We note that in the case of F the degrees of the T_i must be taken into account in computing Euler characteristics. We then have

Theorem 1.1 (Gabber) *With notation as above, we have*

1. $[R(X) : Q(R/\mathfrak{q})] \cdot \chi_R(R/\mathfrak{p}, R/\mathfrak{q}) = \chi_Y(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{K})$, *where* $R(X)$ *is the function field of* $X = \mathrm{Proj}(A/I)$ *and* $Q(R/\mathfrak{q})$ *is the field of fractions of* $R/\mathfrak{q}$.

2. $\chi_Y(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{K}) > 0$ *if and only if* $\chi_{Y'}((\mathcal{O}_Y/\mathcal{I}) \otimes_R k, (\mathcal{O}_Y/\mathcal{K}) \otimes_R k) > 0$, *where we put* $Y' = \mathrm{Proj}(gr_I(A) \otimes_R k)$.

3. $\chi_{Y'}(\mathcal{O}_Y/\mathcal{I} \otimes_R k, \mathcal{O}_Y/\mathcal{K} \otimes_R k) = \chi_Z(\mathcal{O}_Z/\mathcal{I}_F, \mathcal{O}_Z/\mathcal{K}_F) = \chi_W(\mathcal{O}_W/\mathcal{I}_G, \mathcal{O}_W/\mathcal{K}_G)$.

Proof. For the proof of everything except statement 2 we refer to the references Berthelot [2], Hochster [6], or Roberts [15] cited above. Statement 2 is not proven explicitly there. Let r be the codimension of $X = \mathrm{Proj}(A/I)$ in $\mathrm{Proj}(A)$. Then Gabber shows that $\chi_Y(\mathcal{O}_Y/\mathcal{I}, \mathcal{M})$ is nonnegative for modules of dimension at most r such that the support of $\mathcal{M}$ lies over the maximal ideal of R, and that it vanishes for modules of dimension less than r. Hence we have that whether $\chi_Y(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{K}) > 0$ or not depends only on the support of $\mathcal{O}_Y/\mathcal{K}$ (and, in fact, only on the prime ideals of dimension r in the support). Since a power of $\mathfrak{m}$ annihilates $\mathcal{O}_Y/\mathcal{K}$, the support of $\mathcal{O}_Y/\mathcal{K}$ is the same as the support of $(\mathcal{O}_Y/\mathcal{K})\otimes_R k$. This proves the second statement. **q.e.d.**

From the theorem above, it suffices to test the positivity of $\chi_W(\mathcal{O}_W/\mathcal{I}_G, \mathcal{O}_W/\mathcal{K}_G)$. We note that $k[S_i, U_{jk}]$ is a subring of G. In what follows, if J is an ideal of one of the graded rings we are considering, we denote $\overline{J}$ the ideal of elements a such that $(X_0,\dots,X_n)^k a \subseteq J$ for some integer k. The aim of this section is to prove the next theorem:

Theorem 1.2 *With notation as above, the following statements are equivalent:*

1. $\chi_R(R/\mathfrak{p}, R/\mathfrak{q}) > 0$.
2. $\overline{K_G} \cap k[S_1,\dots,S_d,U_{00},\dots U_{nn}] = 0$.

In the remainder of this section, we give a proof of Theorem 1.2.

We recall that we have shown that it suffices to show that the second condition of the theorem is equivalent to the fact that $\chi_W(\mathcal{O}_W/\mathcal{I}_G, \mathcal{O}_W/\mathcal{K}_G) > 0$. We next examine the computation of this Euler characteristic more closely.

Let C denote the ring $k[S_i, U_{jk}]$. We then have a commutative diagram of inclusions of rings:

$$\begin{array}{ccc} G & \leftarrow & B \\ \uparrow & & \uparrow \\ C & \leftarrow & k \end{array}$$

and $G = C\otimes_k B$. These maps induce a commutative diagram of maps of schemes

$$\begin{array}{ccc} \mathrm{Proj}(G) & \rightarrow & \mathrm{Proj}(B) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(C) & \rightarrow & \mathrm{Spec}(k). \end{array}$$

In this diagram the vertical arrows are projective morphisms. For a scheme T we let $K_0(T)$ denote the Grothendieck group of coherent sheaves on T. We then have a corresponding diagram of Grothendieck groups

$$\begin{array}{ccc} K_0(\mathrm{Proj}(G)) & \rightarrow & K_0(\mathrm{Proj}(B)) \\ \downarrow & & \downarrow \\ K_0(\mathrm{Spec}(C)) & \rightarrow & K_0(\mathrm{Spec}(k)). \end{array} \tag{1}$$

The vertical maps are obtained by taking alternating sums of sheaf cohomology. The top horizontal map is obtained by taking a coherent sheaf $\mathcal{M}$ to the alternating sums of $\mathrm{Tor}_i^{\mathcal{O}_W}(\mathcal{O}_W/\mathcal{I}_G, \mathcal{M})$, while the bottom horizontal map is obtained by taking a C-module M to the alternating sum of $\mathrm{Tor}_i^C(C/(S_i, U_{jk}), M)$. To see that the diagram commutes, let $K_\bullet$ be the Koszul complex on the $d+(n+1)^2$ elements S_i, U_{jk} over C, and let $C^\bullet$ be the Cech complex of B with respect to $X_0,\dots,X_n$. Then, since $G = B\otimes_k C$, and since the variables S_i and U_{jk} have degree zero in the first grading on G, the complex $K_\bullet \otimes_k B$ induces a locally free resolution of $\mathcal{O}_W/\mathcal{I}_G$, and $C^\bullet \otimes_k C$ is a Cech complex of G with respect to $X_0, \ldots, X_n$. Hence the commutativity follows from the isomorphism of complexes

$$(K_\bullet \otimes_k B) \otimes_B C^\bullet \cong (C^\bullet \otimes_k C) \otimes_C K_\bullet$$

and a standard spectral sequence argument.

