HILBERT-KUNZ FUNCTIONS OVER RINGS REGULAR IN CODIMENSION ONE

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Abstract. The aim of this manuscript is to discuss the Hilbert-Kunz functions over an excellent local ring regular in codimension one. We study the shape of the Hilbert-Kunz functions of modules and discuss the properties of the coefficient of the second highest term in the function.

Our results extend Huneke, McDermott and Monsky's result about the shape of the Hilbert-Kunz functions in [9] and a theorem of the second author in [13] for rings with weaker conditions. In this paper, for a Cohen-Macaulay ring, we also explore an equivalent condition under which the second coefficient vanishes whenever the Hilbert-Kunz function of the ring is considered with respect to an $\mathfrak{m}$-primary ideal of finite projective dimension. We introduce an additive error of the Hilbert-Kunz functions of modules on a short exact sequence and give an estimate of such error.

1. Introduction

The aim of this paper is to study the stability of the Hilbert-Kunz functions for finitely generated modules over local rings of prime characteristic p. Besides its mysterious leading coefficient, known as the Hilbert-Kunz multiplicity, the behavior of the Hilbert-Kunz function is rather unpredictable. Some of the results presented in this paper are extensions of those in [9] and [13] for rings with weaker

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conditions. Following these improvements, we make further discussions on the properties of the second terms in which the Hilbert-Kunz functions stabilize and estimate the *additive error* of the Hilbert-Kunz functions on short exact sequences (defined in Section 4).

We begin with a brief introduction to the definition of the Hilbert-Kunz function and the motivation of our work presented in this paper.

Throughout the paper, let R be a Noetherian local ring of positive characteristic p and dimension d. We assume also that the residue field of R is perfect. Let I be an ideal primary to the maximal ideal. A *Frobenius n-th power* of I, denoted $I^{[p^n]}$, is the ideal generated by all elements of the form x^{p^n} for any x in I. For simplicity on notation, we write I_n for $I^{[p^n]}$. The length of a finitely generated module, if it exists, is denoted by $\ell(\cdot)$.

Let M be a finitely generated R-module. In 1969 Kunz introduced a map from $\mathbb{N}$ to $\mathbb{Z}_{\geq 0}$, for any positive integer n, defined by

$$\varphi_n^{R,I}(M) = \ell(M/I_nM).$$

We note that the input variable n is written as a subscript in the above expression for the convenience of discussion. This map was named the *Hilbert-Kunz function of M with respect to I* by Monsky [16]. Although the function depends on both M and I, when there is no ambiguity on the ideal I, we simply say the Hilbert-Kunz function of M and denote it by $\varphi_n(M)$. Monsky considered also the limit

$$\lim_{n\to\infty} \frac{\varphi_n(M)}{(p^n)^t} \tag{1.1}$$

where $t = \dim M$ and proved the following results:

Theorem 1.1 (Monsky [16]). *Let $(R, \mathfrak{m})$ be a Noetherian local ring of positive characteristic p and $\dim R = d$. Let I be an $\mathfrak{m}$-primary ideal and M a finitely generated module of dimension $t \leq d$. Then*

(*a*) *The limit in (1.1) always exists as a positive real number, denoted by $e_{HK}(M)$.*

(*b*) *The Hilbert-Kunz function is always of the form*

$$\varphi_n(M) = e_{HK}(M)q^t + O(q^{t-1}),$$

where $q = p^n$*.*

Monsky named the limit in (1.1) the *Hilbert-Kunz multiplicity of* M (*with respect to* I). We recall that a function $f(n)$ is $O(p^{ns})$ for some fixed integer s if there exists a constant C such that $|f(n)| \leq C \cdot p^{ns}$ for all $n \gg 0$. For simplicity, we write $O(q^s)$ for $O(p^{ns})$ with $q = p^n$ as defined in the previous paragraph. And we sometimes write $g(n) = h(n) + O(q^s)$ if $g(n) - h(n)$ is $O(q^s)$.

In this paper, we express the Hilbert-Kunz function of M as

$$\varphi_n(M) = \alpha_I(M) q^d + O(q^{d-1}), \tag{1.2}$$

where $\alpha_I(M)$ equals $e_{HK}(M)$ if $\dim M = d$ and zero otherwise. Huneke, McDermott and Monsky studied further how $\varphi_n(M)$ depends on n as n grows. Their main theorem states:

Theorem 1.2 (Huneke-McDermott-Monsky [9])**.** *Let* $(R, \mathfrak{m})$ *be an excellent local normal domain of characteristic* p *with a perfect residue field and* $\dim R = d$*. Then* $\varphi_n(M) = \alpha_I(M) q^d + \beta_I(M) q^{d-1} + O(q^{d-2})$ *for some real constants* $\alpha_I(M)$ *and* $\beta_I(M)$ *in* $\mathbb{R}$*.*

If the module M and the primary ideal I is clear to the content under consideration, we drop them from the expression and use α and β in place of $\alpha_I(M)$ and $\beta_I(M)$ respectively.

In this paper we say that a local ring satisfies (R1$'$) if the localization of the ring is a field (resp. a DVR) at a prime ideal of dimension d (resp. $d-1$). Here the dimension of a prime ideal $\mathfrak{p}$ means the Krull dimension of $R/\mathfrak{p}$. It is easy to check that (R1$'$) is equivalent to the usual (R1) if R is an excellent local domain.

We observe that a key lemma and certain crucial properties needed to prove Theorem 1.2 either hold or have their natural substitution without the normal condition on the ring, although these extensions are not necessarily obvious (see Section 5 for details). Therefore it is natural to ask: is the normal condition essential for $\varphi_n(M)$ to stabilize at the second term $\beta_I(M) q^{d-1}$?

The goals of this paper are to prove that the shape of the Hilbert-Kunz functions of modules in Theorem 1.2 holds for excellent local rings satisfying (R1'), and to analyze vanishing properties of the second term. The latter extends a theorem of the second author in [13] about the vanishing of $\beta_I(R)$ related to the canonical module of R. We also discuss the vanishing properties of $\beta_I(R)$ in terms of the Todd class of the ring in the numerical Chow group. Furthermore, we prove that associated to each module, there exists another quantity τ which is additive on short exact sequences. The Hilbert-Kunz multiplicity is additive on short exact sequences but the Hilbert-Kunz function is not. Using the additivity of τ, we define an additive error of the Hilbert-Kunz functions, and provide for this error an estimation in terms of the torsion submodules and the τ-value of appropriate modules.

It should be noted that generalization of the work in [9] is also studied independently by Hochster and Yao [8] via a different approach.

An example in Monsky [16] shows that the (R1$'$) condition can not be further relaxed for such stability of the Hilbert-Kunz function to hold. Precisely, take $R = \mathbb{Z}/p[[x,y]]/(x^5 - y^5)$ then R is one-dimensional, has an isolated singularity and its Hilbert-Kunz function is $\varphi_n(R) = 5p^n + \delta_n$ where $\delta_n = -4$ if n is even and -6 if n is odd.

As mentioned also in [9] the normal ring $R = \mathbb{Z}/5\mathbb{Z}[x_1, x_2, x_3, x_4]/(x_1^4+x_2^4+x_3^4+x_4^4)$ of dimension $d = 3$ has the Hilbert-Kunz function $\varphi_n(R) = \frac{168}{61}(5^n)^3 - \frac{107}{61}(3^n)$ which was computed by Han and Monsky [5]. The tail $-\frac{107}{61}(3^n)$ is $O((5^n)^{d-2})$ but not $O((5^n)^{d-3})$. So the existence of the third coefficient in the Hilbert-Kunz function in general is not possible.

We briefly describe the outline of the paper and the machinery used in each section as follows.

Section 2 reviews the definition of the rational equivalence and properties of cycle classes that will be used in the later discussions.

Section 3 contains the main results of this paper about the existence of the second coefficient of the Hilbert-Kunz function and its property. First we prove that the shape of the Hilbert-Kunz function as described in Theorem 1.2 holds for local excellent rings that satisfy (R1$'$) condition in Theorem 3.2. The method of proving Theorem 3.2 is to reduce the general case to normal domains and then apply Theorem 1.2. In the proof of Theorem 3.2, no rational equivalence is involved. We prove also that the vanishing property of the second coefficient of the Hilbert-Kunz function with respect to the maximal ideal is characterized by the canonical module in Theorem 3.3. These generalize the result of Theorem 1.2 and that in [13] respectively. Secondly we investigate the properties of the second coefficient with respect to arbitrary $\mathfrak{m}$-primary ideals of finite projective dimension. This is done in Theorem 3.5 using the techniques developed in [12, 13, 22].

Section 4 presents some possible applications. Theorem 4.1 first proves that each torsion free module is associated with a real number τ and then formally describes τ as a group homomorphism compatible with rational equivalence (extending [9, Corollary 1.10]). Finally we deduce that τ is additive on a short exact sequence and that the *additive error of the Hilbert-Kunz function* always arises from torsion submodules.

Finally in Section 5 we revisit the proof in [9] and show that all delicate analysis in [9] works for an arbitrary local domain that is F-finite and satisfies (R1$'$). This section is listed as an appendix since the result is a special case of Theroem 3.2. Nevertheless the section provides a straightforward proof without the need of taking normalization. We believe that such a generalized proof of [9] consists of interesting arguments and it is worth sharing it with curious readers; especially those who are interested in rational equivalence. This argument, however, has its own limitation. It works for integral domains. The authors do not know how to extend the analysis beyond the case of integral domains.

In the proofs of the main theorems, it is essential that the integral closure of the ring R in its ring of fractions is finite over R. For general results in Sections 3 and 4 where R is not necessarily a domain, we assume R to be excellent. In Section 5 where the ring is an integral domain, we assume R is F-finite which implies R is also excellent.

