ASYMPTOTIC REGULARITY OF POWERS OF IDEALS OF POINTS IN A WEIGHTED PROJECTIVE PLANE

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Dedicated to the memory of Professor Masayoshi Nagata

ABSTRACT. In this paper we study the asymptotic behavior of the regularity of symbolic powers of ideals of points in a weighted projective plane. By a result of Cutkosky, Ein and Lazarsfeld [4], regularity of such powers behaves asymptotically like a linear function, which is deeply related to the Seshadri constant of a blow-up. We study the difference between regularity of such powers and this linear function. Under some conditions, we prove that this difference is bounded, or eventually periodic.

As a corollary we show that, if there exists a negative curve, then the regularity of symbolic powers of a monomial space curve is eventually a periodic linear function. We give a criterion for the validity of Nagata's conjecture in terms of the lack of existence of negative curves.

1. INTRODUCTION

Suppose that H is an ample $\mathbb{Q}$-Cartier divisor on a normal projective variety V, and $\mathcal{I}$ is an ideal sheaf on V. Let $\nu : W \to V$ be the blow up of $\mathcal{I}$. Let E be the effective Cartier divisor on W defined by $\mathcal{O}_W(-E) = \mathcal{I}\mathcal{O}_W$. The s-invariant, $s_{\mathcal{O}_V(H)}(\mathcal{I})$, is defined by

$$s_{\mathcal{O}_V(H)}(\mathcal{I}) = \inf\{s \in \mathbb{R} \mid \nu^*(sH) - E \text{ is an ample } \mathbb{R}\text{-divisor on } W\}.$$

The reciprical, $\frac{1}{s_H(\mathcal{I})}$, is the Seshadri constant of $\mathcal{I}$.

Examples in [3] and [4] show that $s_H(\mathcal{I})$ can be irrational, even when $\mathcal{O}_V(H) \cong \mathcal{O}_{\mathbb{P}}(1)$ on ordinary projective space.

Suppose that K is a field, and $a_0, \ldots, a_{\overline{n}}$ are positive integers. Let $S = K[x_0, \ldots, x_{\overline{n}}]$ be a polynomial ring, graded by the weighting $\mathrm{wt}(x_i) = a_i$ for $0 \le i \le \overline{n}$. Let $\mathfrak{m}$ be the graded maximal ideal of S. Let $\mathbb{P} = \mathbb{P}(a_0, a_1, \ldots, a_{\overline{n}}) = \mathrm{proj}(S)$ be the associated weighted projective space. $\mathbb{P}$ is a normal projective variety. $\mathbb{P}$ is isomorphic to a weighted projective space in which $\gcd(a_0, a_1, \ldots, a_{i-1}, a_{i+1} \ldots, a_{\overline{n}}) = 1$ for $0 \le i \le \overline{n}$ [8], [10].

We will suppose through most of this paper that $a_0, \ldots, a_{\overline{n}} \in \mathbb{Z}_+$ satisfy the condition that $\gcd(a_0, a_1, \ldots, a_{i-1}, a_{i+1} \ldots, a_{\overline{n}}) = 1$ for $0 \le i \le \overline{n}$. With this assumption on the a_i, there exists a Weil divisor H on $\mathbb{P}$ such that $\mathcal{O}_{\mathbb{P}}(r) \cong \mathcal{O}_{\mathbb{P}}(rH)$ is a divisorial sheaf of $\mathcal{O}_{\mathbb{P}}$ modules (reflexive of rank 1) for all $r \in \mathbb{Z}$ [25].

Let M be a finitely generated, graded S-module. The local cohomology modules $H^i_{\mathfrak{m}}(M)$ are naturally graded. The regularity of M is defined ([12]) by

$$\mathrm{reg}(M) = \max\{i + j \mid H^i_{\mathfrak{m}}(M)_j \neq 0\}.$$

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Suppose that $I \subset S$ is a homogeneous ideal. Let $I^{\rm sat}$ be the saturation of I with respect to the graded maximal ideal of I. Let $\mathcal{I}$ be the sheaf associated to I on $\mathbb{P}$. Let $X = X(I) = \mathrm{proj}(\bigoplus_{m\ge 0} \mathcal{I}^m)$ be the blow up of $\mathcal{I}$, with natural projection $f : X \to \mathbb{P}$.

In Section 2, we develop the basic properties of regularity on weighted projective space, and show that the theory of asymptotic regularity on ordinary projective space extends naturally to weighted projective space. For instance, the statement on ordinary projective space, proven in Theorem 1.1 of [5], or in [21], extends to show that $\mathrm{reg}(I^m)$ is a linear function for $m \gg 0$. We also establish the following basic result, which generalizes the statement for ample line bundles proven in Theorem B of [4] to the $\mathbb{Q}$-Cartier Weil divisor $\mathcal{O}_{\mathbb{P}}(1)$ on weighted projective space (with the a_i pairwise relatively prime).

Theorem 1.1. *We have that*

$$\lim_{m\to\infty} \frac{\mathrm{reg}((I^m)^{\rm sat})}{m} = s_{\mathcal{O}_{\mathbb{P}}(1)}(\mathcal{I}).$$

In general, as commented above, this limit is irrational, so $\mathrm{reg}((I^m)^{\rm sat})$ is in general far from being a linear function.

We will write $\lfloor x \rfloor$ for the greatest integer in a real number x. We may define a function $\sigma_I : \mathbb{N} \to \mathbb{Z}$ by

$$\mathrm{reg}((I^m)^{\rm sat}) = \lfloor m s_{\mathcal{O}_{\mathbb{P}}(1)}(\mathcal{I}) \rfloor + \sigma_I(m).$$

By Theorem 1.1, we have that

$$\lim_{m\to\infty} \frac{\sigma_I(m)}{m} = 0.$$

An interesting question is to determine when $\sigma_I(m)$ is bounded. We do not know of an example where $\sigma_I(m)$ is not bounded.

In this paper, we study the case where $I \subset S = K[x,y,z]$ is the ideal of a set of nonsingular points with multiplicity (a "fat point") in a weighted two dimensional projective space $\mathbb{P} = \mathbb{P}(a,b,c)$, with $\mathrm{wt}(x) = a$, $\mathrm{wt}(y) = b$, $\mathrm{wt}(z) = c$ and a,b,c pairwise relatively prime. We also assume that K is algebraically closed. Suppose that $P_1, \ldots, P_r$ are distinct nonsingular closed points in $\mathbb{P}(a,b,c)$, and e_i are positive integers. Let $I_{P_i} \subset S = K[x,y,z]$ be the weighted homogeneous ideal of the point P_i, and let $I = \cap_{i=0}^r I_{P_i}^{e_i}$. Let $\mathcal{I} = \tilde{I}$ be the sheafication of I on $\mathbb{P}$, and define

$$s(I) = s_{\mathcal{O}_{\mathbb{P}}(1)}(\mathcal{I}).$$

Let $u = \sum_{i=1}^r e_i^2$. We have that $s(I) \ge \sqrt{abcu}$. If $s(I) > \sqrt{abcu}$, then $s(I)$ is a rational number.

Nagata's conjecture states that $s(I) = \sqrt{r}$ if $r \ge 9$, $e_i = 1$ for $1 \le i \le r$ and $P_1, \ldots, P_r$ are independent generic points in ordinary projective space $\mathbb{P}^2$. Nagata proved this conjecture in [27] in the case that r is a perfect square, as a critical ingredient in his counterexample to Hilbert's fourteenth problem. A proof of Nagata's conjecture in the case of an r which is not a perfect square would give a set of points in $\mathbb{P}^2$ for which $s(I)$ is not rational. Some recent papers on regularity and s-invariants of points in $\mathbb{P}^2$ are [1], [9], [14], [17] and [20].

Let $I^{(m)}$ be the m-th symbolic power of I, which is also, in our situation, the saturation $(I^m)^{\rm sat}$ of I^m with respect to the graded maximal ideal $\mathfrak{m}$ of S.

We prove the following asymptotic statements about regularity in Section 3.

A function $\sigma : \mathbb{N} \to \mathbb{Z}$ is bounded if there exists $c \in \mathbb{N}$ such that $|\sigma(m)| < c$ for all $m \in \mathbb{N}$. In Theorem 4.6, we prove:

Let I be the ideal of a set of fat points in a weighted projective plane. Then $\sigma_I(m)$ is a bounded function.