Let β and γ denote the vertical map on the left and the bottom horizontal map respectively in diagram (1). The commutativity of diagram (1) implies that we have

$$\chi_W(\mathcal{O}_W/\mathcal{I}_G, \mathcal{O}_W/\mathcal{K}_G) = \gamma\beta(\mathcal{O}_W/\mathcal{K}_G).$$

We recall that the sheaf $\mathcal{O}_W/\mathcal{K}_G$ has support of dimension $r = d + (n+1)^2$.

It follows from results of Serre [16] that for any finitely generated C-module M, if we identify $K_0(\mathrm{Spec}(k))$ with $\mathbb{Z}$, then $\gamma([M])$ is equal to the rank of M as a C-module. Thus we have

$$\gamma\beta(\mathcal{O}_W/\mathcal{K}_G) = \sum (-1)^i \mathrm{rank}_C(H^i(\mathrm{Proj}(G), \mathcal{O}_W/\mathcal{K}_G)).$$

To prove the theorem, we must show that this alternating sum is positive if and only if there is no nonzero element x of C such that $(X_0, \ldots, X_n)^k x \subseteq K_G$ for some k. If there were such an x, it would annihilate $H^i(\mathrm{Proj}(G), \mathcal{O}_W/\mathcal{K}_G)$ for all i, so the ranks of these modules would be zero, and thus we would have $\gamma\beta(\mathcal{O}_W/\mathcal{K}_G) = 0$. This proves one implication.

Conversely, assume that no element x with these properties exists. Let L be the quotient field of C. Then our assumption implies that the ideal generated by $X_0, \ldots, X_n$ in $L \otimes_C G/K_G$ is not nilpotent, so $\mathrm{Proj}(L \otimes_C (G/K_G))$ is not empty. On the other hand, since the dimension of G/K_G is $r = \dim(C)$, we must have that the dimension of $L \otimes_C (G/K_G)$ is zero. Hence $\mathrm{Proj}(L \otimes_C (G/K_G))$ is finite over $\mathrm{Spec}(L)$, and therefore $H^i(\mathrm{Proj}(L \otimes_C G/K_G), [L \otimes_C G/K_G]^\sim) = H^i(\mathrm{Proj}(G), \mathcal{O}_W/\mathcal{K}_G) \otimes_C L$ is zero for $i \neq 0$ and is nonzero for $i = 0$. Thus the rank of $H^i(\mathrm{Proj}(G), \mathcal{O}_W/\mathcal{K}_G)$ is zero for $i \neq 0$ and is positive for $i = 0$, so the alternating sum of ranks is positive. This completes the proof.

Remark 1.3 It is not difficult to check directly that the second condition of Theorem 1.2 is equivalent to the condition that the coefficient of n^r in the Hilbert polynomial P_M is nonzero in Theorem 3 (3) in [15].

Remark 1.4 If the vector bundle N on $\mathrm{Proj}(B)$ defined by the dual of $I/I^2 \otimes_A B$ (in the proof of Theorem 1.2) is *ample* in the sense of 12.1 in Fulton [4], then $\chi_{Y'}(\mathcal{O}_Y \otimes_R k, \mathcal{O}_T) > 0$ for any subscheme T of $Y' = \mathrm{Proj}(gr_I(A) \otimes_R k)$ of dimension r. More generally, if the *disamplitude locus* $\mathrm{Damp}(N)$ is not equal to $\mathrm{Proj}(B)$ (Example 12.1.10 in Fulton [4]), then $\chi_{Y'}(\mathcal{O}_Y \otimes_R k, \mathcal{O}_{T_i}) > 0$ for some component T_i of the support of $\mathcal{O}_Y/\mathcal{K}$ since the support (as a subcone of N) of at least one of T_i's is not contained in $\mathrm{Damp}(N)$ in the case. Therefore, if N is ample or $\mathrm{Damp}(N) \neq \mathrm{Proj}(B)$, then $\chi_R(R/\mathfrak{p}, R/\mathfrak{q}) > 0$ (cf. Theorem 3 (2) in [15]).

If $\mathfrak{q} \not\subset \mathfrak{m}^2$, then it is easy to show that N is not ample. Now we give an example in which N is not ample even though $\mathfrak{q}$ is contained in $\mathfrak{m}^2$.

Let k be a field. Put $R = k[[s_1, \ldots, s_4]]$ and $\mathfrak{q} = (s_1^2 - s_2^2 s_3 s_4)$. Furthermore, we put $A = R[X_0, X_1]$ and

$$I = \mathfrak{q}A + I_2 \begin{pmatrix} X_0 & X_1 & s_2 \\ -s_3 X_1 & -s_4 X_0 & s_1 \end{pmatrix}.$$

Then, $I + \mathfrak{m}A = \mathfrak{m}A$ and $\mathrm{Proj}(B) = \mathbb{P}_k^1$ in this case. It is easy to see that $\mathrm{Proj}(A/I) \to \mathrm{Spec}\, R/\mathfrak{q}$ is a desingularization and the sheaf of sections of N is equal to $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$. Therefore, it is not ample. By definition, $\mathrm{Damp}(N) = \mathrm{Proj}(B)$ in this case.