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2. Preliminary on the Chow Group

In this section, we recall the definition of Chow groups following Roberts [19], and state some properties that will be handy in the later sections. Let R be a Noetherian ring of dimension d. We define $Z_i(R)$ to be the free Abelian group generated by all prime ideals of dimension i in R. The *group of cycles* is the direct sum $Z_*(R) = \oplus_{i=0}^{d} Z_i(R)$. For any prime ideal $\mathfrak{p}$, we write $[R/\mathfrak{p}]$ for the element corresponding to $\mathfrak{p}$ in $Z_*(R)$. Let $\mathfrak{q}$ be a prime ideal of dimension $i+1$ and x an element in R not contained in $\mathfrak{q}$. The *rational equivalence* is an equivalence relation on $Z_*(R)$ by setting $\operatorname{div}(\mathfrak{q}, x) \sim 0$ where

$$\operatorname{div}(\mathfrak{q}, x) = \sum \ell((R/\mathfrak{q})_{\mathfrak{p}}/x(R/\mathfrak{q})_{\mathfrak{p}})[R/\mathfrak{p}]$$

with the summation over all prime ideals in R of dimension i, so $\operatorname{div}(\mathfrak{q}, x)$ is an element in $Z_i(R)$. Note that this is a finite sum since there are only finitely many minimal prime ideals for R/xR. Let $\operatorname{Rat}_i(R)$ be the subgroup of $Z_i(R)$ generated by $\operatorname{div}(\mathfrak{q}, x)$ for all $\mathfrak{q}$ of dimension $i+1$ and all $x \in R - \mathfrak{q}$. The *Chow group* $A_*(R)$ of R is the quotient of $Z_*(R)$ by $\operatorname{Rat}_*(R) = \oplus_{i=0}^{d} \operatorname{Rat}_i(R)$. The Chow group is also decomposed into the direct sum of $\mathrm{A}_i(R) = Z_i(R)/\operatorname{Rat}_i(R)$ for all $i = 0, \ldots, d$. By abuse of notation, we also use $[R/\mathfrak{p}]$ to denote the image of $[R/\mathfrak{p}]$ in $A_*(R)$.

For any finitely generated module M, there exists a prime filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M$$

such that each quotient of consecutive submodules is a cyclic module whose annihilator is a prime ideal; *i.e.*, $M_{i+1}/M_i \cong R/\mathfrak{p}_i$ for some prime $\mathfrak{p}_i$ and for all $i = 0, \ldots, n$. We note that such a prime filtration of a module is not unique. The prime ideals occurring in a filtration and the number of times each prime ideal occurs vary except for the minimal prime ideals. Indeed if $\mathfrak{p}$ is a minimal prime ideal for M, then the number of times that it occurs in a filtration is exactly $\ell(M_{\mathfrak{p}})$. It is proved in [3] that the sums of prime ideals of dimension d and $d-1$ from different filtrations are rationally equivalent. Therefore they define a unique class in the Chow group. Their equivalence class in $A_*(R)$ is called the *cycle class* of M denoted $[M]$. By definition $[M] = [M]_d + [M]_{d-1}$ with $[M]_i \in A_i(R)$ for $i = d, d-1$. Theorem 2.1 lists properties that will be utilized in this paper:

Theorem 2.1 (Chan [3])**.** *Let R be a Noetherian ring of dimension d and let M be a finitely generated module over R. Then the cycle class $[M] = [M]_d + [M]_{d-1}$ in $Z_d(R) \oplus Z_{d-1}(R)$ defined by taking the sum of prime ideals of dimension d and $d-1$ in a prime filtration has the following properties:*

(a) $[M]$ is independent of the choice of filtrations and hence defines a unique class in $\mathrm{A}_*(R)$.

(b) If $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of finitely generated R-modules, then $[M_3]_i - [M_2]_i + [M_1]_i = 0$ in $\mathrm{A}_i(R)$, for $i = d$ and $d-1$.

We make a few remarks on the cycle classes just defined: If R is a domain and M is a module of rank r, then $[M]_d = r[R]$ in $\mathrm{A}_d(R)$. If R is normal, then $\mathrm{A}_{d-1}(R)$ is isomorphic to the divisor class group and $[M]_{d-1}$ corresponds to $\mathrm{cl}(M)$. Sometimes $A_{d-1}(R)$ is called the *non-normal class group.*

3. Main Theorems

In this section, unless otherwise indicated, the ring R is an excellent local ring *regular in codimension one* by which we mean that $R_{\mathfrak{p}}$ is a field for every prime ideal $\mathfrak{p}$ with $\dim R/\mathfrak{p} = d$ and $R_{\mathfrak{p}}$ is a DVR for those $\mathfrak{p}$ with $\dim R/\mathfrak{p} = d-1$. We sometimes denote by $\dim \mathfrak{p}$ the Krull dimension of the ring $R/\mathfrak{p}$. As mentioned in Section 1, in this paper we also use (R1$'$) to denote this condition.

Theorem 3.2 proves, for any finitely generated module M, the existence of the second coefficient of the Hilbert-Kunz function $\varphi_n(M)$ with respect to an arbitrary maximal primary ideal I. This generalizes the main result of Huneke, McDermott and Monsky [9] where R is assumed to be an excellent local normal domain.

Using the singular Riemann-Roch theorem, the second author proved in [13] that in a Noetherian normal local domain R, if the canonical module of R is a torsion element in the divisor class group, then the second coefficient of the Hilbert-Kunz function $\varphi_n(R)$ also vanishes. Theorem 3.3 shows that this result holds also in the more general setting of Theorem 3.2 in which the second coefficient exists.

In preparation for the proofs of the main theorems, we begin with two useful lemmas.

Lemma 3.1. *Let R be a Noetherian local ring and let M_i be finitely generated modules over R for $i = 1, 2, 3, 4$. Assume that $0 \to M_1 \to M_2 \to M_3 \to M_4 \to 0$ is exact and that M_1 and M_4 have dimension at most $d-2$. Then $\varphi_n(M_2) = \varphi_n(M_3) + O(q^{d-2})$.*

Proof. Let N be the image of the map $M_2 \to M_3$. Then we obtain two exact sequences: $0 \to N \to M_3 \to M_4 \to 0$ and $0 \to M_1 \to M_2 \to N \to 0$. It is enough to show $\varphi_n(N) = \varphi_n(M_3) + O(q^{d-2})$ and $\varphi_n(M_2) = \varphi_n(N) + O(q^{d-2})$. The equalities on the Hilbert-Kunz function are the results of the short exact sequences in their respective order. We give a proof for the first one. The second equality is based on a similar argument.

We tensor the short exact sequence $0 \to N \to M_3 \to M_4 \to 0$ by R/I_n and obtain

$$\cdots \to \operatorname{Tor}_1^R(M_4, R/I_n) \to N/I_nN \to M_3/I_nM_3 \to M_4/I_nM_4 \to 0$$

which yields

$$\varphi_n(M_4) - \varphi_n(M_3) + \varphi_n(N) = \ell(K) \geq 0$$

where K denotes the image of the Tor_1-module in N/I_nN. Furthermore $\ell(K)$ is bounded by $\ell(\operatorname{Tor}_1^R(M_4, R/I_n))$. Thus we obtain that $\ell(\operatorname{Tor}_1^R(M_4, R/I_n)) = O(q^t)$ with $t = \dim M_4$ (*c.f.* [9, Lemma1.1] or Section 5 of the current paper) and $\ell(K) = O(q^{d-2})$. Also by the assumption and Theorem 1.1(b), $\varphi_n(M_4) = O(q^{d-2})$. Hence $\varphi_n(N) = \varphi_n(M_3) + O(q^{d-2})$ as desired. $\square$

If N can be viewed as a finitely generated module over R and S simultaneously, $\varphi_n^{R,I}(N)$ (resp. $\varphi_n^{S,IS}(N)$) denotes the Hilbert-Kunz function of N as a module over R (resp. S) and we skip the superscript when there is no ambiguity. In the proof of next theorem, we have to specify the base ring.

We consider the primary decomposition of the zero ideal of R:

$$(0) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_u \cap \mathfrak{q}_{u+1} \cap \cdots \cap \mathfrak{q}_\ell. \tag{3.1}$$

With the (R1$'$) condition on the ring, one observes the following properties of this decomposition:

- Each primary ideal of dimension d in the decomposition is a prime ideal.
- There is not a primary ideal of dimension $d-1$ in the decomposition.
- Any prime ideal of R of dimension $d-1$ contains a unique prime ideal of dimension d.

We assume that $\dim \mathfrak{q}_i = d$ for $i = 1, \ldots, u$ and $\dim \mathfrak{q}_i \leq d-2$ otherwise. We further observe that the Krull dimension of the module $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_u$ is at most $d-2$ because $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_u$ becomes zero when localizing at prime ideals of dimension larger than $d-2$. In fact let $\mathfrak{p}$ be a prime ideal of dimension at least $d-1$. Since $\mathfrak{p}$ contains a unique d-dimensional prime ideal in R, say $\mathfrak{q}_1$, and since $R_\mathfrak{p}$ is a regular local ring, then the localized ideal $\mathfrak{q}_1 R_\mathfrak{p}$ must be the zero ideal.

Theorem 3.2. *Let $(R, \mathfrak{m})$ be an excellent local ring of dimension d and positive characteristic p whose residue field is perfect. Assume that R satisfies the (R1$'$) condition. Let I be an $\mathfrak{m}$-primary ideal. Then there exist constants $\alpha(M)$ and $\beta(M)$ in $\mathbb{R}$ such that the Hilbert-Kunz function of M with respect to I is*

$$\varphi_n(M) = \alpha(M)q^d + \beta(M)q^{d-1} + O(q^{d-2}).$$

Proof. Since all conditions pass through completion and the Hilbert-Kunz function remains the same, we may assume that R is complete.

Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_u$ be the minimal prime ideals of R such that $\dim R/\mathfrak{q}_i = d$. The following sequence is exact

$$0 \longrightarrow K \longrightarrow R \stackrel{\eta}{\longrightarrow} \oplus_{i=1}^u \overline{R/\mathfrak{q}_i} \longrightarrow C \longrightarrow 0, \tag{3.2}$$

where η is the composition of the usual projection of a ring to its quotient followed by the inclusion to the integral closure of the quotient, and K and C are the kernel and cokernel of η. Here we remark that $\overline{R/\mathfrak{q}_i}$ is a local normal domain since $R/\mathfrak{q}_i$ is a complete local domain. For any prime ideal $\mathfrak{p}$ of dimension at least $d-1$, $\mathfrak{p}$ contains exactly one of $\mathfrak{q}_1, \ldots, \mathfrak{q}_u$, say $\mathfrak{q}_1$. Moreover $\mathfrak{q}_1 R_{\mathfrak{p}}$ is the zero ideal since $R_{\mathfrak{p}}$ is regular. This implies $R_{\mathfrak{p}} \cong (\oplus_{i=1}^u \overline{R/\mathfrak{q}_i})_{\mathfrak{p}}$ and so K and C have dimension at most $d-2$.