If the graded K-algebra $\bigoplus_{m\geq 0} I^{(m)}$ is a finitely generated K-algebra, then $\mathrm{reg}(I^{(m)})$ must be a quasi polynomial for large m. A quasi polynomial is a polynomial in m with coefficients which are periodic functions in m. In general, $\bigoplus_{m\geq 0} I^{(m)}$ is not a finitely generated K-algebra. Some examples where this algebra is not finitely generated are given by Nagata's Theorem [27], showing that it is not finitely generated when $r \geq 9$ is a perfect square and r generic points in $\mathbb{P}^2$ are blown up. Goto, Nishida and Watanabe [15] give examples of monomial primes $P(a, b, c)$ such that the symbolic algebra is not finitely generated.

A function $\sigma : \mathbb{N} \to \mathbb{Z}$ is *eventually periodic* if $\sigma(m)$ is periodic for $m \gg 0$. In Theorem 4.7, we prove:

Suppose that $s(I) > \sqrt{abcu}$ and K has characteristic zero or is the algebraic closure of a finite field. Then the function $\sigma_I(m)$ is eventually periodic.

An example, defined over a field K which is of positive characteristic and is transcendental over the prime field, where $s(I) > \sqrt{abcu}$ but $\sigma(m)$ is not eventually periodic, is given in Example 4.4 [5]. In this example, constructed from 17 special points P_i in ordinary projective space $\mathbb{P}^2$, $I = I_{P_1} \cap \cdots \cap I_{P_{13}} \cap I^2_{P_{14}} \cap \cdots \cap I^2_{P_{17}}$. We have $s(I) = \frac{29}{5} > \sqrt{abcu} = \sqrt{29}$ in this example.

Let H be a Weil divisor on $\mathbb{P}$ such that $\mathcal{O}_{\mathbb{P}}(1) \cong \mathcal{O}_{\mathbb{P}}(H)$ and let $A = f^*(H)$. An effective divisor D on X such that $(D \cdot D) < 0$ will be called a negative curve. An effective divisor D such that $D \sim aA - mE$ for some positive integers a and m will be called an E-uniform curve.

We establish in Corollary 4.8 that

Suppose there exists an E-uniform negative curve on $X(I)$, and K has characteristic zero, or is a finite field. Then $s(I)$ is a rational number and the function $\sigma_I(m)$ of Theorem 4.6 is eventually periodic.

An important case of this construction is when $i = 1$, and $I = P(a, b, c)$ is the prime ideal of a monomial space curve. The ideal $P(a, b, c)$ is defined to be the kernel of the K-algebra homomorphism $K[x, y, z] \to K[t]$, given by $x \mapsto t^a$, $y \mapsto t^b$, $z \mapsto t^c$. As a corollary to Theorems 4.6 and 4.7, we have the following application, Corollary 4.9, to monomial space curves.

Suppose that $I = P(a, b, c)$ is the prime ideal of a monomial space curve, and there exists a negative curve on $X(I)$. Then $s(I)$ is a rational number, and the function $\sigma_I(m)$ is eventually periodic.

We do not know of an example of a monomial prime $I = P(a, b, c)$ where there does not exist a negative curve. This interesting problem is discussed in [22].

In Section 5, we give a criterion for the validity of Nagata's conjecture in terms of the lack of existence of uniform negative curves on certain weighted projective planes.

2. Regularity on Weighted Projective Space

In this section we define the regularity of a finitely generated graded module over a non-standard graded polynomial ring.

Let K be a field and $B = K[x_1, \dots, x_s]$ be a graded polynomial ring with $\mathrm{wt}(x_1) = d_1$, $\dots$, $\mathrm{wt}(x_s) = d_s$, where d_1, $\dots$, d_s are positive integers. Set $\mathfrak{m} = (x_1, \dots, x_s)B$.

Definition 2.1. For a finitely generated B-module $M \neq 0$, We define $a_i(M)$, $\mathrm{reg}(M)$, $\mathrm{reg}_i(M)$ and $\mathrm{reg}'(M)$ as follows:

$$\begin{array}{lrcl} & a_i(M) & = & \begin{cases} \max\{j \in \mathbb{Z} \mid H^i_{\mathfrak{m}}(M)_j \neq 0\} & \text{if } H^i_{\mathfrak{m}}(M) \neq 0 \\ -\infty & \text{otherwise} \end{cases} \\ (1) & \mathrm{reg}(M) & = & \max\{i + j \mid H^i_{\mathfrak{m}}(M)_j \neq 0\} = \max\{a_i(M) + i \mid 0 \leq i \leq \dim M\} \\ & \mathrm{reg}_i(M) & = & \begin{cases} \max\{j \in \mathbb{Z} \mid \mathrm{Tor}^B_i(M, B/\mathfrak{m})_j \neq 0\} - i & \text{if } \mathrm{Tor}^B_i(M, B/\mathfrak{m}) \neq 0 \\ -\infty & \text{otherwise} \end{cases} \\ & \mathrm{reg}'(M) & = & \max\{\mathrm{reg}_i(M) \mid i \geq 0\} \end{array}$$

It is not difficult to prove the following theorem (cf. Theorem 3.5 in [7]). We omit a proof.

Theorem 2.2. *With notation as above,*

$$\mathrm{reg}(M) = \mathrm{reg}'(M) + s - \sum_{i=1}^{s} d_i.$$

Remark 2.3. Assume $d_1 = \cdots = d_s = 1$. Let I be a homogeneous ideal of B. Put $\mathbb{P} = \mathrm{proj}(B)$.

The regularity of a coherent $\mathcal{O}_{\mathbb{P}}$ module $\mathcal{F}$ is

$$\mathrm{reg}(\mathcal{F}) = \min\{l \mid H^i(\mathbb{P}, \mathcal{F}(j - i)) = 0 \text{ for all } j \geq l \text{ and } i > 0\}. \tag{2}$$

Let $\mathcal{I}$ be the ideal sheaf on $\mathbb{P}$ associated to I. We have that

$$\mathrm{reg}((I^m)^{\mathrm{sat}}) = \mathrm{reg}(\mathcal{I}^m) \tag{3}$$

for all $m \geq 0$, as follows from Theorem A4.1 [11].

For each $i \geq 0$, $\mathrm{reg}_i(I^m)$ is eventually linear on m by Theorem 3.1 in [5].

Hence, $\mathrm{reg}(I^m)$ is eventually linear on m as in Theorem 1.1 (ii) in [5].

On the other hand, there exists an example that $a_i(I^m)$ is not eventually linear on m as follows. Let I be the ideal in Example 4.4 in [5] (which was refered to in the introduction). Then,

$$\mathrm{reg}((I^m)^{\mathrm{sat}}) = \max\{a_2((I^m)^{\mathrm{sat}}) + 2, a_3((I^m)^{\mathrm{sat}}) + 3\} = \max\{a_2(I^m) + 2, 0\} = a_2(I^m) + 2$$

since $\mathrm{reg}((I^m)^{\mathrm{sat}}) = \mathrm{reg}'((I^m)^{\mathrm{sat}}) > 0$, and $\mathrm{reg}((I^m)^{\mathrm{sat}})$ is not eventually linear in m. Therefore, $a_2(I^m)$ is not eventually linear in m in this case.

In Section 3, we will consider the case when $S = K[x, y, z]$, with $\mathrm{wt}(x) = a$, $\mathrm{wt}(y) = b$ and $\mathrm{wt}(z) = c$ for pairwise relatively prime positive integers a, b, c, and $I = I_{P_1}^{e_1} \cap \cdots \cap I_{P_r}^{e_r}$ with P_i distinct nonsingular points of $\mathbb{P}(a, b, c)$. We will then have that

$$\mathrm{reg}((I^m)^{\mathrm{sat}}) = \max\{a_2((I^m)^{\mathrm{sat}})+2, a_3((I^m)^{\mathrm{sat}})+3\} = \max\{a_2(I^m)+2, 3-a-b-c\} = a_2(I^m)+2$$

since $\mathrm{reg}((I^m)^{\mathrm{sat}}) = \mathrm{reg}'((I^m)^{\mathrm{sat}}) + 3 - a - b - c > 3 - a - b - c$, and (with the notation defined in the introduction)

$$a_2(I^m) = \max\{n \in \mathbb{Z} \mid H^1(X, \mathcal{O}_X(nA - mE)) \neq 0\}.$$

Remark 2.4. Let $B_1 = K[x_1, \ldots, x_s]$ and $B_2 = K[y_1, \ldots, y_s]$ be graded polynomial rings with $\mathrm{wt}(x_i) = d_i$ $(i = 1, \ldots, s)$ and $\mathrm{wt}(y_j) = d'_j$ $(j = 1, \ldots, s)$, where the d_i's and d'_j's are positive integers.