In the terminology of Theorem 1.2 this states that there is an element of $k[S_i, U_{jk}]$ which is annihilated by a power of $(X_0, \ldots, X_n)$ modulo the image of I/I^2 in G. However, this element does not come from an element of an ideal of the form K_G defined by a prime ideal of R satisfying the conditions which we have been considering.

2 A Reduction Lemma

In the previous section we showed that the positivity of intersection multiplicities is equivalent to the condition that a certain ideal $\overline{K_G}$ in the ring $G = B[S_i, U_{jk}]$ has zero intersection with the ring $k[S_i, U_{jk}]$. In this section we refine this condition and show that it is equivalent to the condition that the corresponding ideal $\overline{K_F}$ in F has zero intersection with the smaller subring $k[S_1, \ldots, S_d]$.

We keep the notation of the previous section. Let P be a bigraded prime ideal of F. The situation which arises in the investigation of the Serre positivity conjecture is where the quotient F/P has dimension $n + d + 1$, and we wish to find the Euler characteristic $\chi_Z(\mathcal{O}_Z/\mathcal{I}_F, [F/P]^{\sim})$, where $\mathcal{I}_F$ is the ideal sheaf of $\mathcal{O}_Z$ generated by the S_i and the T_j. We recall that we have a map $\varphi : F \to G$ defined by letting $\varphi(T_i) = \sum_j X_j U_{ij}$. For an ideal J, we let $\overline{J}$ denote the ideal of elements which are annihilated by a power of $(X_0, \ldots, X_n)$ modulo J.

Theorem 2.1 *Let P be a bigraded prime ideal in $B[S_1, \ldots, S_d, T_0, \ldots, T_n]$. Suppose that*

(i) $k[S_1, \ldots, S_d] \cap P = 0$.

(ii) $\overline{\varphi(P)G} \cap k[S_i, U_{jk}] \neq 0$.

Then the dimension of F/P is at most $d + n$.

By the theorem, we shall obtain the following corollary immediately.

Corollary 2.2 *With notation as in Theorem 1.2, the following statements are equivalent:*

1. $\chi_R(R/\mathfrak{p}, R/\mathfrak{q}) > 0$.

2. $\overline{K_F} \cap k[S_1, \ldots, S_d] = 0$.

In order to prove the corollary, we need the fact that $\mathrm{Proj}(F/K_F)$ is equidimensional. It is easy to see since $\mathrm{Proj}(gr_{\tilde{I}}(A/\mathfrak{p}A))$ is equidimensional.

To prove Theorem 2.1, we first reduce to the case in which B is a polynomial ring in $X_0, \ldots, X_n$. To achieve this, we map a graded polynomial ring $k[Y_0, \ldots, Y_n]$, which we denote by B', onto B by mapping Y_i to X_i. The surjective map from B' to B induces surjective maps from $F' = B'[S_i, T_j]$ to F and from $G' = B'[S_i, U_{jk}]$ to G. Furthermore, if we define φ' from F' to G' by mapping T_j to $Y_0U_{j0} + \cdots + Y_nU_{jn}$ in analogy with the definition of φ , we have a commutative diagram

$$\begin{array}{ccccc} B' & \longrightarrow & F' & \xrightarrow{\varphi'} & G' \\ \downarrow & & \downarrow & & \alpha\downarrow \\ B & \longrightarrow & F & \xrightarrow{\varphi} & G \end{array} .$$

Let P' be the inverse image of P in F'. We then have an isomorphism $F'/P' \cong F/P$.

We must check that the theorem is true for P if and only if it is true for P'. Since $F'/P' \cong F/P$, it is clear that $\dim(F/P) \leq d + n$ if and only if $\dim(F'/P') \leq d + n$. We thus only have to check that the hypotheses are also equivalent.

To check the first hypothesis, we note that there is a commutative diagram

$$\begin{array}{ccc} k[S_1, \ldots, S_d] & \rightarrow & F'/P' \\ & \searrow & \downarrow \\ & & F/P \end{array} .$$

Since the first hypothesis is satisfied for P if and only if the diagonal map in this diagram is injective and for P' if and only if the horizontal map is injective, and since the vertical map is an isomorphism, the equivalence of the two conditions is clear. For the equivalence of condition (ii) in the two cases, we use a similar argument; we must show that if we let Q and Q' denote the ideals generated by P and P' in G and G' respectively, we have an isomorphism $G'/Q' \cong G/Q$. We prove this fact directly; let α be the surjective map from G' to G. Since the map induced by α from G'/Q' to G/Q is clearly surjective, it suffices to show that it is injective. Let c' be an element of G' such that $\alpha(c') \in Q$. Since Q is generated by $\varphi(P)$, this means that we can write $\alpha(c') = \sum c_i\varphi(p_i)$ with $c_i \in G$ and $p_i \in P$. We can lift these elements to elements c_i' and p_i' in G' and P', and then $d' = \sum \varphi'(p_i')c_i'$ is in Q' and $\alpha(c' - d') = 0$. Now the kernel of α is generated by the kernel of the map from B' to B, and this kernel is clearly contained in P' and hence the ideal it generates in G' is contained in Q'. This completes the proof that we can reduce to the case of B', F', G', and P'.

We now change notation, replace B, F, G, P, and Q with B', F', G', P', and Q' and assume that $X_0, \ldots X_n$ are algebraically independent over k.