For an arbitrary module M, (3.2) induces the exact sequence

$$0 \longrightarrow K' \longrightarrow R \otimes M \stackrel{\eta\otimes 1}{\longrightarrow} \oplus_{i=1}^u (\overline{R/\mathfrak{q}_i} \otimes M) \longrightarrow C' \longrightarrow 0$$

where $C' = C \otimes M$ and K' is the kernel of $\eta \otimes 1$, and they also have dimension at most $d-2$. By Lemma 3.1, we have

$$\begin{array}{rcl} \varphi_n^{R,I}(M) & = & \sum_{i=1}^u \varphi_n^{R,I}(\overline{R/\mathfrak{q}_i} \otimes M) + O(q^{d-2}) \\ & = & \sum_{i=1}^u g_i \varphi_n^{\overline{R/\mathfrak{q}_i}, I\overline{R/\mathfrak{q}_i}}(\overline{R/\mathfrak{q}_i} \otimes M) + O(q^{d-2}) \end{array}$$

where g_i is defined to be the degree of the field extension $[\kappa(\overline{R/\mathfrak{q}_i}) : \kappa(R)]$ in which $\kappa(\cdot)$ denotes the residue field of the local ring. The second equality holds for if we let N denote the module $\overline{R/\mathfrak{q}_i} \otimes M$ for some i, then N/I_nN has finite length over R which gives $\varphi_n^{R,I}(\overline{R/\mathfrak{q}_i} \otimes M)$. Also N/I_nN has finite length over $\overline{R/\mathfrak{q}_i}$ which gives $\varphi_n^{\overline{R/\mathfrak{q}_i}, I\overline{R/\mathfrak{q}_i}}(\overline{R/\mathfrak{q}_i} \otimes M)$. And the former is g_i times the latter. Now each $\overline{R/\mathfrak{q}_i}$ is a normal local ring so the proof is completed by applying Theorem 1.2. $\square$

For the remaining of the section, we assume that R is a homomorphic image of a regular local ring A. We define the canonical module of R as in [6]. It is well known that $\omega_R = \mathrm{Ext}^c(R, A)$ where $c = \dim A - \dim R$ is the codimension. For $\overline{R}$, we put $\omega_{\overline{R}} = \mathrm{Ext}^c(\overline{R}, A)$ with $c = \dim A - \dim R$. We refer the reader to [6, Satz 5.12] (see also [1] and [2, Remark 3.5.10]).

Theorem 3.3. *Let $(R, \mathfrak{m})$ be as in Theorem 3.2. Assume also that R is the homomorphic image of a regular local ring A. Let ω_R be the canonical module of R. If $[\omega_R]_{d-1} = 0$ in $\mathrm{A}_{d-1}(R)_{\mathbb{Q}}$, then $\beta(R)$ in the Hilbert-Kunz function $\varphi_n(R)$ vanishes.*

Proof. As in the proof of Theorem 3.2, we also replace R by its completion and assume that R is complete. Here note that we can show $[\omega_R \otimes \hat{R}]_{d-1} = 0$ using the (R1$'$) condition of R. We write the Hilbert-Kunz function of R as $\varphi_n^{R,I}(R) = \alpha q^d + \beta q^{d-1} + \mathcal{O}(q^{d-2})$. Recall the proof of Theorem 3.2 and for each $i = 1, \ldots, u$ let α_i and β_i be real numbers in the Hilbert-Kunz function of $\overline{(R/\mathfrak{q}_i)}$ such that

$$\varphi_n^{\overline{(R/\mathfrak{q}_i)}, I\overline{(R/\mathfrak{q}_i)}}(\overline{(R/\mathfrak{q}_i)}) = \alpha_i q^d + \beta_i q^{d-1} + O(q^{d-2}).$$

Then $\beta = \sum_i g_i \beta_i$ where $g_i = [\kappa(\overline{(R/\mathfrak{q}_i)}) : \kappa(R)]$ also as defined in the previous proof. Obviously in order to prove $\beta = 0$, it suffices to prove $\beta_i = 0$ for each i.

Claim: That $[\omega_R]_{d-1} = 0$ *in* $\mathrm{A}_{d-1}(R)_{\mathbb{Q}}$ *implies* $[\omega_{\overline{(R/\mathfrak{q}_i)}}]_{d-1} = 0$ *in* $\mathrm{A}_{d-1}(\overline{(R/\mathfrak{q}_i)})_{\mathbb{Q}}$.

Assume the claim. Since $\overline{(R/\mathfrak{q}_i)}$ is a normal local ring and $[\omega_{\overline{(R/\mathfrak{q}_i)}}]_{d-1}$ vanishes in $\mathrm{A}_{d-1}(\overline{(R/\mathfrak{q}_i)})_{\mathbb{Q}}$, then $\beta_i = 0$ by [13, Corollary 1.4] and the theorem is proved.

Now we prove the above claim. Recall the minimal prime ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_u$ of R and the map η in (3.2). Since R and $\oplus_{i=1}^u \overline{(R/\mathfrak{q}_i)}$ are isomorphic when localizing at prime ideals of dimension $\geq d-1$, η also induces an isomorphism on the $(d-1)$-component of the Chow groups

$$\oplus_{i=1}^u \mathrm{A}_{d-1}(\overline{(R/\mathfrak{q}_i)}) \underset{\xi}{\simeq} \mathrm{A}_{d-1}(R). \tag{3.3}$$

We observe that $\omega_{\overline{(R/\mathfrak{q}_i)}} \cong \mathrm{Hom}_R(\overline{(R/\mathfrak{q}_i)}, \omega_R)$. Indeed let $\mathbb{I}^\bullet$ be an injective resolution of A, then there exists a quasi-isomorphism between the two complexes $\mathrm{Hom}_A(\overline{(R/\mathfrak{q}_i)}, \mathbb{I}^\bullet)$ and $\mathrm{Hom}_R(\overline{(R/\mathfrak{q}_i)}, \mathrm{Hom}_A(R, \mathbb{I}^\bullet))$. Hence by the definition of the canonical modules we have the isomorphism just mentioned. Then we apply $\mathrm{Hom}(-, \omega_R)$ to the exact sequence (3.2) and obtain the following exact sequence for some C'' and K'' of dimension at most $d-2$:

$$0 \longrightarrow C'' \longrightarrow \Pi_{i=1}^u \omega_{\overline{(R/\mathfrak{q}_i)}} \xrightarrow{\theta} \omega_R \longrightarrow K'' \longrightarrow 0. \tag{3.4}$$

Since both ξ in (3.3) and θ in (3.4) are induced by η, we may conclude that

$$\xi(\oplus[\omega_{\overline{(R/\mathfrak{q}_i)}}]_{d-1}) = [\omega_R]_{d-1}.$$

Hence $[\omega_R]_{d-1} = 0$ if and only if $[\omega_{\overline{(R/\mathfrak{q}_i)}}]_{d-1} = 0$ for each $i = 1, \ldots, u$ and the proof of the claim is completed. $\square$

Let $C(R)$ be the category of bounded complexes of free R-modules with support in $\{\mathfrak{m}\}$. We recall the definition of numerical equivalence in Kurano [12]. Define a subgroup of $\mathrm{A}_*(R)_{\mathbb{Q}}$

$$N\,\mathrm{A}_*(R)_{\mathbb{Q}} = \{\gamma \in \mathrm{A}_*(R)_{\mathbb{Q}} |\, \mathrm{ch}(\alpha) \cap \gamma = 0 \text{ for any } \alpha \in C(R)\}$$

in which $\mathrm{ch}(\alpha) \cap \gamma$ is the intersection of the localized Chern character of α with γ in the sense of [4] or [19]. An element in $\mathrm{A}_*(R)_{\mathbb{Q}}$ is said to be *numerically equivalent* to zero if it is an element in $N\,\mathrm{A}_*(R)_{\mathbb{Q}}$. The group modulo the numerical equivalence, denoted $\overline{\mathrm{A}_*(R)_{\mathbb{Q}}}$, is called *the numerical Chow group*. It is proved in [13] that $N\,\mathrm{A}_*(R)_{\mathbb{Q}}$ maintains the grading that comes from the dimension of cycles; that is,

$$N\,\mathrm{A}_*(R)_{\mathbb{Q}} = \oplus_i N\,\mathrm{A}_i(R)_{\mathbb{Q}}.$$

The singular Riemann-Roch theorem was first applied to study the Hilbert-Kunz function in Kurano [12, 13]. In the following Lemma 3.4, we restate a result proved by a computational technique presented in [13, Example 3.1(3)] (*c.f.* also [21, Proposition 1]) .

We first recall the definition of the Frobenius map $f : R \to R$. For any x in R, the Frobenius map is defined to be $f(x) = x^p$. A ring R is said to be *F-finite* if it is a finitely generated module via f. (Some more properties about the Frobenius map are stated in the Appendix, prior to Theorem 5.4.)