Let $\delta : B_1 \to B_2$ be a flat K-algebra graded homomorphism. Assume that $B_2/(x_1, \ldots, x_s)B_2$ is of finite length.

Set $\mathfrak{m}_1 = (x_1, \ldots, x_s)B_1$ and $\mathfrak{m}_2 = (y_1, \ldots, y_s)B_2$.

Let M be a finitely generated graded B_1-module. Then,

$$H^i_{\mathfrak{m}_1}(M) \otimes_{B_1} B_2 = H^i_{\mathfrak{m}_1 B_2}(M \otimes_{B_1} B_2) = H^i_{\mathfrak{m}_2}(M \otimes_{B_1} B_2). \tag{4}$$

Here, set

$$\xi = \max\{n \in \mathbb{Z} \mid [B_2/\mathfrak{m}_1 B_2]_n \neq 0\}.$$

By (4), we obtain

$$a_i(M \otimes_{B_1} B_2) = a_i(M) + \xi$$

for $i = 0, \ldots, \dim M$. Therefore,

$$\mathrm{reg}(M \otimes_{B_1} B_2) = \mathrm{reg}(M) + \xi. \tag{5}$$

From the above Remark, we see that the statement on ordinary projective space, proven in Theorem 1.1 of [5], or in [21], extends to show that $\mathrm{reg}(I^m)$ is a linear function for $m \gg 0$, when I is a homogeneous ideal with respect to our weighting.

Remark 2.5. Let $B_1 = K[x_1, \ldots, x_s]$ and $B_2 = K[y_1, \ldots, y_s]$ be graded polynomial rings with $\mathrm{wt}(x_i) = d_i$ $(i = 1, \ldots, s)$ and $\mathrm{wt}(y_j) = 1$ $(j = 1, \ldots, s)$, where the d_i's are positive integers satisfying the condition $\gcd(d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_s) = 1$ for $1 \leq i \leq s$.

Let $\delta : B_1 \to B_2$ be the K-algebra graded homomorphism satisfying $\delta(x_i) = y_i^{d_i}$ for $i = 1, \ldots, s$. Remark that δ is flat, $B_2/(x_1, \ldots, x_s)B_2$ is of finite length and

$$\xi = \max\{n \in \mathbb{Z} \mid [B_2/(x_1, \ldots, x_s)B_2]_n \neq 0\} = \sum_{i=1}^{s} d_i - s.$$

Let I be a homogeneous ideal of B_1. Then,

$$(I^m)^{\mathrm{sat}} \otimes_{B_1} B_2 = (I^m)^{\mathrm{sat}} B_2 = (I^m B_2)^{\mathrm{sat}}$$

for any $m > 0$. Thus, using (5), we have that

$$\mathrm{reg}((I^m B_2)^{\mathrm{sat}}) = \mathrm{reg}((I^m)^{\mathrm{sat}}) + \sum_{i=1}^{s} d_i - s$$

for any $m > 0$. Therefore,

$$\lim_{m \to \infty} \frac{\mathrm{reg}((I^m B_2)^{\mathrm{sat}})}{m} = \lim_{m \to \infty} \frac{\mathrm{reg}((I^m)^{\mathrm{sat}})}{m}. \tag{6}$$

Here, set $X = \mathrm{proj}(B_1)$, $Z = \mathrm{proj}(B_2)$. By our condition on the d_i, there exists a Weil divisor H on X such that $\mathcal{O}_X(r) \cong \mathcal{O}_X(rH)$ is a reflexive, rank 1 sheaf of $X_{\mathbb{P}}$ modules for all $r \in \mathbb{Z}$ [25]. Let Y (resp. W) be the blow-ups of X (resp. Z) along the ideal sheaf $\mathcal{I} = \tilde{I}$ (resp. $\mathcal{I}\mathcal{O}_Z = \widetilde{IB_2}$). Then, we have the following Cartesian diagram:

$$\begin{array}{ccc} W & \longrightarrow & Z \\ f \downarrow & & \downarrow \\ Y & \xrightarrow{\pi} & X \end{array}$$

Set $E = \pi^{-1}(\mathrm{proj}(B_1/I))$. Let ℓ be a positive integer such that $\mathcal{O}_X(\ell)$ is invertible (we can take $\ell = \mathrm{lcm}(d_1, \ldots, d_s)$). By the projection formula, for any positive integers α and

β, $\mathcal{O}_Y(-\ell\beta E)\otimes\pi^*\mathcal{O}_X(\ell\alpha)$ is nef if and only if so is $f^*(\mathcal{O}_Y(-\ell\beta E)\otimes\pi^*\mathcal{O}_X(\ell\alpha))$. Hence, $\alpha/\beta\ge s_{\mathcal{O}_Z(1)}(\mathcal{I}\mathcal{O}_Z)$ if and only if $\alpha/\beta\ge s_{\mathcal{O}_X(1)}(\mathcal{I})$. Therefore, we obtain

$$s_{\mathcal{O}_Z(1)}(\mathcal{I}\mathcal{O}_Z)=s_{\mathcal{O}_X(1)}(\mathcal{I}). \tag{7}$$

On the other hand,

$$\lim_{m\to\infty}\frac{\operatorname{reg}((I^mB_2)^{\mathrm{sat}})}{m}=s_{\mathcal{O}_Z(1)}(\mathcal{I}\mathcal{O}_Z) \tag{8}$$

by Theorem B in [4]. By (6), (7) and (8), we obtain

$$\lim_{m\to\infty}\frac{\operatorname{reg}((I^m)^{\mathrm{sat}})}{m}=s_{\mathcal{O}_X(1)}(\mathcal{I}), \tag{9}$$

which is the statement of Theorem 1.1.

3. Blow ups of a Weighted Projective Plane

Suppose that G is a subgroup of $\mathbb{R}$. Then G_+ will denote the semigroup of positive elements of G, and $G_{\ge 0}$ will denote the semigroup of nonnegative elements of G.

In this section, we will suppose that K is an algebraically closed field, and $a,b,c\in\mathbb{Z}_+$ are pairwise relatively prime. Let $\mathbb{P}=\mathbb{P}(a,b,c)$ be the corresponding weighted projective space. Suppose that $P_1,\ldots,P_r$ are distinct nonsingular closed points in $\mathbb{P}(a,b,c)$, and $e_1,\ldots,e_r\in\mathbb{Z}_+$.

The coordinate ring of $\mathbb{P}(a,b,c)$ is the graded polynomial ring

$$S=K[x,y,z]=\bigoplus_{n\ge 0}H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(n)),$$

which is graded by $\mathrm{wt}(x)=a$, $\mathrm{wt}(y)=b$, $\mathrm{wt}(z)=c$. Let $\mathfrak{m}=(x,y,z)$ be the graded maximal ideal of S.

Some references on the geometry of weighted projective spaces are [10] and [25]. We have that $\mathbb{P}(a,b,c)$ is a normal surface, which is nonsingular, except possibly at the three points $Q_1=V(x,y)$, $Q_2=V(x,z)$ and $Q_3=V(y,z)$. Since a,b,c are pairwise relatively prime, there exists a Weil divisor H on $\mathbb{P}$ such that $\mathcal{O}_{\mathbb{P}}(r)\cong\mathcal{O}_{\mathbb{P}}(rH)$ is a reflexive, rank 1 sheaf of $\mathcal{O}_{\mathbb{P}}$ modules for all $r\in\mathbb{Z}$ [25]. The canonical divisor on $\mathbb{P}$ is $\mathcal{O}_{\mathbb{P}}(-a-b-c)$. We have that $\mathcal{O}_{\mathbb{P}}(\ell)$ is an ample invertible sheaf if $\ell=\operatorname{lcm}(a,b,c)$.