We next show that it suffices to prove that under the given hypotheses the height of P is at least equal to $n+1$. Since we have

$$\mathrm{ht}(P) + \dim(F/P) = \dim(F) = \dim(B) + n + d + 1,$$

we have that $\mathrm{ht}(P) \geq \dim(B) + 1$ if and only if

$$\dim(F/P) \leq \dim(F) - \dim(B) - 1 = n + d.$$

Thus it suffices to show that the height of P is at least the dimension of B (or of $\mathrm{Proj}(B)$) plus 1, which, now that we have reduced to the case in which B is the polynomial ring $k[X_0, \ldots, X_n]$, is $n+1$.

We next localize at the multiplicatively closed set S consisting of nonzero elements of $k[S_1, \ldots, S_d]$ and reduce the problem to a question on graded rings over a field. Let L denote the quotient field of $k[S_1, \ldots, S_d]$. Hypothesis (i) of the theorem implies that P defines a prime ideal of the localization, and its height will remain the same. Furthermore, the second hypothesis implies that there is a non-zero element in $L[U_{00}, \ldots, U_{nn}]$ such that some power of the ideal generated by $X_0, \ldots, X_n$ annihilates this element modulo the image of P after localization at S. Changing notation, we use F, G, P and Q to denote the localizations of the original rings F and G and ideals P and Q at S. In the localization we consider the grading induced by the grading in the first component; that is, elements of L have degree zero, the degrees of the X_i and T_i are 1, and the degrees of U_{ij} are zero. It suffices to prove the following lemma.

Lemma 2.3 *Let L be a field, and let P be a graded prime ideal of the polynomial ring $F = L[X_0, \ldots X_n, T_0, \ldots, T_n]$. Let Q be the ideal generated by $\varphi(P)$ in $G = L[X_0, \ldots, X_n, U_{00}, \ldots, U_{nn}]$, and let*

$$\overline{Q} = \{c \in G | \exists k \text{ with } (X_0, \ldots, X_n)^k c \subseteq Q\}.$$

If $\overline{Q} \cap L[U_{00}, \ldots, U_{nn}] \neq 0$, then the height of P is at least $n+1$.

If P contains $(X_0, \ldots, X_n)$, then the height of P is clearly at least $n+1$. Therefore, we may assume $P \not\supseteq (X_0, \ldots, X_n)$. Since the ideal P is graded, it suffices to localize at one of the X_i, which we can take to be X_0, take the part of degree zero, and show that the height of the resulting prime is at least $n+1$. After localization, the ideals $\overline{Q}$ and Q are equal, so $\overline{Q}$ will be generated by $\varphi(P)$. Denote X_i/X_0 by a_i and denote T_i/X_0 by t_i. Then the part of degree zero of the localization of F is the polynomial ring $L[a_1, \ldots, a_n, t_0, \ldots, t_n]$. The part of degree 0 of the localization of G is $L[a_1, \ldots, a_n, U_{00}, \ldots, U_{nn}]$, and the localization of the map defined by φ sends a_i to itself and sends t_i to $U_{i0} + a_1 U_{i1} + \cdots + a_n U_{in}$. We identify t_i with its image in G, and we now have

$$U_{i0} = t_i - a_1 U_{i1} - \cdots - a_n U_{in}$$

for each i.

Replace F (resp. G) by the part of degree 0 of the localization of F (resp. G). Using this substitution, we may identify F with a subring of G, and G is now a polynomial ring over F on the set of U_{ij} with $0 \leq i \leq n$ and $1 \leq j \leq n$. Since G is a polynomial ring over F, it is easy to describe the ideal Q generated by P in G. If we write an element of G as a polynomial in the U_{ij} with $1 \leq j \leq n$ with coefficients in F, then this polynomial is in Q if and only if every coefficient lies in P. We now reverse the point of view, and start with an element c of G. If we write c as a polynomial with coefficients in F as above, then its coefficients generate an ideal $\mathfrak{a}$ of F, and if c is in Q, then P must contain $\mathfrak{a}$. We refer to $\mathfrak{a}$ as the minimal ideal of F defined by c. To find $\mathfrak{a}$, we substitute $t_i - a_1 U_{i1} - \cdots - a_n U_{in}$ for U_{i0} and take the ideal generated by the coefficients of the resulting polynomial in the U_{ij} with $1 \leq j \leq n$. The proof of Lemma 2.3 will be completed by showing that if c is a nonzero element of $L[U_{00}, U_{01},, \ldots, U_{nn}]$, then the minimal ideal of F defined by c has height at least $n+1$.

Let c be as above, and let $\mathfrak{a}$ be the minimal ideal defined by c. We will show that $\mathfrak{a}$ has height at least $n+1$ by showing that for any subset of the set of variables $\{a_1, \ldots, a_n, t_0, \ldots t_n\}$ consisting of n elements, there is a nonzero element of $\mathfrak{a}$ which does not involve these elements. Thus, it will follow that such an element is a nonzero polynomial in the remaining $n+1$ elements of the set. Hence no subset of $\{a_1, \ldots, a_n, t_0, \ldots, t_n\}$ consisting of $n+1$ elements is algebraically independent over L in F/P, so the transcendence degree of F/P over L is at most n for any prime ideal P containing $\mathfrak{a}$. Thus the dimension (as a ring) of F/P is at most n and the height of P is at least $n+1$.

Let $c = f(U_{ij})$ be a nonzero polynomial with coefficients in L in the variables U_{ij} with $i = 0, \ldots, n$ and $j = 0, \ldots, n$. Choose a subset of $\{a_1, \ldots, a_n, t_0 \ldots, t_n\}$ consisting of n elements; renumbering, we may assume that these elements are $t_0, \ldots, t_m$ and $a_{m+2}, \ldots, a_n$. The strategy is to take a monomial in the U_{ij} with nonzero coefficient in f for which a certain linear function of the exponents (we define this function below) is maximal and show that the coefficient of a certain monomial in the expansion after substituting $t_i - a_1 U_{i1} - \cdots - a_n U_{in}$ for U_{i0} for each i is nonzero and does not involve $t_0, \ldots, t_m$ or $a_{m+2}, \ldots, a_n$.