Lemma 3.4 ([13, 21]). *Let R be the homomorphic image of a regular local ring. We assume that R is an F-finite Cohen-Macaulay ring such that the residue class field is perfect. Let I be an $\mathfrak{m}$-primary ideal of finite projective dimension. Then the Hilbert-Kunz function of R with respect to I is a polynomial in p^n.*

Precisely, if we let $\mathbb{G}_\bullet$ be a finite free resolution of R/I and let c_i be in $\mathrm{A}_i(R)_{\mathbb{Q}}$ such that the Todd class $\mathrm{td}([R])$ is $c_d + c_{d-1} + \cdots + c_1 + c_0$ in $\mathrm{A}_*(R)_{\mathbb{Q}}$. Then

$$\varphi_n^{R,I}(R) = \sum_{i=0}^{d} (\mathrm{ch}_i(\mathbb{G}_\bullet) \cap c_i) q^i.$$

Proof. By definition the Euler characteristic is the alternating sum of the length of homology modules; that is

$$\chi_{\mathbb{G}_\bullet}(R^{\frac{1}{p^n}}) = \sum_i (-1)^i \ell(H_i(\mathbb{G}_\bullet \otimes R^{\frac{1}{p^n}})).$$

Since the Frobenius functor is exact and when applied to a complex, we simply raise the boundary maps by the corresponding Frobenius power, therefore

$$\begin{aligned}\chi_{\mathbb{G}_\bullet}(R^{\frac{1}{p^n}}) &= \ell(H_0(\mathbb{G}_\bullet \otimes R^{\frac{1}{p^n}}))\\ &= \ell(R/I^{[p^n]}) = \varphi_n^{R,I}(R).\end{aligned}$$

On the other hand by the singular Riemann-Roch theorem, we have

$$\chi_{\mathbb{G}_\bullet}(R^{\frac{1}{p^n}}) = \mathrm{ch}(\mathbb{G}_\bullet)(\mathrm{td}([R^{\frac{1}{p^n}}]))$$

It is known that $\mathrm{td}([R^{\frac{1}{p^n}}])$ decomposes in the Chow group and that we have for the i-th component the formula $\mathrm{td}_i([R^{\frac{1}{p^n}}]) = p^{in}\,\mathrm{td}_i([R]) = p^{in}c_i$ ([12, Lemma 2.2 (*iii*)]). Then

$$\begin{aligned}\chi_{\mathbb{G}_\bullet}(R^{\frac{1}{p^n}}) &= \mathrm{ch}(\mathbb{G}_\bullet)(\mathrm{td}([R^{\frac{1}{p^n}}]))\\ &= \mathrm{ch}(\mathbb{G}_\bullet)(p^{dn}c_d + \cdots + p^n c_1 + c_0)\\ &= \textstyle\sum_{i=0}^d (p^n)^i\,\mathrm{ch}_i(\mathbb{G}_\bullet) \cap c_i.\end{aligned}$$

Notice that $\mathrm{ch}_i(\mathbb{G}_\bullet) \cap c_i$ is in $\mathrm{A}_*(R/\mathfrak{m})_{\mathbb{Q}} \simeq \mathbb{Q}$. It is clear now that the Hilbert-Kunz function of R with respect to I is a polynomial in $q = p^n$ with coefficients described by the intersection of Todd classes and localized Chern characters

$$\varphi_n^{R,I}(R) = \chi_{\mathbb{G}_\bullet}(R^{\frac{1}{p^n}}) = \sum_{i=0}^d (\mathrm{ch}_i(\mathbb{G}_\bullet) \cap c_i)(p^n)^i = \sum_{i=0}^d (\mathrm{ch}_i(\mathbb{G}_\bullet) \cap c_i) q^i.$$

□

Theorem 3.5. *Let R be the homomorphic image of a regular local ring. We assume that R is a Cohen-Macaulay ring and is F-finite such that the residue class field is perfect. Then the following statements are equivalent:*

(a) $\beta_I(R) = 0$ for any $\mathfrak{m}$-primary ideal I of finite projective dimension.

(b) $\mathrm{td}_{d-1}([R]) = 0$ in $\overline{\mathrm{A}_{d-1}(R)_{\mathbb{Q}}}$ where $\mathrm{td}_{d-1}([R])$ is the $(d-1)$-component of the Todd class of $[R]$.

In addition if $R_{\mathfrak{p}}$ is Gorenstein for all minimal prime ideals $\mathfrak{p}$ of R, then (a) and (b) are equivalent to (c):
(c) $[\omega_R]_{d-1} = [R]_{d-1}$ *in* $\overline{\mathrm{A}_{d-1}(R)_{\mathbb{Q}}}$.

Proof. The equivalence of (a) and (b) will be proved by using the method in the proof of Theorem 6.4(2) in [12]. We give a complete proof here. Lemma 3.4 gives a presentation for the second coefficient of $\varphi_n^{R,I}(R)$:

$$\beta_I(R) = \mathrm{ch}_{d-1}(\mathbb{G}_\bullet) \cap c_{d-1}.$$

Assume (b); *i.e.*, $c_{d-1} = \mathrm{td}_{d-1}([R]) = 0$ in $\overline{\mathrm{A}_*(R)_{\mathbb{Q}}}$. Then by the definition of the numerical equivalence, $\mathrm{ch}_{d-1}(\mathbb{G}_\bullet) \cap c_{d-1} = 0$ since $\mathbb{G}_\bullet \in C(R)$. Hence $\beta_I(R) = 0$ and this proves (b) $\Rightarrow$ (a).

Conversely, assume (a). Let $\mathrm{K}_0(C)$ be the Grothendieck group of the category C of R-modules that have finite projective dimension and finite length. For any M in C, there exist finitely many ideals $I_1, \ldots, I_\ell$ generated by maximal regular sequences and an $\mathfrak{m}$-primary ideal I of finite projective dimension such that

$$[M] + \sum_{i=1}^{\ell} [R/I_i] = [R/I] \tag{3.5}$$

in $\mathrm{K}_0(C)$.

We see (3.5) following Kumar's proof for Hochster's theorem on $\mathrm{K}_0(C)$ ([22, Lemma 9.10]): let $\mathfrak{a}$ be the annihilator of M, then there exists a regular sequence $f_1, \ldots, f_d$ of maximal length in $\mathfrak{a}$. If M is not cyclic, we assume M is minimally generated by $n \geq 2$ elements, and let x_1 and x_2 be part of a minimal generating set of M. We may construct a homomorphism $\varphi : (f_1, \ldots, f_d) \to M$ such that $\varphi(f_i) = x_i$ for $i = 1, 2$. This can be done since the f_i's are in the annihilator of M. Let N be the push-out defined by φ; that is, $N = (M \oplus R)/B$ where B is the submodule generated by $(\varphi(f_i), -f_i)$ for all $i = 1, \ldots, d$. Then N is generated at most by $n-1$ elements and φ induces the following exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow R/(f_1, \ldots, f_\ell) \longrightarrow 0,$$

which shows $[M]+[R/(f_1,\ldots,f_d)]=[N]$ in $\mathrm{K}_0(C)$. Apply the above procedure on N repeatedly until it reduces to a cyclic module. We then obtain (3.5) with some $\mathfrak{m}$-primary ideal I which is the annihilator of the final cyclic module.

Let $\mathbb{G}^i_\bullet$, $\mathbb{G}_\bullet$ and $\mathbb{M}_\bullet$ be the resolutions of R/I_i, R/I and M respectively. By assumption, the second coefficients of the Hilbert-Kunz function of R with respect to I_i and I all vanish. From the above computation of the Hilbert-Kunz functions, we obtain $\mathrm{ch}_{d-1}(\mathbb{G}^i_\bullet)\cap c_{d-1}=0$ and $\mathrm{ch}_{d-1}(\mathbb{G}_\bullet)\cap c_{d-1}=0$. This implies $\mathrm{ch}_{d-1}(\mathbb{M}_\bullet)\cap c_{d-1}=0$ by (3.5). The theory of Roberts and Srinivas [20] states that $\mathrm{K}_0(C(R))=\mathrm{K}_0(C)$. Since M is arbitrary in the category C, we have that

$$\mathrm{ch}_{d-1}(\mathbb{F}_\bullet)\cap c_{d-1}=0$$

for all $\mathbb{F}_\bullet$ in the category $C(R)$. By definition, this means $c_{d-1}=0$ in $\overline{\mathrm{A}_{d-1}(R)_{\mathbb{Q}}}$ and completes the proof of the implication (a) $\Rightarrow$ (b).

Next in addition to the existing conditions, assuming that $R_\mathfrak{p}$ is Gorenstein for any $\mathfrak{p}$ in Min(R), the set of all minimal prime ideals of R, we prove that (c) is equivalent to (a) and (b).

Let Min(R) $=\{\mathfrak{q}_1,\ldots,\mathfrak{q}_\ell\}$ which is also the set of all associated prime ideals of R since R is Cohen-Macaulay. Then the total quotient ring $Q(R)$ is a zero-dimensional semi-local ring. Thus $Q(R)$ is complete and is a direct product of complete local rings. Precisely we have $Q(R)\simeq R_{\mathfrak{q}_1}\times\cdots\times R_{\mathfrak{q}_\ell}$. Then

$$\begin{aligned}\omega_R\otimes_R Q(R) &\simeq (\omega_R)_{\mathfrak{q}_1}\times\cdots\times(\omega_R)_{\mathfrak{q}_\ell}\\ &\simeq \omega_{R_{\mathfrak{q}_1}}\times\cdots\times\omega_{R_{\mathfrak{q}_\ell}}\\ &\simeq R_{\mathfrak{q}_1}\times\cdots\times R_{\mathfrak{q}_\ell}\\ &\simeq Q(R)\end{aligned}$$

The second last isomorphism is due to the assumption that $R_{\mathfrak{q}_i}$ is Gorenstein.

Claim: There exists an embedding from ω_R into R.

Proof of Claim. It is well-known that ω_R is a maximal Cohen-Macaulay module. Therefore, there is an embedding from ω_R to $\omega_R\otimes Q(R)$ hence an embedding to $Q(R)$ since $\omega_R\otimes Q(R)\simeq Q(R)$. On the other hand, R is a subring of $Q(R)$ and ω_R is a finitely generated R-module. Thus by clearing finitely many denominators from a generating set of ω_R in $Q(R)$, we have an embedding from ω_R to R by multiplying with some nonzerodivisor. This completes the proof of the Claim.

The above claim leads to a short exact sequence $0 \to \omega_R \to R \to R/\omega_R \to 0$ and thus the equality $[R/\omega_R]_{d-1} = [R]_{d-1} - [\omega_R]_{d-1}$ in $\mathrm{A}_{d-1}(R)$. On the other hand recall that $\mathrm{td}([R]) = c_d + c_{d-1} + \cdots + c_1 + c_0 \in \mathrm{A}_*(R)_{\mathbb{Q}}$, then $\mathrm{td}([\omega_R]) = c_d - c_{d-1} + \cdots + (-1)^d c_0$ since R is Cohen-Macaulay. This implies that $\mathrm{td}([R/\omega_R]) = 2c_{d-1} + 2c_{d-3} + \cdots$ since Todd class is additive. The dimension of R/ω_R is at most $d-1$ because for any minimal prime ideal $\mathfrak{q}_i$ of R, $(R/\omega_R)_{\mathfrak{q}_i} = 0$ since $R_{\mathfrak{q}_i}$ is Gorenstein. Therefore by the top term property of the Todd class which states, in the current terminology, that $\mathrm{td}_{d-1}([R/\omega_R]) = [R/\omega_R]_{d-1}$, we have shown that $[R/\omega_R]_{d-1} = 2c_{d-1}$.