Suppose that L is a finitely generated graded S-module. Recall (Section 2) that the regularity, reg$L$, of L is the largest integer t such that there exists an index j such that $H^j_{\mathfrak{m}}(L)_{t-j}\ne 0$. We will denote the sheaf associated to L on $\mathbb{P}$ by $\tilde{L}$.

For all $j\in\mathbb{Z}$, we have a natural morphism of sheaves of $\mathcal{O}_{\mathbb{P}}$ modules

$$\Lambda:\tilde{L}(j):=\tilde{L}\otimes\mathcal{O}_{\mathbb{P}}(j)\to\widetilde{L(j)}$$

which is an isomorphism whenever $\mathcal{O}_{\mathbb{P}}(j)$ is Cartier.

Let $I_{P_i}\subset K[x,y,z]$ be the weighted homogeneous ideal of the point P_i, and let $I=\cap_{i=0}^r I_{P_i}^{e_i}$.

An important case is when $i=1$, and $I=P(a,b,c)$ is the prime ideal of a monomial space curve.

Let $\mathcal{I}=\tilde{I}$ be the sheaf associated to I on $\mathbb{P}$. For $m\in\mathbb{N}$ and $n\in\mathbb{Z}$,

$$\mathcal{I}^m(n)\cong\widetilde{I^m(n)}\cong\mathcal{I}^m\otimes\mathcal{O}_{\mathbb{P}}(n),$$

since $\mathcal{O}_{\mathbb{P}}(1)$ is locally free at the support of $\mathcal{O}/\mathcal{I}$.

Let $I^{(m)}$ be the m-th symbolic power of I, which is also, in our situation, the saturation $(I^m)^{\mathrm{sat}}$ of I^m with respect to the graded maximal ideal $\mathfrak{m}$ of S. We have, as follows from Theorem A4.1 [11], that the graded local cohomology satisfies

$$H^0_{\mathfrak{m}}(I^{(m)}) = H^1_{\mathfrak{m}}(I^{(m)}) = 0 \tag{10}$$

for all $m \in \mathbb{N}$, and for $i \geq 1$, we have graded isomorphisms

$$H^{i+1}_{\mathfrak{m}}(I^{(m)}) \cong \bigoplus_{n\in\mathbb{Z}} H^i(\mathbb{P}, \mathcal{I}^m(n)) \cong \bigoplus_{n\in\mathbb{Z}} H^i(\mathbb{P}, \mathcal{I}^m \otimes \mathcal{O}_{\mathbb{P}}(n)). \tag{11}$$

Let $f : X = X(I) \to \mathbb{P}(a, b, c)$ be the blow up of these points. Let E_i be the exceptional curves mapping to P_i for $1 \leq i \leq r$, and let $E = e_1E_1 + \cdots + e_rE_r$. Let A be a Weil divisor on X such that $\mathcal{O}_X(A) \cong f^*\mathcal{O}_{\mathbb{P}}(1)$ (recall that $\mathcal{O}_{\mathbb{P}}(1)$ is locally free at points where f is not an isomorphism).

Since f is the blow up of the nonsingular points $P_1, \ldots, P_r$, for $m \geq 0$, $f_*\mathcal{O}_X(-mE) \cong \mathcal{I}^m$ and $R^if_*\mathcal{O}_X(-mE) = 0$ for $i > 0$ and $m \geq 0$ (for instance by Proposition 10.2 [24]). Since $\mathcal{O}_{\mathbb{P}}(1)$ is locally free above all points on $\mathbb{P}$ where f is not an isomorphism, by the projection formula,

$$f_*\mathcal{O}_X(nA - mE) \cong \mathcal{I}^m \otimes \mathcal{O}_{\mathbb{P}}(n) \cong \mathcal{I}^m(n)$$

for all $m \in \mathbb{N}$ and $n \in \mathbb{Z}$. By the Leray spectral sequence, we have that

$$H^i(X, \mathcal{O}_X(nA - mE)) \cong H^i(\mathbb{P}, \mathcal{I}^m(n)) \tag{12}$$

for all $m \in \mathbb{N}$, $n \in \mathbb{Z}$ and $i \geq 0$.

The m-th symbolic power of I can be computed as

$$I^{(m)} = \bigoplus_{n\geq 0} H^0(X, \mathcal{O}_X(nA - mE)).$$

Let $u = \sum_{i=1}^r e_i^2$.

We have

$$(A \cdot A) = \frac{1}{abc},\ (E_i \cdot E_i) = -1,\ (A \cdot E_i) = 0,\ (E \cdot E) = -u \text{ and } (A \cdot E) = 0. \tag{13}$$

Let $\mathrm{Div}(X)$ be the group of Weil divisors on X. There is an intersection theory on $\mathrm{Div}(X)$, developed in [26], which associates to Weil divisors D_1 and D_2 on X a rational number $(D_1 \cdot D_2)$. Divisors D_1 and D_2 are numerically equivalent, written $D_1 \equiv D_2$ if $(D_1 \cdot C) = (D_2 \cdot C)$ for every Weil divisor C on X. A $\mathbb{Q}$-divisor D on X is called numerically ample if $(D \cdot C) > 0$ for all curves C on X and $(D \cdot D) > 0$. Let $N_1(X) = (\mathrm{Div}(X)/\equiv) \otimes \mathbb{R}$. We will write $\overline{D}$ to denote the class in $N_1(X)$ of a Weil divisor D on X.

Let L be the real vector subspace of $N_1(X)$ spanned by (the classes of) E and A. Let $NL = \overline{\mathrm{NE}}(X) \cap L$ and let $AL = \overline{\mathrm{AMP}}(X) \cap L$, where $\overline{\mathrm{NE}}(X)$ is the closure of the cone of curves on X, and $\overline{\mathrm{AMP}}(X)$ is the closure of the ample cone on X.

We now make a sketch of the cones NL and AL. NL is a cone with boundary rays $E\mathbb{R}_{\geq 0}$ and $R = (\tau A - E)\mathbb{R}_{\geq 0}$ for some $\tau = \tau(I) \in \mathbb{R}$. AL is a cone with boundary rays $A\mathbb{R}_{\geq 0}$ and $T = (sA - E)\mathbb{R}_{\geq 0}$, where $s = s(I) = s_{\mathcal{O}_{\mathbb{P}(1)}}(\mathcal{I}) \in \mathbb{R}$ is the s-invariant of I.

Let $g : Y \to X$ be the minimal resolution of singularities. Let $\mathcal{L}$ be a line bundle on X. X has rational singularities implies $H^i(Y, \mathcal{M}) = H^i(X, \mathcal{L})$ where $\mathcal{M} = g^*(\mathcal{L})$. By the Riemann Roch Theorem on Y, ($\chi(\mathcal{O}_Y) = 1$ since Y is rational)

$$\chi(\mathcal{L}) = \chi(\mathcal{M}) = \frac{1}{2}(\mathcal{M} \cdot \mathcal{M} \otimes \omega_Y^{-1}) + 1.$$

By the projection formula,

$$\chi(\mathcal{L}) = \frac{1}{2}(\mathcal{L} \cdot \mathcal{L} \otimes \omega_X^{-1}) + 1. \tag{14}$$

Since X has rational singularities, $g_*\omega_Y = \omega_X = \mathcal{O}_X(-(a+b+c)A + E_1 + \cdots + E_r)$.