We note that when $t_i - a_1 U_{i1} - \cdots - a_n U_{in}$ is substituted for U_{i0} in a monomial with factor U_{i0}^k, the result has several terms, including one with factor t_i^k and one with factor $(-a_j U_{ij})^k$ for each j. We will choose the monomial under consideration and the term of the expansion so that the only terms we need to consider are those involving these factors for certain i and j which we can determine.

We choose a monomial $M = \prod U_{ij}^{k_{ij}}$ for which

$$\sum_{i=0}^{m} (k_{i0} + k_{i,i+1}) - \sum_{i=m+1}^{n} (k_{i1} + k_{i2} + \cdots + k_{in})$$

is maximal among all monomials with nonzero coefficients in f. We then consider the monomial in the expansion of M with exponents which are equal to k_{ij} for all $i = 0, \ldots, n$ and $j = 1, \ldots, n$ except that the exponent of $U_{i,i+1}$ is $k_{i0} + k_{i,i+1}$ for $i = 0, \ldots, m$. Denote this monomial N. We claim that the coefficient of N in the expansion of f is nonzero and does not involve $t_0, \ldots, t_m$ or $a_{m+2}, \ldots, a_n$.

Let $M' = \prod U_{ij}^{n_{ij}}$ be a monomial whose expansion contributes to this coefficient. Let i be an integer with $0 \leq i \leq m$. Then, since $U_{i,i+1}$ occurs with exponent $k_{i0} + k_{i,i+1}$ in N, and since the largest exponent of $U_{i,i+1}$ which can possibly occur in any term of the expansion of M' is $n_{i0} + n_{i,i+1}$, we must have

$$n_{i0} + n_{i,i+1} \geq k_{i0} + k_{i,i+1}. \tag{2}$$

For i between $m+1$ and n, we consider the total degree of N in the variables $U_{i1}, \ldots, U_{in}$. By our construction and choice of N, this degree is $k_{i1} + \cdots + k_{in}$. On the other hand, the degree of any monomial in the expansion of M' in these variables is at least equal to $n_{i1} + \cdots + n_{in}$. Hence if the monomial N occurs in the expansion, we must have

$$n_{i1} + \cdots + n_{in} \leq k_{i1} + \cdots + k_{in}. \tag{3}$$

If any of the inequalities (2) or (3) were strict, we would then have that

$$\sum_{i=0}^{m}(n_{i0} + n_{i,i+1}) - \sum_{i=m+1}^{n}(n_{i1} + n_{i2} + \cdots + n_{in})$$

was strictly greater than

$$\sum_{i=0}^{m}(k_{i0} + k_{i,i+1}) - \sum_{i=m+1}^{n}(k_{i1} + k_{i2} + \cdots + k_{in}).$$

However, the monomial was chosen to maximize this quantity. Thus all the inequalities (2) and (3) are equalities.

We now compute the contribution of the monomial M' to the coefficient of N. Let i be between 0 and m. Then, as shown above, we have

$$n_{i0} + n_{i,i+1} = k_{i0} + k_{i,i+1}.$$

To find the coefficient of N, we need to take the part of the expansion of

$$(t_i - a_1 U_{i1} - \cdots - a_{i+1} U_{i,i+1} - \cdots - a_n U_{in})^{n_{i0}} U_{i1}^{n_{i1}} \cdots U_{i,i+1}^{n_{i,i+1}} \cdots$$

for which the exponent of $U_{i,i+1}$ is $n_{i0} + n_{i,i+1}$. Every term will have exponent strictly lower than $n_{i0} + n_{i,i+1}$ except one, and that is the one coming from $(-a_{i+1} U_{i,i+1})^{n_{i0}}$. Hence the only terms which end up contributing to the coefficient of N are obtained by substituting $(-a_{i+1} U_{i,i+1})$ for U_{i0}.

We next consider i between $m+1$ and n. As shown above, we have

$$n_{i1} + \cdots + n_{in} = k_{i1} + \cdots + k_{in}.$$

This time the only terms which can appear in the coefficient of N are those which contain the power $t_i^{n_{i0}}$, since any other term has degree in $U_{i1}, \ldots, U_{in}$ larger than that of N. Thus the only terms which contribute are obtained by substituting t_i for U_{i0}.

We now summarize the situation. To contribute to the coefficient of N, the monomial must have $n_{i0} + n_{i,i+1} = k_{i0} + k_{i,i+1}$ for $i = 0, \ldots, m$ and $n_{ij} = k_{ij}$ for all other i, j with $j \geq 1$. The resulting coefficient is then obtained by substituting $-a_{i+1}U_{i,i+1}$ for U_{i0} for $i = 0, \ldots, m$ and substituting t_i for U_{i0} for $i = m+1, \ldots, n$. The resulting contribution to the coefficient is the monomial $\prod_{i=0}^{m}(-a_{i+1})^{n_{i0}} \prod_{i=m+1}^{n} t_i^{n_{i0}}$. Our conditions imply that all of the ordered sequences of exponents are distinct and that there is at least one with non-zero coefficient. Thus, since a_i, t_j are algebraically independent the resulting coefficient is nonzero and does not involve $t_0, \ldots, t_m$ or $a_{m+2}, \ldots, a_n$. This proves Lemma 2.3.