Hence in $\overline{\mathrm{A}_{d-1}(R)_{\mathbb{Q}}}$, $c_{d-1} = \mathrm{td}_{d-1}([R]) = 0$ if and only if $[R/\omega_R]_{d-1} = 0$ which is equivalent to $[R]_{d-1} = [\omega_R]_{d-1}$. This proves the equivalence of (b) and (c). $\square$

If R is a domain, then $[R]_{d-1} = 0$ in $\mathrm{A}_{d-1}(R)$ by definition. Therefore the condition (c) in Theorem 3.5 is equivalent to $[\omega_R]_{d-1} = 0$ in $\overline{\mathrm{A}_{d-1}(R)_{\mathbb{Q}}}$. But if R is not a domain, then the condition (c) does not imply $[\omega_R]_{d-1} = 0$ as shown in the following example in which we construct a Gorenstein ring R of dimension 4. The ring R satisfies $[\omega_R]_3 = [R]_3$ but it does not define a zero class in $\overline{\mathrm{A}_3(R)_{\mathbb{Q}}}$.

We first recall the definition of the idealization of a module. Let $(S, \mathfrak{n})$ be a local ring with the maximal ideal $\mathfrak{n}$ and N an S-module. The *idealization* of N is a ring, denoted $S \ltimes N$, as an S-module, is equal to $S \oplus N$. For any two elements (a, n) and (b, m) in $S \oplus N$, we define the multiplication $(a, n)(b, m) = (ab, am + bn)$. It is straightforward to check that this multiplication gives a ring structure on $S \oplus N$ in which $0 \oplus N$ becomes an ideal. Moreover $S \ltimes N$ is a local ring with maximal ideal $\mathfrak{n} \oplus N$ and the ideal $0 \oplus N$ is nilpotent since by definition $(0, m)(0, n) = (0, 0)$ for any $m, n \in N$. So $\dim S \ltimes N = \dim S$.

Example 3.6. Let $S = k[\{x_i : i = 1, \ldots, 6\}]/I$ where I is the ideal generated by the maximal minors of the matrix $\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}$. It is known that S is

Cohen-Macaulay of dimension 4 with a canonical module ω_S which is isomorphic to the ideal generated by x_1 and x_4 over S. Viewing ω_S as an S-module, we let R be the idealization $S \ltimes \omega_S$ as defined above. Since $R = S \oplus \omega_S$ is a maximal Cohen-Macaulay S-module, $\dim R = \dim S = \operatorname{depth} S = \operatorname{depth} R$. So R is Cohen-Macaulay as a ring. To verify that R is Gorenstein, we see that R is finite over S and the canonical module $\omega_R \simeq \operatorname{Hom}_S(R, \omega_S) \simeq \operatorname{Hom}_S(S \oplus \omega_S, \omega_S) \simeq \operatorname{Hom}_S(S, \omega_S) \oplus \operatorname{Hom}_S(\omega_S, \omega_S) \simeq \omega_S \oplus S$. Such viewpoint of ω_R comes with a natural R-module structure and so $\omega_S \oplus S$ at the other end of the isomorphism sequence is also an R-module. Precisely for any $a \in S$ and $y \in \omega_S$, the multiplications on $S \ltimes \omega_S$ by $(a, 0)$ and $(0, y)$ induce respective multiplications on $\operatorname{Hom}_S(S \ltimes \omega_S, \omega_S)$. One can carefully check that these coincide with the multiplications on $\omega_S \oplus S$ as a module over R as an idealization of ω_S. Hence $\omega_R \simeq R$ as R-modules and so R is Gorenstein.

The ring R just constructed is Gorenstein of dimension 4 but is not reduced. We consider the short exact sequence of R-modules:

$$0 \to \omega_S \to S \ltimes \omega_S \to S \to 0.$$

Then $[S \ltimes \omega_S]_3 = [\omega_S]_3 + [S]_3$ in $\mathrm{A}_3(S \ltimes \omega_S)$.

Furthermore since ω_S is nilpotent in $R = S \ltimes \omega_S$, the quotient ring $R/\omega_S \simeq S$ and R have exactly the same prime ideals and rational equivalence relation. Thus $\mathrm{A}_*(R)$ is isomorphic to $\mathrm{A}_*(S)$ and we have $[S \ltimes \omega_S]_3 = [\omega_S]_3 + [S]_3$ holds in $\mathrm{A}_3(S)$. But $[S]_3 = 0$ since S is a domain so $[S \ltimes \omega_S]_3 = [\omega_S]_3 = -[S/\omega_S]_3$ in $\mathrm{A}_3(S) = \mathrm{A}_3(R)$. As mentioned above ω_S is isomorphic to the ideal generated by x_1 and x_4. We also know that $\mathrm{A}_3(S)_{\mathbb{Q}}$ has dimension one and can be generated by the class of $S/(x_1, x_4)$. This shows that $-[S/\omega_S]_3 = -[S/(x_1, x_4)]$ is nonzero in $\mathrm{A}_3(S)_{\mathbb{Q}}$. Furthermore, $\mathrm{A}_3(S)_{\mathbb{Q}} = \overline{\mathrm{A}_3(S)_{\mathbb{Q}}}$ by [12, Example 7.9]. Hence $[R]_3 \neq 0$ in $\overline{\mathrm{A}_3(R)_{\mathbb{Q}}}$ as wished to show in this example.

4. Additive Error of the Hilbert-Kunz Function

This section discusses some observations that grow out of the proof in Section 5. First we define an additive error of the Hilbert-Kunz function on a short exact sequence. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of finitely generated R-modules. The alternating sum $\varphi_n(M_3) - \varphi_n(M_2) + \varphi_n(M_1)$ is called the *additive error of the Hilbert-Kunz function*. It is known from [16, Theorem 1.6] that

$$\varphi_n(M_3) - \varphi_n(M_2) + \varphi_n(M_1) = O(q^{d-1}) \tag{4.1}$$

and hence the leading coefficient $\alpha(M_i)$ is additive [16, Theorem 1.8]. We give this error a more precise approximation in Corollary 4.3.

We will see in Theorem 4.1 below that, in order to estimate $\varphi_n(M) - \mathrm{rank}_R M \cdot \varphi_n(R)$, each torsion free module M is associated with a real number $\tau([M]_{d-1})$ and this leads to a group homomorphism from $\mathrm{A}_{d-1}(R)$ to $\mathbb{R}$. Then we deduce that τ is additive on short exact sequences and use this fact to estimate the additive error of the Hilbert-Kunz function.

In the case where R is normal, the τ-value for torsion free module is established in [9, Theorem 1.9] where it is proved that for a finitely generated torsion free module M of rank r, there exists a real constant $\tau(M)$ such that $\varphi_n(M) - r\varphi_n(R) = \tau(M)q^{d-1} + O(q^{d-2})$. As a corollary to the referred theorem, there exists a homomorphism from the divisor class group to $\mathbb{R}$. The vanishing properties of τ, in the normal case, is also discussed in [9]. In the proof of the following Theorem 4.1, we take a normalization of the ring and achieve the desired map by the compositions of a sequence of indued maps. We note that the τ-value in Theorem 4.1 can also be obtained as shown in Lemma 5.5 without taking normalization. One can also verify that this value is compatible with the rational equivalence by definition and hence it induces a group homomorphism from $\mathrm{A}_{d-1}(R)$ to $\mathbb{R}$.

Theorem 4.1. *Let R be an excellent local domain regular in codimension one of characteristic p. Assume that the residue field of R is perfect and* $\dim R = d$. *Then* *there exists a group homomorphism* $\tau : \mathrm{A}_{d-1}(R) \to \mathbb{R}$ *such that for a torsion free* R*-module* M *of rank* r, $\varphi_n(M) = r\varphi_n(R) + \tau([M]_{d-1})q^{d-1} + O(q^{d-2})$.

Proof. Take the integral closure $\overline{R}$ of R in its quotient field. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_s$ be the maximal ideals of $\overline{R}$. Let $g_i = [\overline{R}/\mathfrak{m}_i : R/\mathfrak{m}]$ be the degree of the field extensions. The completion of $\overline{R}$ is again normal and it is a product of the completions of $\overline{R}_{\mathfrak{m}_i}$; denote $\hat{\overline{R}} = R_1 \times \cdots \times R_s$ with $R_i = \widehat{\overline{R}_{\mathfrak{m}_i}}$ (*c.f.* [17, Theroem 17.7, p. 56] or [23, Section 4.3]). Then the following equalities of Hilbert-Kunz functions hold:

$$
\begin{aligned}
\varphi_n^{R,I}(M) &= \varphi_n^{R,I}(M \otimes_R \overline{R}) + O(q^{d-2}) \\
&= \textstyle\sum_{i=1}^{s} g_i \varphi_n^{\overline{R}_{\mathfrak{m}_i}, I\overline{R}_{\mathfrak{m}_i}}(M \otimes_R \overline{R}_{\mathfrak{m}_i}) + O(q^{d-2}) \\
&= \textstyle\sum_{i=1}^{s} g_i \varphi_n^{R_i, IR_i}(M \otimes_R R_i) + O(q^{d-2}).
\end{aligned}
\tag{4.2}
$$

The first equality holds because of the exact sequence $0 \to K \to M \to M \otimes \overline{R} \to C \to 0$ obtained by tensoring M to the inclusion $R \hookrightarrow \overline{R}$ for some K and C. Since R and $\overline{R}$ are isomorphic to each other when localized at prime ideals of dimension $\geq d-1$, we know that K and C have dimension at most $d-2$. Then we apply Lemma 3.1 to establish the equality.

Notice that the residue field $\kappa(R_i)$ of R_i is a finite field extension of that of $R/\mathfrak{m}$, so $g_i < \infty$. Because R_i is complete with perfect residue field, R_i is F-finite.