Proposition 3.1. *We have vanishing of cohomology $H^2(X, \mathcal{O}_X(\alpha A - \beta E)) = 0$ if $\beta \geq 0$ and $\alpha > -(a+b+c)$.*

Proof. We have $H^2(X, \mathcal{O}_X(\alpha A - \beta E)) \cong H^2(\mathbb{P}, \mathcal{I}^\beta(\alpha))$. From the exact sequence

$$0 \to \mathcal{I}^\beta(\alpha) \to \mathcal{O}_{\mathbb{P}}(\alpha) \to (\mathcal{O}_{\mathbb{P}}/\mathcal{I}^\beta)(\alpha) \to 0$$

and the fact that $(\mathcal{O}_{\mathbb{P}}/\mathcal{I}^\beta)(\alpha)$ has zero dimensional support, we have that

$$H^2(\mathbb{P}, \mathcal{I}^\beta(\alpha)) \cong H^2(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(\alpha)) = 0$$

for $\alpha > -(a+b+c)$. $\square$

From (13), (14) and Proposition 3.1 we see that $\sqrt{abcu}A - E \in NL$, since

$$((\sqrt{abcu}A - E) \cdot (\sqrt{abcu}A - E)) = 0.$$

Thus

$$0 < \tau(I) \leq \sqrt{abcu} \leq s(I). \tag{15}$$

Suppose that D is a Weil divisor on X. $g^*(D)$ is defined in [26] as the $\mathbb{Q}$-divisor on Y which agrees with the strict transform of D away from the exceptional locus of g, and has intersection number 0 with all exceptional curves. If $F = \sum \alpha_i E_i$ is a $\mathbb{Q}$-divisor on Y (with $\alpha_i \in \mathbb{Q}$), then we define a $\mathbb{Z}$-divisor by $\lfloor F \rfloor = \sum \lfloor \alpha_i \rfloor E_i$. If $\mathcal{L}$ is a line bundle on X, then as for instance follows from the projection formula of Theorem 2.1 of [28],

$$H^0(X, \mathcal{O}_X(D) \otimes \mathcal{L}) = H^0(Y, \mathcal{O}_Y(\lfloor g^*(D) \rfloor) \otimes g^*\mathcal{L}). \tag{16}$$

Lemma 3.2. *Suppose $\mathcal{F}$ is a coherent sheaf on X and $\mathcal{B}$ is a line bundle on X.*

1. *The Euler characteristic $\chi(\mathcal{F} \otimes \mathcal{B}^n)$ is a polynomial in n for $n \in \mathbb{N}$.*
2. *If $(\mathcal{B} \cdot \mathcal{O}_X(A)) > 0$ then $H^2(X, \mathcal{F} \otimes \mathcal{B}^n) = 0$ for $n \gg 0$.*

Proof. We first prove 1. Let $\mathcal{M}$ be an ample line bundle on the projective surface X. Thus there is a composition series of $\mathcal{F}$ by $\mathcal{O}_X$ modules $\mathcal{O}_{Z_i} \otimes \mathcal{M}^{e_i}$ with $1 \leq i \leq m$, where m is a positive integer, Z_i are (integral) subvarieties of X and $e_i \in \mathbb{Z}$ (c.f. Section 7 of Chapter 1 of [19]). Thus

$$\chi(\mathcal{F} \otimes \mathcal{B}^n) = \sum_{i=1}^{m} \chi(\mathcal{O}_{Z_i} \otimes \mathcal{M}^{e_i} \otimes \mathcal{B}^n). \tag{17}$$

If $Z_i = X$ we have that

$$\chi(\mathcal{O}_{Z_i} \otimes \mathcal{M}^{e_i} \otimes \mathcal{B}^n)$$

is polynomial in n by the Riemann Roch formula (14). If Z_i is an (integral) curve, then we have an exact sequence

$$0 \to \mathcal{O}_{Z_i} \to \mathcal{O}_{\overline{Z}_i} \to \mathcal{G}_i \to 0$$

of $\mathcal{O}_{Z_i}$ modules, where $\overline{Z}_i$ is the normalization of Z_i. Since $\mathcal{G}_i$ has finite support, $\chi(\mathcal{O}_{Z_i} \otimes \mathcal{M}^{e_i} \otimes \mathcal{B}^n)$ is a polynomial in n by the Riemann Roch theorem on the nonsingular projective curve $\overline{Z}_i$. In the case when Z_i is a point, $\chi(\mathcal{O}_{Z_i} \otimes \mathcal{M}^{e_i} \otimes \mathcal{B}^n) = \chi(\mathcal{O}_{Z_i}) = 1$ for all n.

Now we prove 2. By the consideration of the composition sequence constructed in the first part of the proof, we are reduced to showing that for $1 \leq i \leq m$, $H^2(Z_i, \mathcal{O}_{Z_i} \otimes \mathcal{M}^{e_i} \otimes$

$\mathcal{B}^{\otimes n}) = 0$ for $n \gg 0$. If $Z_i = X$ this follows from Proposition 3.1. Otherwise, Z_i has dimension smaller than 2, so the vanishing must hold. $\square$

4. Regularity of Symbolic Powers

We continue with the assumptions of Section 3.

Proposition 4.1. *There exist positive integers b_0 and t_0 such that if D is a Weil divisor on X such that $\overline{D}$ is in the translation of* AL *by $t_0(b_0abcA - E) + abcA$, then $H^i(X, D) = 0$ for $i > 0$.*

Proof. Let $\mathcal{F}_m = \mathcal{O}_X(mA)$ for $0 \le m < abc$. There exists a positive integer b_0 such that $b_0abcA - E$ is ample on X. Let $\mathcal{M} = \mathcal{O}_X(b_0abcA - E)$.

We use the following vanishing theorem, proven in Theorem 5.1 of [13]. Let $\mathcal{F}$ be coherent on a projective scheme Y, and $\mathcal{M}$ be an ample line bundle. Then there exists an integer t such that $H^i(Y, \mathcal{F} \otimes \mathcal{L} \otimes \mathcal{M}^t) = 0$ for all nef line bundles $\mathcal{L}$ on Y and for all $i > 0$.

Choose t_0 so that t_0 satisfies the condition for t in the above vanishing theorem, for $\mathcal{F}_m$ with $0 \le m < abc$ and for all $i > 0$.

Suppose $\overline{D}$ is in the translation of AL by $t_0(b_0abcA - E) + abcA$. Then $D \sim \alpha A - \beta E + t_0(b_0abcA - E) + abcA$, with $\alpha \ge s\beta$. Expand $\alpha = nabc + m$ with $0 \le m < abc$. Then we have

$$\mathcal{O}_X(D) \cong \mathcal{F}_m \otimes \mathcal{L} \otimes \mathcal{M}^{t_0}$$

where $\mathcal{L} = \mathcal{O}_X((n+1)abcA - \beta E)$ is nef. Thus the conclusions of the Proposition hold for D. $\square$

Suppose that D is a divisor on X. Define

$$\overline{D}^{\perp} = \{\varphi \in N_1(X) \mid (D \cdot \varphi) = 0\}.$$

An effective divisor D such that $(D \cdot D) < 0$ will be called a negative curve. An effective divisor D such that $D \sim aA - mE$ for some positive integers a and m will be called an E-uniform curve.

Lemma 4.2. *Either $T = (\sqrt{abcu}A - E)\mathbb{R}_{\ge 0}$, or $T = \overline{C}^{\perp} \cap AL$, where C is an irreducible negative curve.*

Proof. Recall from (15), that $T = (s(I)A - E)\mathbb{R}_{\ge 0}$, with $s = s(I) \ge \sqrt{abcu}$. Suppose that $s > \sqrt{abcu}$. There exists $\alpha \in \mathbb{Q}$ such that $s > \alpha > \sqrt{abcu}$. Write $\alpha = \frac{c}{d}$ where $c, d \in \mathbb{Z}_+$. By (13), (14) and Proposition 3.1, we have that $h^0(X, \mathcal{O}_X(m(cA - dE))) > 0$ for $m \gg 0$. Thus there exist only a finite number of irreducible curves $C_1, \ldots, C_t$ on X such that $(C_i \cdot (\alpha A - E)) < 0$.

Suppose that $(C \cdot (sA - E)) > 0$ for all irreducible curves C on X. In particular, $(C_i \cdot (sA - E)) > 0$ for all $1 \le i \le t$. This implies that there exists a real number β with $\alpha < \beta < s$ such that $(C_i \cdot (\beta A - E)) > 0$ for $1 \le i \le t$. If C is an irreducible curve on X other than one of the C_i, then we have $(C \cdot (\alpha A - E)) \ge 0$ and $(C \cdot A) \ge 0$, so that $(C \cdot (\beta A - E)) \ge 0$. Thus $\beta A - E \in AL$, a contradiction. Thus there exists an irreducible curve C on X such that $(C \cdot (sA - E)) = 0$ (the only irreducible curves C on X with $(C \cdot A) = 0$ are the E_i).

$\square$

Theorem 4.3. *Let $u = \sum_{i=1}^{r} e_i^2$. We have that $s(I) \ge \sqrt{abcu}$. If $s(I) > \sqrt{abcu}$, then $s(I)$ is a rational number.*

Proof. The proof is immediate from (15) and Lemma 4.2. $\square$

Negative curves and E-uniform curves are defined before Lemma 4.2.