3 Positivity and Symbolic Powers

For a prime ideal $\mathfrak{q}$ of a commutative ring R, $\mathfrak{q}^{(k)} = \mathfrak{q}^k R_{\mathfrak{q}} \cap R$ is called the k-th *symbolic power* of $\mathfrak{q}$. The symbolic power can also be defined as the set of $r \in R$ such that there exists $s \notin \mathfrak{p}$ with $sr \in \mathfrak{p}^n$.

In this section we discuss the following conjecture:

Conjecture 3.1 Let $(R, \mathfrak{m})$ be a regular local ring. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R that satisfy $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ and $\mathrm{ht}_R\, \mathfrak{p} + \mathrm{ht}_R\, \mathfrak{q} = d$. Then

$$\mathfrak{p} \cap \mathfrak{q}^{(k)} \subseteq \mathfrak{m}^{k+1}$$

for any $k > 0$.

We show below that in the case where $R \supset \mathbb{Q}$, we can easily solve the conjecture affirmatively.

The aim of the section is to prove the following theorem:

Theorem 3.2 *Let (R, m) be a regular local ring that is equicharacteristic or ramified. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R that satisfy the following three conditions;*

1. $\sqrt{\mathfrak{p}+\mathfrak{q}} = m$,
2. $\mathrm{ht}_R\, \mathfrak{p} + \mathrm{ht}_R\, \mathfrak{q} = d$,
3. $\chi_R(R/\mathfrak{p}, R/\mathfrak{q}) > 0$.

Then we have $\mathfrak{p} \cap \mathfrak{q}^{(k)} \subseteq m^{k+1}$ *for any positive integer* k.

If $(R, \mathfrak{m})$ is equicharacteristic in Theorem 3.2, then the third assumption for $\mathfrak{p}$ and $\mathfrak{q}$ follows from the first and second ones by the positivity theorem due to Serre [16]. Therefore, if $(R, \mathfrak{m})$ is equicharacteristic, Conjecture 3.1 is true.

By Theorem 3.2, Serre's positivity conjecture implies Conjecture 3.1 in the case where R is ramified.

Since the connection between the positivity conjecture and Conjecture 3.1 depends on the construction of section 1, which requires that R is equicharacteristic or ramified, we do not know whether Theorem 3.2 is true in the case of an unramified regular local ring of mixed characteristic. Conjecture 3.1 is open in that case.

Before proving Theorem 3.2, we give a remark on symbolic powers.

Remark 3.3 Let $(R, \mathfrak{m})$ be a regular local ring and $\mathfrak{q}$ a prime ideal of R.

(a) For any $k > 0$, $\mathfrak{q}^{(k)} \subseteq m^k$ (see Theorem (38.3) in Nagata [12]).

(b) Assume that $\mathfrak{q} \subseteq \mathfrak{m}^2$. If R contains a field of characteristic 0 (resp. $p > 0$), then $\mathfrak{q}^{(k)} \subseteq \mathfrak{m}^{k+1}$ for any $k > 0$ (resp. $0 < k < p$) by Proposition 3.4 below.

On the other hand, we now present an example in positive characteristic in which $\mathfrak{q} \subseteq \mathfrak{m}^2$ but $\mathfrak{q}^{(p)} \not\subseteq \mathfrak{m}^{p+1}$.

Let E be a 3-dimensional regular local ring with regular system of parameters x, y, z, and let

$$\Phi \ : \ R = E[[S, T, U, V]] \longrightarrow E[[W]]$$

be the ring homomorphism of formal power series rings over E defined by letting $\Phi(S) = x^3 W$, $\Phi(T) = y^3 W$, $\Phi(U) = z^3 W$, and $\Phi(V) = (xyz)^2 W$. Put $\mathfrak{q} = \mathrm{Ker}(\Phi)$. Then, $\mathfrak{q}$ is generated by the following 7 elements:

$$\begin{array}{lll} y^3S - x^3T, & z^3T - y^3U, & x^3U - z^3S, \\ xV - y^2z^2S, & yV - z^2x^2T, & zV - x^2y^2U, \\ V^2 - xyz^4ST & & \end{array}$$

Therefore $\mathfrak{q}$ is contained in $\mathfrak{m}^2$, where $\mathfrak{m} = (x, y, z, S, T, U, V)R$.

If E contains a field of characteristic 0, then $\mathfrak{q}^{(k)} \subseteq \mathfrak{m}^{k+1}$, but it can be shown (see [14]) that $\mathfrak{q}^{(k)} \not\subseteq \mathfrak{m}^{k+2}$ for any $k > 0$, and it follows that the *symbolic Rees ring* $\oplus_{k \geq 0} \mathfrak{q}^{(k)}$ is not Noetherian in this case. It is shown in [14] that a certain subring of $\oplus_{k \geq 0} \mathfrak{q}^{(k)}$ gives a counterexample to the Hilbert's fourteenth problem which is in many ways simpler than that of Nagata [11].

If E contains a field of characteristic $p > 0$, then $\mathfrak{q}^{(k)} \subseteq \mathfrak{m}^{k+1}$ for $0 < k < p$, but $\mathfrak{q}^{(p)} \not\subseteq \mathfrak{m}^{p+1}$ for each prime integer p (see [9]; this example is similar to the one described below in mixed characteristic). In this case, an element in $\mathfrak{q}^{(p)} \setminus \mathfrak{m}^{p+1}$ makes the symbolic Rees ring $\oplus_{k \geq 0} \mathfrak{q}^{(k)}$ Noetherian for each p ([8], [9]).