Let M be a torsion free module of rank r. Then $M \otimes_R R_i$ has rank r and its torsion submodule is of dimension at most $d-2$. Let $T(M \otimes_R R_i)$ denote the torsion submodule of $M \otimes_R R_i$. Hence by [9, Corollary 1.10], $\tau([M \otimes R_i/T(M \otimes R_i)]_{d-1})$ exists. Continuing the computation in (4.2), we see that for each n

$$
\begin{aligned}
& \varphi_n^{R,I}(M) - r\varphi_n^{R,I}(R) \\
=\ & \textstyle\sum_{i=1}^{s} g_i(\varphi_n^{R_i, IR_i}(M \otimes R_i) - r\varphi_n^{R_i,I}(R_i)) + O(q^{d-2}) \\
=\ & \textstyle\sum_{i=1}^{s} g_i \tau([M \otimes R_i/T(M \otimes R_i)]_{d-1})q^{d-1} + O(q^{d-2}).
\end{aligned}
\tag{4.3}
$$

Next we consider the following composition of maps on the $(d-1)$-components of the Chow groups

$$
\mathrm{A}_{d-1}(R) \xrightarrow{\delta^{-1}} \mathrm{A}_{d-1}(\overline{R}) \xrightarrow{\gamma} \mathrm{A}_{d-1}(\hat{\overline{R}}) = \mathrm{A}_{d-1}(R_1) \times \cdots \times \mathrm{A}_{d-1}(R_s) \longrightarrow \mathbb{R}.
$$

The first map δ^{-1} is the inverse of the isomorphism $\delta : \mathrm{A}_{d-1}(\overline{R}) \to \mathrm{A}_{d-1}(R)$ induced by $R \hookrightarrow \overline{R}$ since R satisfies (R1). The second map γ is as defined in [10, Definition 2.2] for the completion is a flat map. Each R_i is a complete normal local ring so there exists a map $\tau_i : \mathrm{A}_{d-1}(R_i) \to \mathbb{R}$ by [9, Corollary 1.10]. Now the desired map τ from $\mathrm{A}_{d-1}(R)$ to $\mathbb{R}$ can be obtained by taking appropriate composition of maps: $(\sum_{i=1}^{s} g_i\tau_i) \circ \gamma \circ \delta^{-1}$. Here note that $\gamma \circ \delta^{-1}([M]_{d-1}) = ([M \otimes R_1/\tau(M \otimes R_1)]_{d-1}, \ldots, [M \otimes R_s/\tau(M \otimes R_s)]_{d-1})$. According to the computation in (4.3), for any torsion free module M of finite rank, $\tau([M]_{d-1}) = \sum_i g_i\tau_i([M \otimes R_i/T(M \otimes R_i)]_{d-1})$. $\square$

Remark 4.2. Assume that R is as in Theorem 4.1. In the rest of this paper, for an R-module M, we denote $\tau([M]_{d-1})$ simply by $\tau(M)$.
(a) The map τ is additive on a short exact sequence. This is an immediate consequence of Theorem 2.1(b) and Theorem 4.1.
(b) For an arbitrary module M of rank r, let T be the torsion submodule of M and M' be the quotient M/T. Then

$$\varphi_n(M) = r\varphi_n(R) + (r\beta(R) + \tau(M'))q^{d-1} + \varphi_n(T) + (q^{d-2}). \tag{4.4}$$

This can be done by taking normalization as done in the proof of Theorem 4.1 and applying [9, Lemma 1.5]. (Or see a straightforward argument in Theorem 5.6 of the next section.)

Corollary 4.3. *Assume that R is as in Theorem 4.1. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of finitely generated R-modules. For each i let T_i be the torsion submodule of M_i and M_i' the torsion free module M_i/T_i. Then*

$$\varphi_n(M_3) - \varphi_n(M_2) + \varphi_n(M_1) = \varphi_n(T_3/\delta_2(T_2)) - \tau(T_3/\delta_2(T_2))q^{d-1} + O(q^{d-2})$$

where δ_2 is the induced map from T_2 to T_3.

Proof. For each i, letting r_i be the rank of M_i, we have

$$\varphi_n(M_i) = r_i\alpha(R)q^d + (r_i\beta(R) + \tau(M_i'))q^{d-1} + \varphi_n(T_i) + O(q^{d-2})$$

by (4.4). Therefore

$$\begin{aligned}
& \varphi_n(M_3) - \varphi_n(M_2) + \varphi_n(M_1) \\
= \quad & (\varphi_n(T_3) - \varphi_n(T_2) + \varphi_n(T_1)) + (\tau(M_3') - \tau(M_2') + \tau(M_1'))q^{d-1} + O(q^{d-2}).
\end{aligned}$$

Next we inspect $\varphi_n(T_i)$ and the alternating sum of $\tau(M_i)$ in order to estimate the additive error of the Hilbert-Kunz function on a short exact sequence.

The submodules T_i and M_i' give the following exact sequences:

$$\begin{array}{ccccc}
0 \longrightarrow T_1 & \xrightarrow{\delta_1} & T_2 & \xrightarrow{\delta_2} & T_3 \\
0 \longrightarrow M_1' & \xrightarrow{f_1} & M_2' & & \\
 & & M_2' & \xrightarrow{f_2} & M_3' \to 0.
\end{array}$$

The kernel of the map from M_2' to M_3' contains $f_1(M_1')$ but is not necessarily equal. Therefore T_i and M_i' do not form their own short exact sequences. By the standard diagram chasing, one sees $T_3/\delta_2(T_2) \cong \operatorname{Ker} f_2/\operatorname{Im} f_1$.

For the simplicity of notation, we identify T_1 with its image in T_2 via δ_1. The above exact sequence of T_i induces an injection from T_2/T_1 to T_3. Note that $0 \longrightarrow T_2/T_1 \longrightarrow T_3 \longrightarrow T_3/\delta_2(T_2) \longrightarrow 0$ is exact. By tensoring with R/I_n, we have the following exact sequences

$$\cdots \longrightarrow \operatorname{Tor}_1(T_2/T_1, R/I_n) \xrightarrow{\eta_1} T_1 \otimes R/I_n \longrightarrow T_2 \otimes R/I_n \longrightarrow T_2/T_1 \otimes R/I_n \longrightarrow 0$$

$$\cdots \longrightarrow \operatorname{Tor}_1(T_3/\delta_2(T_2), R/I_n) \xrightarrow{\eta_2} T_2/T_1 \otimes R/I_n \longrightarrow T_3 \otimes R/I_n \longrightarrow T_3/\delta_2(T_2) \otimes R/I_n \longrightarrow 0.$$

This implies

$$\varphi_n(T_2/T_1) - \varphi_n(T_2) + \varphi_n(T_1) - \ell(H_1) = 0$$

and

$$\varphi_n(T_3/\delta_2(T_2)) - \varphi_n(T_3) + \varphi_n(T_2/T_1) - \ell(H_2) = 0$$

where H_1 and H_2 are the images of η_1 and η_2 respectively. Thus

$$\begin{aligned}
\varphi_n(T_3) - \varphi_n(T_2) + \varphi_n(T_1) &= \varphi_n(T_3/\delta_2(T_2)) + \varphi_n(T_2/T_1) - \ell(H_2) - \varphi_n(T_2/T_1) + \ell(H_1) \\
&= \varphi_n(T_3/\delta_2(T_2)) + \ell(H_1) - \ell(H_2).
\end{aligned}$$

Let S be the quotient of R modulo the annihilator of T_2. Then $\dim S = \dim T_2 \leq d-1$. It is straightforward to see that $\varphi_n^S(T_2) = \varphi_n^R(T_2)$. This implies, by (4.1),

$$\begin{aligned} \ell(H_1) &= \varphi_n(T_2/T_1) - \varphi_n(T_2) + \varphi_n(T_1) \\ &= \varphi_n^S(T_2/T_1) - \varphi_n^S(T_2) + \varphi_n^S(T_1) \\ &= O(q^{\dim T_2 - 1}). \end{aligned}$$

Similarly we have $\ell(H_2) = O(q^{\dim T_2 - 1})$. Hence

$$\varphi_n(T_3) - \varphi_n(T_2) + \varphi_n(T_1) = \varphi_n(T_3/\delta_2(T_2)) + O(q^{\dim T_2 - 1}).$$

For the torsion free part, let K be the module such that $0 \longrightarrow K \longrightarrow M_2' \xrightarrow{f_2} M_3' \longrightarrow 0$ is exact. By Theorem 2.1(b), $[M_2']_{d-1} - [M_3']_{d-1} = [K]_{d-1}$. Then

$$[M_3']_{d-1} - [M_2']_{d-1} + [M_1']_{d-1} = -[K]_{d-1} + [M_1']_{d-1} = -[T_3/\delta_2(T_2)]_{d-1}.$$

The second last equality holds because $T_3/\delta_2(T_2) \cong \operatorname{Ker} f_2/\operatorname{Im} f_1$, and $\operatorname{Ker} f_2 = K$ and $\operatorname{Im} f_1 \cong M_1'$. Theorem 4.1 shows that τ is a group homomorphism so we obtain

$$\begin{aligned} \tau(M_3') - \tau(M_2') + \tau(M_1') &= \tau([M_3']_{d-1}) - \tau([M_2']_{d-1}) + \tau([M_1']_{d-1}) \\ &= \tau([M_1']_{d-1} - [K]_{d-1}) \\ &= -\tau([T_3/\delta_2(T_2)]_{d-1}) = -\tau(T_3/\delta_2(T_2)). \end{aligned}$$

It is clear now that

$$\begin{aligned} & \varphi_n(M_3) - \varphi_n(M_2) + \varphi_n(M_1) \\ = \; & (\varphi_n(T_3) - \varphi_n(T_2) + \varphi_n(T_1)) + (\tau(M_3) - \tau(M_2') + \tau(M_1'))q^{d-1} + O(q^{d-2}) \\ = \; & \varphi_n(T_3/\delta_2(T_2)) - \tau(T_3/\delta_2(T_2))q^{d-1} + O(q^{d-2}). \end{aligned}$$

□

As an immediate corollary, if R is an integral domain satisfying the assumption in Corollary 4.3 and $\mathrm{A}_{d-1}(R) = 0$, then the additive error of the Hilbert-Kunz function is measured by $\varphi_n(T_3/\delta_2(T_2))$ up to $O(q^{d-2})$.