Lemma 4.4. *Suppose that there exists an E-uniform negative curve F. Then $s(I) > \sqrt{abcu}$, and $T = \overline{C}^{\perp} \cap AL$, where C is an irreducible negative curve in the support of F.*

Proof. Let F be an E-uniform negative curve. $F \sim mA - nE$ for some $m, n \in \mathbb{Z}_+$. Since $(F^2) < 0$, and F is effective, there exists an irreducible curve C in the support of F such that $(C \cdot F) < 0$. Since $((sA - E) \cdot (\tau A - E)) \geq 0$ and $\tau < \sqrt{abcu}$, we have $s > \sqrt{abcu}$. We have $T = (\alpha A + F)\mathbb{R}_{\geq 0}$ for some $\alpha > 0$. Since T is a boundary ray of AL, for all $\varepsilon > 0$, there exists an irreducible curve C_ε on X such that $(C_\varepsilon \cdot ((\alpha - \varepsilon)A + F)) < 0$. Since $(C_\varepsilon \cdot A) \geq 0$, we must have $(C_\varepsilon \cdot F) < 0$, so that C_ε is in the support of F. Since F has only a finite number of irreducible components, we have that $T = \overline{C}^{\perp}$ for some irreducible component C of F. $\square$

Proposition 4.5. *There exist $t_1 > 0$ and $m_0 > 0$ such that $m \geq m_0$ implies there exists a Cartier divisor $D = \alpha A - mE$ such that D lies between the rays T and the translation of T by $-t_1 A$ such that $h^1(X, \mathcal{O}_X(D)) \neq 0$.*

Proof. Let $\gamma = \sqrt{abcu}A - E$. Let
$$d = (e_1 + \cdots + e_r)\sqrt{\frac{abc}{u}}.$$
Observe that $\sqrt{abcu} \in \mathbb{Q}$ if and only if $d \in \mathbb{Q}$. Suppose that $n \in \mathbb{Z}_+$ and $\alpha(n) \in \mathbb{R}_{\geq 0}$ are such that $n\gamma - \alpha(n)A$ is a Cartier divisor. Then by (13) and (14),

$$\begin{aligned}(18)\qquad &\chi(\mathcal{O}_X(n\gamma - \alpha(n)A)) \\ &= n\tfrac{\sqrt{u}}{2\sqrt{abc}}\left((a+b+c) - d - 2\alpha(n)\right) + \tfrac{1}{2abc}\left(\alpha(n)^2 - \alpha(n)(a+b+c)\right) + 1.\end{aligned}$$

By (15), we always have $s \geq \tau$, so that we reduce to establishing the Proposition in the two cases $s = \tau$ and $s > \tau$.

Case 1 Assume that $s = \tau$ so that by (15), $T = R = \gamma\mathbb{R}_{\geq 0}$. For $n \in \mathbb{Z}_+$, choose $\alpha(n)$ in (18) so that $2abc \leq \alpha(n) < 3abc$. Then $(a+b+c) - d - 2\alpha(n) < 0$, so that
$$-h^1(X, \mathcal{O}_X(n\gamma - \alpha(n)A)) \leq \chi(\mathcal{O}_X(n\gamma - \alpha(n)A)) < 0$$
for $n \gg 0$.

Case 2 Assume that the boundary ray $R = (\tau A - E)\mathbb{R}_{\geq 0}$ of NL and the boundary ray $T = (sA - E)\mathbb{R}_{\geq 0}$ of NA satisfy $s > \tau$.

We must have $s > \sqrt{abcu}$ with these assumptions, for otherwise, $\gamma \in AL$, and there would be an effective divisor $F \equiv \alpha A - \beta E$ such that $\frac{\alpha}{\beta} < \sqrt{abcu}$, so that F is an E-uniform negative curve. We would then have that $(F \cdot \gamma) < 0$, a contradiction.

By Lemma 4.2, we have $T = \overline{C}^{\perp}$, where C is an irreducible negative curve. Let $p_a(C)$ be the arithmetic genus of C. Let $\delta = sA - E$. Let
$$\beta = \max\{0, \frac{1 - p_a(C)}{(C \cdot A)}\}.$$
For $n \in \mathbb{Z}_+$, let $\alpha(n)$ be such that
$$(19)\qquad \beta < \alpha(n) \leq \beta + abc$$

and $n\delta - \alpha(n)A$ is a Cartier divisor.

We have an exact sequence of $\mathcal{O}_X$ modules

$$0 \to \mathcal{O}_X(n\delta - \alpha(n)A - C) \to \mathcal{O}_X(n\delta - \alpha(n)A) \to \mathcal{O}_X(n\delta - \alpha(n)A) \otimes \mathcal{O}_C \to 0. \tag{20}$$

Since $s > \sqrt{abcu}$, be have that $h^0(X, \mathcal{O}_X(n\delta - \alpha(n)A)) > 0$ for $n \gg 0$. We further have that $((n\delta - \alpha(n)A) \cdot C) < 0$, so since C is an integral curve, for $n \gg 0$,

$$h^0(X, \mathcal{O}_X(n\delta - \alpha(n)A - C)) = h^0(X, \mathcal{O}_X(n\delta - \alpha(n)A)) > 0.$$

We have that

$$h^2(X, \mathcal{O}_X(n\delta - \alpha(n)A - C)) = h^2(X, \mathcal{O}_X(n\delta - \alpha(n)A)) = 0$$

by Proposition 3.1.

From (20) we now have

$$\begin{aligned} & h^1(X, \mathcal{O}_X(n\delta - \alpha(n)A - C)) - h^1(X, \mathcal{O}_X(n\delta - \alpha(n)A)) \\ & \quad = \chi(\mathcal{O}_X(n\delta - \alpha(n)A)) - \chi(\mathcal{O}_X(n\delta - \alpha(n)A - C)) \\ & \quad = \chi(\mathcal{O}_X(n\delta - \alpha(n)A) \otimes \mathcal{O}_C) \\ & = (C \cdot (n\delta - \alpha(n)A)) + 1 - p_a(C) < 0 \end{aligned}$$

where the last equality is by the Riemann Roch theorem for the curve C, and (19). Thus $h^1(X, \mathcal{O}_X(n\delta - \alpha(n)A)) > 0$ for $n \gg 0$.

$\square$

Recall that $\lfloor x \rfloor$ is the greatest integer in a real number x.

Theorem 4.6. *There exists a bounded function $\sigma_I : \mathbb{N} \to \mathbb{Z}$ such that*

$$\operatorname{reg}(I^{(m)}) = \lfloor s(I)m \rfloor + \sigma_I(m)$$

for all $m \in \mathbb{N}$.

Proof. For $i \geq 2$,

$$H^i_{\mathfrak{m}}(I^{(m)})_n = H^{i-1}(X, \mathcal{O}_X(nA - mE))$$

by equations (11) and (12). The theorem now follows from Propositions 4.1, 3.1 and 4.5 for large m, and thus the theorem is true for all m. $\square$

A function $\sigma : \mathbb{N} \to \mathbb{Z}$ is *eventually periodic* if $\sigma(m)$ is periodic for $m \gg 0$.

Recall (Theorem 4.3) that $s(I) \geq \sqrt{abcu}$, and if $s(I) > \sqrt{abcu}$, then $s(I)$ is a rational number.

Theorem 4.7. *Suppose that $s(I) > \sqrt{abcu}$ and K has characteristic zero or is an algebraic closure of a finite field. Then the function $\sigma_I(m)$ of Theorem 4.6 is eventually periodic.*

Proof. By Propositions 4.1, 3.1 and 4.5, we need only compute $h^1(X, \mathcal{O}_X(nA - mE))$ for $nA - mE$ between the rays T translated up by $(t_0 b_0 + 1)abcA$ and T translated down by $-t_1 A$. Call this region Δ.

By Theorem 4.3, there exists a numerically effective Cartier divisor G such that $T = \overline{G}\mathbb{R}_{\geq 0}$. We have $(G^2) > 0$. Since $\overline{G}$ is rational, there exists a finite number of Weil divisors D_i with $\overline{D}_i \in \Delta$ such that every divisor D with $\overline{D} \in \Delta$ can be written as $D \sim D_i + nG$ for some i and some $n \in \mathbb{N}$. Let $\mathcal{L} = \mathcal{O}_X(G)$.