We give an example in the case where E has mixed characteristic. Assume that E is a regular local ring of mixed characteristic such that its residue class field is of characteristic 2 and that x divides 2. Since $\mathfrak{q} \cap E = 0$, we have

$$\begin{aligned} & V^2 - 2x^{-1}y^2z^2SV + xy^4zSU + xyz^4ST - x^4yzTU \\ = \ & \frac{1}{x^2}\left\{(xV - y^2z^2S)^2 + yz(y^3S - x^3T)(x^3U - z^3S)\right\} \in \mathfrak{q}^{(2)}. \end{aligned}$$

Therefore $\mathfrak{q}^{(2)} \not\subseteq \mathfrak{m}^3$ in this case.

(c) Eisenbud-Mazur [3] studied conditions that imply $\mathfrak{m}\mathfrak{q}^{(k)} \supseteq \mathfrak{q}^{(k+1)}$.

The next proposition implies Conjecture 3.1 in the case where $R \supset \mathbb{Q}$.

Proposition 3.4 *Let R be a regular local ring containing a field and $\mathfrak{q}$ a prime ideal of R. Suppose that $\mathfrak{q}^{(k)} \subseteq \mathfrak{m}^l$ for some positive integers k and l.*

a) If the characteristic of R is 0, then $\mathfrak{q}^{(k+1)} \subseteq \mathfrak{m}^{l+1}$.

b) If the characteristic of R is $p > 0$ and if $l < p$, then $\mathfrak{q}^{(k+1)} \subseteq \mathfrak{m}^{l+1}$.

Proof. Assume the contrary. Then there exist $\alpha \in \mathfrak{m}^l \setminus \mathfrak{m}^{l+1}$ and $\beta \in R \setminus \mathfrak{q}$ such that $\alpha\beta \in \mathfrak{q}^{k+1}$. We denote by $\hat{R}$ the completion of R and put $I = \mathfrak{q}\hat{R}$. Then, for any $t > 0$, we have

$$I^k :_{\hat{R}} \beta^t = (\mathfrak{q}^k :_R \beta^t)\hat{R} \subseteq \mathfrak{q}^{(k)}\hat{R} \subseteq \mathfrak{m}^l\hat{R} \tag{4}$$

since $\beta^t \notin \mathfrak{q}$. We will use this fact later.

Put $\hat{R} = k[[s_1, \ldots, s_d]]$, where k is the coefficient field of $\hat{R}$. Since $\alpha \in \mathfrak{m}^l\hat{R} \setminus \mathfrak{m}^{l+1}\hat{R}$ (and $l < p$ in the case where $\mathrm{ch}(k) = p > 0$), there exists i such that

$$\partial\alpha/\partial s_i \in \mathfrak{m}^{l-1}\hat{R} \setminus \mathfrak{m}^l\hat{R}. \tag{5}$$

Put $\alpha' = \partial\alpha/\partial s_i$, $\beta' = \partial\beta/\partial s_i$ and $(\alpha\beta)' = \partial(\alpha\beta)/\partial s_i$. Since $(\alpha\beta)' = \alpha'\beta + \alpha\beta'$, we have

$$\beta^2\alpha' = -\beta'\alpha\beta + \beta(\alpha\beta)'.$$

On the other hand, since $\alpha\beta \in \mathfrak{q}^{k+1}\hat{R} = I^{k+1}$, we have $(\alpha\beta)' \in I^k$. Therefore, we have $\beta^2\alpha' \in I^k$. Hence, $\alpha' \in I^k :_{\hat{R}} \beta^2$. By (4), we obtain $\alpha' \in \mathfrak{m}^l\hat{R}$, which contradicts (5). **q.e.d.**

It follows immediately from Proposition 3.4 that if $R \supset \mathbb{Q}$ and $\mathfrak{q}$ is contained in $\mathfrak{m}^2$, then $\mathfrak{p} \cap \mathfrak{q}^{(k)} \subseteq \mathfrak{m}^{k+1}$. The general case in which $R \supset \mathbb{Q}$ can be reduced to this case by induction on $\dim_k(\mathfrak{q} + \mathfrak{m}^2)/\mathfrak{m}^2$.

In order to prove Theorem 3.2, we need the following stronger version of the existence of regular alterations:

Proposition 3.5 *Assume that R is a regular local ring essentially of finite type over a field or a complete discrete valuation ring. Then, there exists a graded prime ideal $I \subset A = R[X_0, \ldots, X_n]$ for some n, where each X_i is of degree 1, that satisfies the following four conditions:*

1. $I \cap R = \mathfrak{q}$.
2. $\mathrm{Proj}(A/I)$ *is regular.*
3. $\mathrm{Proj}(A/I) \to \mathrm{Spec}\, R/\mathfrak{q}$ *is generically finite.*
4. $\mathrm{Proj}(A/I + \mathfrak{m}A)$ *is an effective Cartier divisor of* $\mathrm{Proj}(A/I)$.

This statement is identical to that of section 1 except that we have added the condition that $I + \mathfrak{m}A$ defines a Cartier divisor in $\mathrm{Proj}(A/I)$. This condition will be needed to show that certain maps on graded rings which we define below agree. The stronger version also follows from the theorem of de Jong [7], this time applied to the proper transform of $\mathrm{Spec}(R/\mathfrak{q})$ in the blowup of $\mathrm{Spec}(R)$ at the maximal ideal of R.

