TOPICS ON SEQUENTIALLY COHEN-MACAULAY MODULES

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ABSTRACT. In this paper, we study the two different topics related to sequentially Cohen-Macaulay modules. The questions are when the sequentially Cohen-Macaulay property preserve the localization and the module-finite extension of rings.

CONTENTS

1. INTRODUCTION

Throughout this paper, unless otherwise specified, let R be a commutative Noetherian ring, $M \neq (0)$ a finitely generated R-module of dimension d. Then there exists the largest R-submodule M_n of M with $\dim_R M_n \leq n$ for every $n \in \mathbb{Z}$ (here $\dim_R(0) = -\infty$ for convention). Let

$$\begin{aligned} \mathcal{S}(M) &= \{\dim_R N \mid N \text{ is an } R\text{-submodule of } M, N \neq (0)\} \\ &= \{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Ass}_R M\}. \end{aligned}$$

Put $\ell = \sharp\mathcal{S}(M)$ and write $\mathcal{S}(M) = \{d_1 < d_2 < \cdots < d_\ell = d\}$. Then *the dimension filtration of M* is a chain

$$\mathcal{D} : D_0 := (0) \subsetneq D_1 \subsetneq D_2 \subsetneq \ldots \subsetneq D_\ell = M$$

of R-submodules of M, where $D_i = M_{d_i}$ for every $1 \leq i \leq \ell$. Notice that the notion of dimension filtration is due to S. Goto, Y. Horiuchi and H. Sakurai ([GHS]) and a little different from that of the original one given by P. Schenzel ([Sch]). However, let us adopt the above definition throughout this paper. We say that M is *a sequentially Cohen-Macaulay R-module*, if the quotient module $C_i = D_i/D_{i-1}$ of D_i is Cohen-Macaulay for every $1 \leq i \leq \ell$. In particular, the ring R is called *a sequentially Cohen-Macaulay ring*, if $\dim R < \infty$ and R is a sequentially Cohen-Macaulay module over itself.

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The aim of this paper is to investigate the following questions. The first one is whether the sequentially Cohen-Macaulay property is inherited from localizations, which has been already studied by Cuong-Goto-Truong ([CGT]) and Cuong-Nhan ([CN]) in the case of local rings. In Section 2, we shall probe into a possible generalization of their results ([CGT, Proposition 2.6], [CN, Proposition 4.7]). More precisely, we prove the following.

Theorem 1.1. *Suppose that* $\dim R/\mathfrak{p} = \dim R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m}$ *for every* $\mathfrak{p} \in \mathrm{Ass}_R M$ *and for every maximal ideal* $\mathfrak{m}$ *of* R *such that* $\mathfrak{p} \subseteq \mathfrak{m}$*. Then the following conditions are equivalent.*

(1) *M is a sequentially Cohen-Macaulay R-module.*
(2) *M_P is a sequentially Cohen-Macaulay R_P-module for every $P \in \mathrm{Supp}_R M$.*

Another one is the question of whether the sequentially Cohen-Macaulay property preserves the module-finite extension of rings or not. In Section 3, especially, we will give the characterization of sequentially Cohen-Macaulay local rings obtained by the idealization (that is, trivial extension), which is stated as follows.

Theorem 1.2 (Corollary 3.9)**.** *Suppose that R is a local ring. Let $R \ltimes M$ denote the idealization of M over R. Then the following conditions are equivalent.*

1. *$R \ltimes M$ is a sequentially Cohen-Macaulay local ring.*
2. *$R \ltimes M$ is a sequentially Cohen-Macaulay R-module.*
3. *R is a sequentially Cohen-Macaulay local ring and M is a sequentially Cohen-Macaulay R-module.*

2. Localization of sequentially Cohen-Macaulay modules

The purpose of this section is mainly to prove Theorem 1.1.

Proof of Theorem 1.1. (1) $\Rightarrow$ (2) We may assume that $\ell > 1$ and the assertion holds for $\ell - 1$. Thanks to [CGT, Proposition 2.6], it is enough to show the case where P is a maximal ideal of R. Then we get the exact sequence $0 \to N \to M \to C \to 0$ of R-modules where $N = D_{\ell-1}$ and $C = C_\ell$. We may also assume $N_P \neq (0)$, $C_P \neq (0)$ and $\dim_{R_P} N_P = \dim_{R_P} M_P$, since N_P is sequentially Cohen-Macaulay and C_P is Cohen-Macaulay. By using the hypothesis, we get $\dim_{R_P} M_P = \dim_{R_P} C_P$ (see also [Sch, Corollary 2.3]).

Let
$$E_0 = (0) \subsetneq E_1 \subsetneq \cdots \subsetneq E_{t-1} \subsetneq E_t = N_P$$
be the dimension filtration of N_P. Then M_P/E_{t-1} is a Cohen-Macaulay R_P-module of dimension $\dim_{R_P} M_P$, because N_P/E_{t-1} and M_P/N_P are Cohen-Macaulay R_P-modules of same dimension $\dim_{R_P} M_P$. Hence M_P is a sequentially Cohen-Macaulay R_P-module.