Since $(g^*\mathcal{O}_X(abc) \cdot g^*\mathcal{L}) > 0$, Serre duality on Y implies

$$h^2(Y, \mathcal{O}_Y(\lfloor g^*(D_i) \rfloor) \otimes g^*\mathcal{L}^n) = 0 \tag{21}$$

for all i and for $n \gg 0$.

By 2 of Lemma 3.2, (16) and (21), for all i and for $n \gg 0$ we have

$$\begin{aligned} h^1(X, \mathcal{O}_X(D_i) \otimes \mathcal{L}^n) &= h^0(X, \mathcal{O}_X(D_i) \otimes \mathcal{L}^n) - \chi(\mathcal{O}_X(D_i) \otimes \mathcal{L}^n) \\ &= h^0(Y, \mathcal{O}_Y(\lfloor g^*(D_i) \rfloor) \otimes g^*(\mathcal{L}^n)) - \chi(\mathcal{O}_X(D_i) \otimes \mathcal{L}^n) \\ &= h^1(Y, \mathcal{O}_Y(\lfloor g^*(D_i) \rfloor) \otimes g^*(\mathcal{L}^n)) \\ &\quad + \chi(\mathcal{O}_Y(\lfloor g^*(D_i) \rfloor) \otimes g^*(\mathcal{L}^n)) - \chi(\mathcal{O}_X(D_i) \otimes \mathcal{L}^n). \end{aligned}$$

By the Riemann Roch Theorem on Y and 1 of Lemma 3.2 we have that

$$\chi(\mathcal{O}_Y(\lfloor g^*(D_i) \rfloor) \otimes g^*(\mathcal{L}^n)) - \chi(\mathcal{O}_X(D_i) \otimes \mathcal{L}^n)$$

is a polynomial in n for all i.

By Proposition 13 of [6], there exists an effective divisor C on Y such that $g^*(\mathcal{L}) \otimes \mathcal{O}_C$ is numerically trivial, and the restriction maps

$$H^1(Y, \mathcal{O}_Y(\lfloor g^*(D_i) \rfloor) \otimes g^*(\mathcal{L}^n)) \to H^1(C, \mathcal{O}_C \otimes \mathcal{O}_Y(\lfloor g^*(D_i) \rfloor) \otimes g^*(\mathcal{L}^n))$$

are isomorphisms for $n \gg 0$ and all i.

In the case when K has characteristic zero, Theorem 8 of [6], shows that for all i, $h^1(C, \mathcal{O}_C \otimes \mathcal{O}_Y(\lfloor g^*(D_i) \rfloor) \otimes g^*(\mathcal{L}^n))$ is eventually periodic in n for all i. In the case when K is an algebraic closure of a finite field, then the numerically trivial invertible sheaf $\mathcal{O}_C \otimes g^*(\mathcal{L})$ must be torsion, so some power is isomorphic to $\mathcal{O}_C$. Thus we trivially have that $h^1(C, \mathcal{O}_C \otimes \mathcal{O}_Y(\lfloor g^*(D_i) \rfloor) \otimes g^*(\mathcal{L}^n))$ is eventually periodic in n for all i.

In either case of K, $h^1(Y, \mathcal{O}_Y(\lfloor g^*(D_i) \rfloor) \otimes g^*(\mathcal{L}^n))$ is eventually periodic as a function of n. Thus $\sigma(m)$ is eventually periodic. □

When K is a field of positive characteristic which has positive transcendence degree over the prime field, the conclusions of Theorem 4.7 may fail. An example of a set of points in ordinary projective space $\mathbb{P}^2$ where $\sigma(m)$ is not eventually periodic is given in Example 4.4 [5].

Corollary 4.8. *Suppose there exists an E-uniform negative curve, and K has characteristic zero, or is a finite field. Then the function $\sigma_I(m)$ of Theorem 4.6 is eventually periodic.*

Proof. This follows from Lemma 4.4 and Theorem 4.7. □

An important case of this construction is when $i = 1$, and $I = P(a, b, c)$ is the prime ideal of a monomial space curve. As a corollary to Theorems 4.6 and 4.7, we have the following application to monomial space curves.

Corollary 4.9. *Suppose that $I = P(a, b, c)$ is the prime ideal of a monomial space curve, and there exists a negative curve on $X(I)$. Then $s(I)$ is a rational number, and the function $\sigma_I(m)$ of Theorem 4.6 is eventually periodic.*

5. Uniform negative curves and Nagata's conjecture

Let S be a polynomial ring as in Section 3. Let $C = K[u, v, w]$ be a polynomial ring with $\mathrm{wt}(u) = \mathrm{wt}(v) = \mathrm{wt}(w) = 1$.

Consider the K-algebra homomorphism

$$\delta : S \longrightarrow C$$

defined by $\delta(x) = u^a$, $\delta(y) = v^b$, $\delta(z) = w^c$, where a, b, c are pairwise relatively prime positive integers.

Let

$$I = I_{P_1}^{e_1} \cap \cdots \cap I_{P_r}^{e_r}$$

be an ideal of S as in Section 3.

Consider the following two conditions:

(A1) K is an algebraically closed field such that $\mathrm{ch}(K)$ is 0 or $\mathrm{ch}(K)$ does not divide abc.

(A2) $I_{P_i} \not\ni xyz$ for $i = 1, \ldots, r$.

Lemma 5.1. *Assume the conditions (A1) and (A2) as above. There are distinct prime ideals Q_{i1}, Q_{i2}, $\ldots$, $Q_{i,abc}$ of C such that*

$$I_{P_i}^{(m)} C = \bigcap_{j=1}^{abc} Q_{ij}^{(m)}$$

for $m > 0$ and $i = 1, \ldots, r$.

Proof. Let Q be a prime ideal of C lying over I_{P_i}. Then, there exists a point $(\alpha : \beta : \gamma) \in \mathbb{P}_K^2$ such that

$$Q = I_2 \begin{pmatrix} u & v & w \\ \alpha & \beta & \gamma \end{pmatrix},$$

where $I_2(\ )$ is the ideal generated by all the 2×2-minors of the given matrix. Remark that Q is the kernel of the K-algebra homomorphism

$$\varphi_{(\alpha:\beta:\gamma)} : C \longrightarrow K[t]$$

defined by $\varphi_{(\alpha:\beta:\gamma)}(u) = \alpha t$, $\varphi_{(\alpha:\beta:\gamma)}(v) = \beta t$, $\varphi_{(\alpha:\beta:\gamma)}(w) = \gamma t$. Then, I_{P_i} is the kernel of the K-algebra homomorphism

$$\varphi = \varphi_{(\alpha:\beta:\gamma)} \delta : S \longrightarrow K[t]$$

defined by $\varphi(x) = \alpha^a t^a$, $\varphi(y) = \beta^b t^b$, $\varphi(z) = \gamma^c t^c$. Let ζ_q be a primitive q-th root of 1 for a positive integer q.

Set

$$Q_{n_1,n_2,n_3} = I_2 \begin{pmatrix} u & v & w \\ \zeta_a^{n_1}\alpha & \zeta_b^{n_2}\beta & \zeta_c^{n_3}\gamma \end{pmatrix}.$$

It is the kernel of the K-algebra homomorphism $\varphi_{(\zeta_a^{n_1}\alpha:\zeta_b^{n_2}\beta:\zeta_c^{n_3}\gamma)}$. For any n_1, n_2 and n_3, Q_{n_1,n_2,n_3} is a prime ideal of C lying over I_{P_i} since $\varphi_{(\zeta_a^{n_1}\alpha:\zeta_b^{n_2}\beta:\zeta_c^{n_3}\gamma)}\delta = \varphi$. By our assumption (A2), all of α, β and γ are not zero. By (A1),

$$\{(\zeta_a^{n_1}\alpha : \zeta_b^{n_2}\beta : \zeta_c^{n_3}\gamma) \in \mathbb{P}_K^2 \mid n_1 = 0, \ldots, a-1;\ n_2 = 0, \ldots, b-1;\ n_3 = 0, \ldots, c-1\}$$

are distinct abc points in $\mathbb{P}_K^2$. Therefore,

$$\{Q_{n_1,n_2,n_3} \mid n_1 = 0, \ldots, a-1;\ n_2 = 0, \ldots, b-1;\ n_3 = 0, \ldots, c-1\}$$

are distinct prime ideals of C lying over I_{P_i}. Here, we have

$$\begin{aligned} & \mathrm{Ass}_C(C/I_{P_i}C) \\ = \ & \mathrm{Min}_C(C/I_{P_i}C) \\ = \ & \{Q \in \mathrm{Spec}(C) \mid Q \cap S = I_{P_i}\} \\ \supseteq \ & \{Q_{n_1,n_2,n_3} \mid n_1 = 0, \ldots, a-1;\ n_2 = 0, \ldots, b-1;\ n_3 = 0, \ldots, c-1\}. \end{aligned}$$