We now prove Theorem 3.2. Suppose that the theorem is not true, and that there exists a k such that $\mathfrak{p} \cap \mathfrak{q}^{(k)}$ is not contained in $\mathfrak{m}^{k+1}$. By the approximation theorem due to M. Artin [1], we may assume that R is a regular local ring essentially of finite type over a field or a complete discrete valuation ring. Applying Proposition 3.5, we can find a regular alteration of $\mathrm{Spec}(R/\mathfrak{q})$ satisfying the fourth condition in the proposition.

We use the criterion proven there for positivity (Theorem 1.2). We show that the assumption just made implies that there is a nonzero element in $\overline{K_F} \cap k[S_1, \ldots, S_d]$. (It is easy to see that the second condition of Theorem 1.2 implies $\overline{K_F} \cap k[S_1, \ldots, S_d] = 0$.)

For a graded ideal J of A contained in $I + \mathfrak{m}A$, we define a map from J/J^2 to $\Omega_A \otimes_A B$ induced by the differential of A. Since J annihilates B, the map above is A-linear and induces the map of symmetric algebras

$$\beta_J \;:\; \mathrm{Sym}_{A/J}(J/J^2) \longrightarrow \mathrm{Sym}_B(\Omega_A \otimes_A B).$$

Since $\mathfrak{m}A$ is generated by an A-regular sequence, the natural map

$$\mathrm{Sym}_{A/\mathfrak{m}A}(\mathfrak{m}A/\mathfrak{m}^2A) \to gr_{\mathfrak{m}A}(A)$$

is an isomorphism. Therefore $\beta_{\mathfrak{m}A}$ induces $\overline{\beta_{\mathfrak{m}A}} : gr_{\mathfrak{m}A}(A) \to \mathrm{Sym}_B(\Omega_A \otimes_A B)$.

Take $x \in \mathfrak{p} \cap \mathfrak{q}^{(k)} \setminus \mathfrak{m}^{k+1}$. By Remark 3.3 (a), x is contained in $\mathfrak{m}^k$. We denote by $\overline{x}$ the image of x in $\mathfrak{m}^k A/\mathfrak{m}^{k+1}A$. By definition, $\overline{\beta_{\mathfrak{m}A}}(\overline{x})$ is a nonzero element in $k[S_1, \ldots, S_d]$, which is the subring of $F = \mathrm{Sym}_B(\Omega_A \otimes_A B)$. To complete the proof, we will show that $\overline{\beta_{\mathfrak{m}A}}(\overline{x}) \in \overline{K_F}$, which is equivalent to $\overline{\beta_{\mathfrak{m}A}}(\overline{X_i^r x}) \in K_F$ for $i = 0, \ldots, n$ and $r \gg 0$.

We denote by $[*]_{(s,t)}$ the homogeneous component of degree s (resp. t) with respect to the first grading (resp. the second one). Let J be a graded ideal of

A contained in $I+\mathfrak{m}A$ such that $\mathrm{Proj}(A/J)$ is locally a complete intersection in $\mathrm{Proj}(A)$. Then the kernel of the natural map $\mathrm{Sym}_{A/J}(J/J^2) \to gr_J(A)$ is annihilated by a power of $(X_0, \ldots, X_n)$. Therefore

$$[\beta_J]_{(r,k)} \ : \ [\mathrm{Sym}^k_{A/J}(J/J^2)]_r \to [\mathrm{Sym}^k_B(\Omega_A \otimes_A B)]_r$$

induces

$$\overline{[\beta_J]_{(r,k)}} \ : \ [J^k/J^{k+1}]_r \to [\mathrm{Sym}^k_B(\Omega_A \otimes_A B)]_r$$

for $r \gg 0$. Let $J_1 \subseteq J_2$ be ideals of A, and $\gamma_{J_2,J_1} : gr_{J_1}(A) \to gr_{J_2}(A)$ the ring homomorphism induced by the natural inclusion $J_1 \subseteq J_2$. If both $\overline{[\beta_{J_1}]_{(r,k)}}$ and $\overline{[\beta_{J_2}]_{(r,k)}}$ are defined, then we obtain

$$\overline{[\beta_{J_2}]_{(r,k)}} \cdot \gamma_{J_2,J_1} = \overline{[\beta_{J_1}]_{(r,k)}}$$

immediately.

Since $\mathrm{Proj}(A/\mathfrak{m}A)$, $\mathrm{Proj}(A/I)$ and $\mathrm{Proj}(A/I+\mathfrak{m}A)$ are locally complete intersections in $\mathrm{Proj}(A)$ (by the fourth condition in Proposition 3.5), $\overline{[\beta_{\mathfrak{m}A}]_{(r,k)}}$, $\overline{[\beta_I]_{(r,k)}}$ and $\overline{[\beta_{I+\mathfrak{m}A}]_{(r,k)}}$ can be defined for $r \gg 0$. Then, we get

$$\overline{\beta_{\mathfrak{m}A}}(\overline{X_i^r x}) = \overline{[\beta_{\mathfrak{m}A}]_{(r,k)}}(\overline{X_i^r x}) = \overline{[\beta_{I+\mathfrak{m}A}]_{(r,k)}}(\overline{X_i^r x}) = \overline{[\beta_I]_{(r,k)}}(\overline{X_i^r x}),$$

here we assume that $X_i^r x \in I^k$ since $x \in \mathfrak{q}^{(k)} \subseteq I^{(k)}$ and $\mathrm{Proj}(A/I)$ is regular.

Since $x \in \mathfrak{p}$, it is easy to see

$$\overline{[\beta_I]_{(r,k)}}(\overline{X_i^r x}) \in K_F$$

for $r \gg 0$.

This completes the proof of Theorem 3.2.

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