Example 4.4. Let $R = k[x_1, \ldots, x_6]/I_2$ where I_2 indicates the ideals generated by the 2×2 minors of the matrix $\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}$. The Hilbert-Kunz function of

R is $\varphi_n(R) = \frac{1}{8}(13q^4 - 2q^3 - q^2 - 2q)$ computed by K.-i. Watanabe [1]. We consider the exact sequence

$$0 \to (x_1, x_2, x_3) \to R \to R/(x_1, x_2, x_3) \to 0.$$

It is known that $\omega_R = (x_1, x_4)$ is its canonical module and $[R/(x_1, x_4)]$ and $-[R/(x_1, x_2, x_3)]$ are rationally equivalent in $A_3(R)$. Thus $\tau([R/(x_1, x_2, x_3)]) = \tau(\omega_R) = \frac{1}{2}$ since $-\frac{1}{2}\tau(\omega_R) = \beta(R)$ by Kurano [13]. By Corollary 4.3, the additive error of the above short exact sequence is

$$\begin{aligned} & \varphi_n(T_3/\delta_2(T_2)) - \tau(T_3/\delta_2(T_2))q^2 + O(q^2) \\ = \ & \varphi_n(R/(x_1, x_2, x_3)) - \tau(R/(x_1, x_2, x_3))q^3 + O(q^2) \\ = \ & q^3 - \tfrac{1}{2}q^3 + O(q^2) \\ = \ & \tfrac{1}{2}q^3 + O(q^2). \end{aligned}$$

5. Appendix: F-finite Integral Domain with (R1′)

We have seen, in Section 3, the existence of the second coefficient of the Hilbert-Kunz function. In the current section, we revisit the proof of Huneke, McDermott and Monsky [9], in which the second coefficient is analyzed in details. Each lemma in [9] is of its own interest in terms of the result and the proof.

The proof in [9] utilizes the divisor class group and a key lemma that states that if T is a finitely generated torsion module over a local ring $(R, \mathfrak{m})$ of characteristic p, then $\ell(\mathrm{Tor}_1^R(T, R/I_n)) = O(q^s)$ where I is an $\mathfrak{m}$-primary ideal and $s = \dim T$ (*c.f.* [9, Lemma 1.1]). Note that this key lemma holds for general local rings of positive characteristic. When a ring is no longer normal, the $d-1$-component of the Chow group is often a natural replacement of the divisor class group. Also the corresponding divisor class is additive on short exact sequences in the Chow group (see Theorem 2.1) and this fact is used in proving several lemmas in [9].

[1] There is no reference. One can prove it calculating the Hilbert polynomials of Segre embeddings.

Hence the purpose of this section is twofold. We present a generalized proof of [9] for an integral domain under the assumption that the domain is F-finite and regular in codimension one. To do so, the Chow group is utilized in place of the divisor class group. Thus our purpose is also to demonstrate how rational equivalence can be applied in this study. Although the proof presented here follows a similar structure as [9], the argument is a nontrivial extension. The limitation of this argument is that the authors do not know if it is possible to extend each lemma for a ring that is not necessarily an integral domain. Hence the main result Theorem 5.6 requires stronger assumption on the ring than Theorem 3.2.

The idea of the proof goes as follows: we first focus on torsion free modules and prove that their Hilbert-Kunz function is different from that of free modules of the same rank by a function of the form $\tau q^{d-1} + O(q^{d-2})$. The constant τ depends on the cycle class of the module M and is zero if M defines a zero class. (Note that this constant τ is the same as $\tau(M)$ discussed in Remark 4.2.) Then for an arbitrary module M, we put together the information for the functions of the torsion submodule T and the torsion free module M/T to obtain $\varphi_n(M)$. We remind the readers to recall the definition of rational equivalence and cycle classes from Section 2.

For the remaining of the paper, we assume that R is a local integral domain with perfect residue class field such that the Frobenius map $f : R \to R$ is finite and R is regular in codimension one as defined in the introduction. By Kunz's theorem [11], F-finiteness implies that R is excellent hence the integral closure $\overline{R}$ is finite over R.

Lemma 5.1. *Let R be an F-finite local domain regular in codimension one with perfect residue class field. Let J be a nonzero ideal in R. Assume J defines the zero class in* $\mathrm{A}_{d-1}(R)$*;* i.e., $[J]_{d-1} = 0$. *Then* $\varphi_n(J) = \varphi_n(R) + O(q^{d-2})$.

Proof. By assumption $[J]_{d-1} = 0$, there exist nonzero elements a and b in R such that $[J]_{d-1} = \mathrm{div}(\mathfrak{o}; b) - \mathrm{div}(\mathfrak{o}; a)$ where $\mathfrak{o}$ denotes the zero ideal. By definition of divisors, this implies

$$\ell(R_{\mathfrak{p}}/JR_{\mathfrak{p}}) + \ell(R_{\mathfrak{p}}/aR_{\mathfrak{p}}) = \ell(R_{\mathfrak{p}}/bR_{\mathfrak{p}}) \tag{5.1}$$

for any prime ideal $\mathfrak{p}$ of dimension $d-1$. Since R is regular in codimension one, $R_{\mathfrak{p}}$ is a DVR and every ideal is a power of the maximal ideal. We have that $\ell(R_{\mathfrak{p}}/JR_{\mathfrak{p}}) + \ell(R_{\mathfrak{p}}/aR_{\mathfrak{p}}) = \ell(R_{\mathfrak{p}}/aJR_{\mathfrak{p}})$ so (5.1) is equivalent to $aJR_{\mathfrak{p}} = bR_{\mathfrak{p}}$.

Thus as ideals in R, aJ and bR have the same primary decomposition up to codimension one. Namely, there exist prime ideals $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ of height one and ideals Q, Q' of height two such that

$$\begin{aligned} aJ &= \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s \cap Q \\ bR &= \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s \cap Q'. \end{aligned}$$

Note that $\dim(\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s / aJ) \le d-2$ and similarly for the ideal bR. Moreover since a and b are both nonzerodivisor, $aJ \cong J$ and $bR \cong R$. By Lemma 3.1

$$\begin{aligned} \varphi_n(J) = \varphi_n(aJ) &= \varphi_n(\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s) + O(q^{d-2}) \\ &= \varphi_n(bR) + O(q^{d-2}) \\ &= \varphi_n(R) + O(q^{d-2}). \end{aligned}$$

□

The following lemma shows that if a torsion free module has zero cycle class, then its Hilbert-Kunz function is compatible with that of the free module of the same rank.

Lemma 5.2. *Let R be as in Lemma 5.1. Let M be a finitely generated torsion free module of rank r. Assume that $[M]_{d-1} = 0$ in $\mathrm{A}_{d-1}(R)$. Then $\varphi_n(M) = r\varphi_n(R) + O(q^{d-2})$.*

Proof. Since the rank of M is r, M contains an R-submodule N such that N is isomorphic to R^r and M/N is a torsion module. Then, there exists a non-zero element $a \in R$ such that $aM \subseteq N$. Since $aM \simeq M$, we may assume that M is a submodule of R^r. Consider $M \subseteq R^r \subseteq \overline{R}^r$. Let $m_1, \dots, m_s$ be a set of generators for M and let $\overline{M}$ denote the submodule $\overline{R}m_1 + \cdots + \overline{R}m_s$ of $\overline{R}^r$. We observe that $\dim \overline{M}/M \leq d-2$ since $\overline{M}_{\mathfrak{p}} = M_{\mathfrak{p}}$ for any prime of dimension d or $d-1$. In the normal ring $\overline{R}$, there exists an ideal $\mathfrak{a}$ that results the following short exact sequence

$$0 \longrightarrow \overline{R}^{r-1} \longrightarrow \overline{M} \longrightarrow \mathfrak{a} \longrightarrow 0. \tag{5.2}$$

Indeed $\mathfrak{a}$ is a Bourbaki ideal of $\overline{M}$ with respect to a free submodule of rank $r-1$. All modules in (5.2) are finitely generated R-modules and the sequence remains exact as the homomorphisms are viewed over R. Tensoring (5.2) by R/I_n over R, one obtains an exact sequence

$$0 \longrightarrow L \longrightarrow (\overline{R}/I_n\overline{R})^{r-1} \xrightarrow{\phi} \overline{M}/I_n\overline{M} \longrightarrow \mathfrak{a}/I_n\mathfrak{a} \longrightarrow 0$$

where L indicates the kernel of the homomorphism ϕ. This shows

$$\varphi_n(\mathfrak{a}) - \varphi_n(\overline{M}) + (r-1)\varphi_n(\overline{R}) = \ell(L) \geq 0. \tag{5.3}$$

Recall that $\dim \overline{R}/R \leq d-2$ and $\dim \overline{M}/M \leq d-2$. Thus applying Theorem 2.1(2), we have $[\overline{R}]_{d-1} = [R]_{d-1} = 0$ and $[\overline{M}]_{d-1} = [M]_{d-1} = 0$ in $\mathrm{A}_{d-1}(R)$ which further implies $[\mathfrak{a}]_{d-1} = [\overline{M}]_{d-1} - (r-1)[\overline{R}]_{d-1} = 0$. Initially $\mathfrak{a}$ is an ideal of the integral domain $\overline{R}$ so as an R-module, $\mathfrak{a}$ is also finitely generated torsion free of rank one. Also there exists an element c in R such that $c\mathfrak{a} \subseteq R$ as an ideal and $\varphi_n(c\mathfrak{a}) = \varphi_n(\mathfrak{a})$ since $\mathfrak{a}$ is isomorphic to $c\mathfrak{a}$. Moreover $[c\mathfrak{a}]_{d-1} = [\mathfrak{a}]_{d-1} = 0$. By (5.3), Lemmas 5.1 and 3.1, we have

$$\varphi_n(\overline{M}) \leq \varphi_n(\mathfrak{a}) + (r-1)\varphi_n(\overline{R}) = r\varphi_n(R) + O(q^{d-2})$$

and

$$\varphi_n(M) \leq r\varphi_n(R) + O(q^{d-2}). \tag{5.4}$$

On the other hand there exists a short exact sequence $0 \to L \to R^{r+s} \to M \to 0$ by resolution. The module L is torsion free of rank s and $[L]_{d-1} = 0$ again by

Theorem 2.1(b). We obtain another exact sequence by tensoring with R/I_n:

$$0 \longrightarrow L' \longrightarrow L/I_nL \longrightarrow (R/I_n)^{r+s} \longrightarrow M/I_nM \longrightarrow 0.$$

By (5.4), $\varphi_n(L) \leq s\varphi_n(R) + O(q^{d-2})$. A similar computation as above shows

$$\varphi_n(M) \geq (r+s)\varphi_n(R) - \varphi_n(L) \geq r\varphi_n(R) + O(q^{d-2}).$$

Hence $\varphi_n(M) = r\varphi_n(R) + O(q^{d-2})$ and the proof is completed. □

Lemma 5.3. *Let R be as in Lemma 5.1. Let M be a finitely generated torsion free R-module.*

(*a*) *If N is a finitely generated torsion free R-module such that $[N] = [M]$ in $\mathrm{A}_d(R) \oplus \mathrm{A}_{d-1}(R)$, then $\varphi_n(M) = \varphi_n(N) + O(q^{d-2})$.*

(*b*) $\ell(\mathrm{Tor}_1(M, R/I_n)) = O(q^{d-2})$.