Since C is an S-free module of rank abc, $C/I_{P_i}C$ is an S/I_{P_i}-free module of rank abc. Then, it is easy to see

$$abc = \mathrm{rank}_{S/I_{P_i}}(C/I_{P_i}C) = \sum_{\substack{Q\in \mathrm{Spec}(C)\\ Q\cap S = I_{P_i}}} \ell_C(C_Q/I_{P_i}C_Q)\cdot \mathrm{rank}_{S/I_{P_i}}(C/Q).$$

Therefore,

$$\{Q\in \mathrm{Spec}(C) \mid Q\cap S = I_{P_i}\} = \{Q_{n_1,n_2,n_3} \mid n_1 = 0,\ldots,a-1;\ n_2 = 0,\ldots,b-1;\ n_3 = 0,\ldots,c-1\}$$

and

$$\ell_C(C_{Q_{n_1,n_2,n_3}}/I_{P_i}C_{Q_{n_1,n_2,n_3}}) = 1$$

for each n_1, n_2, n_3. It follows from the equation as above that

$$I_{P_i}C_{Q_{n_1,n_2,n_3}} = Q_{n_1,n_2,n_3}C_{Q_{n_1,n_2,n_3}}$$

for each n_1, n_2, n_3.

Since

$$\begin{aligned} & \mathrm{Ass}_C(C/I_{P_i}^{(m)}C) \\ = & \mathrm{Min}_C(C/I_{P_i}^{(m)}C) \\ = & \mathrm{Min}_C(C/I_{P_i}C) \\ = & \{Q_{n_1,n_2,n_3} \mid n_1 = 0,\ldots,a-1;\ n_2 = 0,\ldots,b-1;\ n_3 = 0,\ldots,c-1\}, \end{aligned}$$

we have

$$I_{P_i}^{(m)}C = \bigcap_{n_1,n_2,n_3}\left(I_{P_i}^{(m)}C_{Q_{n_1,n_2,n_3}}\cap C\right) = \bigcap_{n_1,n_2,n_3}(Q^m_{n_1,n_2,n_3}C_{Q_{n_1,n_2,n_3}}\cap C) = \bigcap_{n_1,n_2,n_3} Q^{(m)}_{n_1,n_2,n_3}$$

□

By Lemma 5.1,

$$I^{(m)}C = \bigcap_{i=1}^{r} I_{P_i}^{(e_im)}C = \bigcap_{i=1}^{r}(\bigcap_{j=1}^{abc} Q_{ij}^{(e_im)}).$$

In this case,

$$\max\{n\in\mathbb{Z} \mid [C/(u^a,v^b,w^c)C]_n \neq 0\} = a+b+c-3.$$

Then, by (5),

$$\mathrm{reg}(I^{(m)}C) = \mathrm{reg}(I^{(m)}) + a+b+c-3. \tag{22}$$

By (22), we may assume $a = b = c = 1$ in Theorem 4.6 and Theorem 4.7 if (A1) and (A2) are satisfied.

Let q_1, …, q_n be independent generic points in $\mathbb{P}^2_{\mathbb{C}}$. Suppose that $n \geq 10$. Nagata conjectured that

$$[I_{q_1}^m\cap\cdots\cap I_{q_n}^m]_d = 0$$

if $d \leq \sqrt{n}m$. Nagata [27] solved it affirmatively when n is a square.

Consider the following two conditions:

(A0) $K = \mathbb{C}$, the field of complex numbers, and
(A3) $I = \sqrt{I}$, that is $e_1 = e_2 = \cdots = e_r = 1$ and $E = E_1+\cdots+E_r$.

Proposition 5.2. *Suppose that n is a positive integer which has a factorization $n = abcr$ by positive integers with a, b, c pairwise relatively prime. If there exist distinct points $P_1, \ldots, P_r$ on the weighted projective space $\mathbb{P}_{\mathbb{C}}(a, b, c)$ satisfying (A2), such that there does not exist an E-uniform negative curve on the blow up of $\mathbb{P}_{\mathbb{C}}(a, b, c)$ defined by (A3), then Nagata's conjecture is true for $abcr$ general points in $\mathbb{P}^2_{\mathbb{C}}$.*

Proof. Assume that Nagata's conjecture is not true for $abcr$ general points in $\mathbb{P}^2_{\mathbb{C}}$. If $abcr$ is a square, Nagata solved the conjecture affirmatively. Therefore, we may assume that $\sqrt{abcr}$ is not a rational number.

Let $q_1, \ldots, q_{abcr}$ be independent generic points in $\mathbb{P}^2_{\mathbb{C}}$. I_{q_i} is the defining ideal of q_i. By our assumption, there exist positive integers m_0 and d_0 such that

$$d_0 \le \sqrt{abcr} m_0 \quad \text{and} \quad [I_{q_1}^{m_0} \cap \cdots \cap I_{q_{abcr}}^{m_0}]_{d_0} \neq 0. \tag{23}$$

Since $\sqrt{abcr}$ is not a rational number, we have

$$d_0 < \sqrt{abcr} m_0.$$

Assume that there does not exist an E-uniform negative curve for some P_i's satisfying (A0), (A2) and (A3). Then we have

$$[I^{(m)}]_d = [I_{P_1}^{(m)} \cap \cdots \cap I_{P_r}^{(m)}]_d = 0$$

if $d < \sqrt{abcr} m$. Considering the $\mathbb{C}$-algebra homomorphism

$$\delta : S = \mathbb{C}[x, y, z] \longrightarrow C = \mathbb{C}[u, v, w],$$

we obtain

$$0 = [I^{(m)} C]_d = [\bigcap_{i=1}^{r} I_{P_i}^{(m)} C]_d = [\bigcap_{i=1}^{r} (\bigcap_{j=1}^{abc} Q_{ij}^{(m)})]_d$$

for $d < \sqrt{abcr} m$. This contradicts to (23) since we can specialize $\{q_1, \ldots, q_{abcr}\}$ to

$$\{Q_{ij} \mid i = 1, \ldots, r; \ j = 1, \ldots, abc\}.$$

□

Remark 5.3. Let K be a field and a, b, c be pairwise relatively prime integers. Let $P_K(a, b, c)$ be the kernel of the K-algebra homomorphism

$$\delta : S = K[x, y, z] \to K[t]$$

defined by $\delta(x) = t^a$, $\delta(y) = t^b$, $\delta(z) = t^c$.

Let $X_K(a, b, c)$ be the blow-up of the weighted projective space $\mathbb{P}(a, b, c)$ at the point corresponding to $P_K(a, b, c)$.

Assume that K is of positive characteristic. If there exists a negative curve on $X_K(a, b, c)$, then the symbolic Rees ring

$$S \oplus P_K(a, b, c) \oplus P_K(a, b, c)^{(2)} \oplus P_K(a, b, c)^{(3)} \oplus \cdots \tag{24}$$

is Noetherian by [2].

Here, assume that the symbolic Rees ring (24) is not Noetherian for some a_0, b_0, c_0 over some field K_0 of positive characteristic. Since $P_{K_0}(a, b, c)$ is not a complete intersection, we may assume $3 \le a_0 < b_0 < c_0$. In particular, $a_0 b_0 c_0 \ge 60 > 10$. Then by [2], there is no negative curve on $X_{K_0}(a_0, b_0, c_0)$. By a standard method of mod p reduction, there is no negative curve on $X_{\mathbb{C}}(a_0, b_0, c_0)$. Then, by Proposition 5.2, Nagata's conjecture is true for $a_0 b_0 c_0$.

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