Proof. The assumption $[M] = [N]$ indicates that M and N have the same rank and $[M]_{d-1} = [N]_{d-1}$ in $A_{d-1}(R)$. We write $[M]_{d-1} = \sum_{i=1}^{s}[R/\mathfrak{p}_i]$. Then the module $M \oplus (\oplus_i \mathfrak{p}_i)$ is a torsion free module of rank $r+s$ and determines the zero class in $\mathrm{A}_{d-1}(R)$ since $[\mathfrak{p}_i]_{d-1} = -[R/\mathfrak{p}_i]$. By Lemma 5.2, $M \oplus (\oplus_i \mathfrak{p}_i)$ has the Hilbert-Kunz function in the form of $(r+s)\varphi_n(R) + O(q^{d-2})$ and similarly for N. Thus we have

$$\varphi_n(M) = (r+s)\varphi_n(R) - \sum_{i=1}^{s}\varphi_n(\mathfrak{p}_i) + O(q^{d-2}) = \varphi_n(N) + O(q^{d-2}).$$

To prove (b), we consider a short exact sequence $0 \to G \to F \to M \to 0$ where F is a free module. This induces the exact sequence

$$0 \longrightarrow \mathrm{Tor}_1(M, R/I_n) \longrightarrow G/I_nG \longrightarrow F/I_nF \longrightarrow M/I_nM \longrightarrow 0.$$

Since $[F]_{d-1} = [G \oplus M]_{d-1}$ and both modules are torsion free, by the result of Part (a), we conclude that

$$\ell(\mathrm{Tor}_1(M, R/I_n)) = \varphi_n(G) - \varphi_n(F) + \varphi_n(M) = O(q^{d-2}).$$

□

Lemma 5.3 plays an important role in the discussion below that leads to Theorem 5.6.

To better analyze the Hilbert-Kunz functions for all finitely generated modules, the following definition is given to a torsion free module M of rank r:

$$\delta_n(M) = \varphi_n(M) - r\varphi_n(R).$$

The function $\delta_n(M)$ mainly measures the difference between the Hilbert-Kunz function of M and that of a free module of the same rank. Notice that $\delta_n(M)$ is a function of $O(q^{d-1})$ and it is of $O(q^{d-2})$ if $[M]_{d-1} = 0$ by Lemma 5.2. In particular $\delta_n(R) = 0$ and $\delta_n(M \oplus N) = \delta_n(M) + \delta_n(N)$. If $[M] = [N]$ in $\mathrm{A}_d(R) \oplus \mathrm{A}_{d-1}(R)$, then $\delta_n(M) = \delta_n(N) + O(q^{d-2})$ by Lemma 5.3 (a). In fact as already proved in (4.3), $\delta_n(M) = \tau q^{d-1} + O(q^{d-2})$ for some constant τ by taking normalization. Here we show that this can be achieved independently within the current framework using the next Theorem 5.4 which gives a recursive relation on $\delta_n(M)$ for a given M via the Frobenius map.

We recall the Frobenius map $f : R \to R$ on a ring R of characteristic p assuming R is complete with perfect residue class field, for any x in R, $f(x) = x^p$. For any R-module M, ${}^1M$ denotes the same additive group M with an R-module structure via f. Since f is a finite map, ${}^1R$ is a torsion free R-module of rank p^d. If M is a module of rank r then ${}^1M$ has rank $p^d r$ over R. The map on the Chow group $f^* : \mathrm{A}_{d-1}(R) \to \mathrm{A}_{d-1}(R)$ induced by the Frobenius map is multiplication by p^{d-1}. We observe $I_n \cdot_f M = (I^{[p^n]})^{[p]}M = I_{n+1}M$. Therefore $\varphi_n({}^1M) = \varphi_{n+1}(M)$.

Theorem 5.4. *Let R be an F-finite local domain regular in codimension one with perfect residue class field. Let M be a torsion free R-module of rank r. Then $\delta_{n+1}(M) = p^{d-1}\delta_n(M) + O(q^{d-2})$.*

Proof. First we claim that $[{}^1M]_{d-1} = p^{d-1}[M]_{d-1} + r[{}^1R]_{d-1}$. Indeed there exists an embedding of a free module F of rank r into M such that M/F is a torsion module over R. The sequence $0 \to {}^1F \to {}^1M \to {}^1(M/F) \to 0$ remains exact as R-modules via f by restriction. We have $[{}^1M]_{d-1} = [{}^1(M/F)]_{d-1} + r[{}^1R]_{d-1}$ in $\mathrm{A}_{d-1}(R)$. Notice that ${}^1F$ also has rank $p^d r$ so ${}^1(M/F)$ is torsion, and $[{}^1(M/F)]_{d-1} = f^*([M/F]_{d-1})$ since $\dim {}^1(M/F) \leq d-1$. Furthermore $[M/F]_{d-1} = [M]_{d-1}$ by the above exact sequence since $[R]_{d-1} = 0$. This shows $[{}^1(M/F)]_{d-1} = f^*([M]_{d-1}) = p^{d-1}[M]_{d-1}$ and $[{}^1M]_{d-1} = p^{d-1}[M]_{d-1} + r[{}^1R]_{d-1}$ as claimed.

Now we consider ${}^1M \oplus R^{p^{d-1}r}$ and $(M^{p^{d-1}}) \oplus ({}^1R)^r$. An extra copy of a free module is added to ${}^1M$ so that both modules have the same rank. These two modules define the same cycle class in $\mathrm{A}_d(R) \oplus \mathrm{A}_{d-1}(R)$ by the above claim. Since M is torsion free, obviously both modules above are torsion free and so their Hilbert-Kunz functions coincide up to $O(q^{d-2})$ by Lemma 5.3(a). The expected result follows straightforward computations and that $\varphi_n({}^1M) = \varphi_{n+1}(M)$:

$$\begin{aligned}
\varphi_n({}^1M) + p^{d-1}r\varphi_n(R) &= p^{d-1}\varphi_n(M) + r\varphi_n({}^1R) + O(q^{d-2}) \\
\varphi_{n+1}(M) - r\varphi_{n+1}(R) &= p^{d-1}\varphi_n(M) - p^{d-1}r\varphi_n(R) + O(q^{d-2}).
\end{aligned}$$

Hence $\delta_{n+1}(M) = p^{d-1}\delta_n(M) + O(q^{d-2})$. □

Lemma 5.5. *Let R be as in Theorem 5.4. Assume R has perfect residue field and $\dim R = d$. Let M be a torsion free module of rank r. Then there is a real constant $\tau(M)$ such that $\delta_n(M) = \tau(M)q^{d-1} + O(q^{d-2})$.*

The proof of Lemma 5.5 using Theorem 5.4 is very similar to the original one [9, Theorem 1.9]. A quick sketch can be done by setting $v_n(M) = \dfrac{\delta_n(M)}{q^{d-1}}$. Notice that $q = p^n$ varies as n does. By careful yet straightforward computations, one shows that $v_{n+1}(M) - v_n(M) = O(\frac{1}{p^n})$. Hence $v_n(M)$ converges to some constant, denoted $\tau(M)$, as $n \longrightarrow \infty$. As indicated earlier in Section 4, the value $\tau(M)$ is the same as $\tau([M]_{d-1})$ in Theorem 4.1. Discussions on the properties of $\tau(M)$ and its application can also be found in Section 4.

We conclude this appendix by describing the first and second coefficients of $\varphi_n(M)$ for an arbitrary finitely generated module. For an arbitrary torsion free module M of rank r, we have a comparison of its Hilbert-Kunz function with that of a free module of the same rank as an immediate corollary of Lemma 5.5:

$$\varphi_n(M) = r\varphi_n(R) + \tau(M)q^{d-1} + O(q^{d-2}). \tag{5.5}$$

Theorem 5.6. *Let R be as in Theorem 5.4. Let T be the torsion submodule of M. Then the coefficients $\alpha(M)$ and $\beta(M)$ in the Hilbert-Kunz function $\varphi_n(M)$ of M have the following properties:*
(a) $\alpha(M) = r\alpha(R)$;
(b) $\beta(M) = r\beta(R) + \beta(T) + \tau(M/T)$.

Proof. We set $M' = M/T$. The short exact sequence $0 \to T \to M \to M' \to 0$ induces the following long exact sequence

$$\cdots \longrightarrow \mathrm{Tor}_1(M', R/I_n) \longrightarrow T/I_nT \longrightarrow M/I_nM \longrightarrow M'/I_nM' \longrightarrow 0.$$

In the above M' is a torsion free module. By Lemma 5.3(b), we have $\varphi_n(M) = \varphi_n(T) + \varphi_n(M') + O(q^{d-2})$. Note that M' is a torsion free module of rank r and $\dim T \leq d-1$ since T is torsion. Thus using Monsky's original work [16] for T, (5.5) for M' and Theorem 3.2 for R, we have

$$\begin{aligned}
\varphi_n(M) &= \varphi_n(T) + r\varphi_n(R) + \tau(M')q^{d-1} + O(q^{d-2}) \\
&= \beta(T)q^{d-1} + r\alpha(R)q^d + r\beta(R)q^{d-1} + \tau(M')q^{d-1} + O(q^{d-2}) \\
&= r\alpha(R)q^d + (r\beta(R) + \beta(T) + \tau(M'))q^{d-1} + O(q^{d-2}).
\end{aligned}$$

Hence the constants $\alpha(M)$ and $\beta(M)$ have the desired forms. □

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