(2) $\Rightarrow$ (1) Suppose that C_i is not Cohen-Macaulay for some $1 \leq i \leq \ell$. Then there exists a maximal ideal P of R such that $[C_i]_P \neq (0)$, $[C_i]_P$ is not a Cohen-Macaulay R_P-module. Let $\alpha = \dim_{R_P}[C_i]_P$. We choose $\mathfrak{p} \in \mathrm{Ass}_R C_i$ such that $\mathfrak{p} \subseteq P$, $\alpha = \dim R_P/\mathfrak{p}R_P$. Then
$$\alpha = \dim R_P/\mathfrak{p}R_P = \dim R/\mathfrak{p} = d_i$$
by using the hypothesis and $\mathfrak{p} \in \mathrm{Ass}_R C_i$. Therefore $d_i \in \mathcal{S}(M_P) \subseteq \mathcal{S}(M)$, since
$$\mathcal{S}(M_P) = \{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Ass}_R M,\ \mathfrak{p} \subseteq P\} \subseteq \mathcal{S}(M).$$

Let $q = \sharp \mathcal{S}(M_P)$. We write $\mathcal{S}(M_P) = \{n_1 < n_2 < \cdots < n_q\}$. Then $d_i = n_j$ for some $1 \le j \le q$. Let $(0) = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_R M} M(\mathfrak{p})$ be a primary decomposition of (0) in M. In this case

$$(0) = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_R M,\ \mathfrak{p} \subseteq P} [M(\mathfrak{p})]_P$$

forms a primary decomposition of (0) in M_P.

Claim 2.1. *The following assertions hold true.*
(1) $[D_i]_P = D_j(M_P)$
(2) $[D_{i-1}]_P = D_{j-1}(M_P)$
where $\{D_j(M_P)\}_{0 \le j \le q}$ *stands for the dimension filtration of* M_P.

Proof of Claim 2.1. (1) We may assume that $i < \ell$. Then

$$D_i = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_R M,\ \dim R/\mathfrak{p} > d_i} M(\mathfrak{p}),$$

so that

$$[D_i]_P = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_R M,\ \dim R/\mathfrak{p} > n_j,\ \mathfrak{p} \subseteq P} [M(\mathfrak{p})]_P.$$

We now assume that $\mathfrak{p} \not\subseteq P$ for every $\mathfrak{p} \in \mathrm{Ass}_R M$ with $\dim R/\mathfrak{p} > d_i$. Then $M(\mathfrak{p})_P = M_P$, so that $[D_i]_P = M_P$. Then $\dim_{R_P} M_P = d_i$, since $\dim_{R_P} M_P \le d_i \in \mathcal{S}(M_P)$. Therefore $d_i = n_q$, $j = q$ and

$$[D_i]_P = M_P = D_q(M_P) = D_j(M_P).$$

Thus we may assume that $\mathfrak{p} \subseteq P$ for some $\mathfrak{p} \in \mathrm{Ass}_R M$ with $\dim R/\mathfrak{p} > d_i$. Thus

$$n_q = \dim_{R_P} M_P \ge \dim_{R_P} R_P/\mathfrak{p}R_P = \dim R/\mathfrak{p} > d_i = n_j$$

whence $1 \le j < q$. Hence

$$[D_i]_P = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_R M,\ \dim R_P/\mathfrak{p}R_P > n_j,\ \mathfrak{p} \subseteq P} [M(\mathfrak{p})]_P = D_j(M_P).$$

(2) We get

$$[D_{i-1}]_P = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_R M,\ \dim R/\mathfrak{p} \ge d_i,\ \mathfrak{p} \subseteq P} [M(\mathfrak{p})]_P,$$

since $D_{i-1} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_R M,\ \dim R/\mathfrak{p} \ge d_i} M(\mathfrak{p})$. We may assume $j > 1$. Then $d_i = n_j > n_{j-1}$, so that

$$\begin{aligned}[D_{i-1}]_P &= \bigcap_{\mathfrak{p} \in \mathrm{Ass}_R M,\ \dim R/\mathfrak{p} \ge d_i,\ \mathfrak{p} \subseteq P} [M(\mathfrak{p})]_P = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_R M,\ \dim R_P/\mathfrak{p}R_P > n_{j-1}} [M(\mathfrak{p})]_P \\ &= D_{j-1}(M_P).\end{aligned}$$

□

Hence $C_j(M_P) = D_j(M_P)/D_{j-1}(M_P)$ is Cohen-Macaulay, since M_P is sequentially Cohen-Macaulay, whence $[C_i]_P$ is a Cohen-Macaulay R_P-module, a contradiction. □

If the base ring R is a finitely generated algebra over a field, then the assumption of Theorem 1.1 is automatically satisfied and we get the following.

Corollary 2.2. *Let R be a finitely generated algebra over a field, $M \neq (0)$ a finitely generated R-module. Then the following conditions are equivalent.*

(1) *M is a sequentially Cohen-Macaulay R-module.*
(2) *M_P is a sequentially Cohen-Macaulay R_P-module for every $P \in \mathrm{Supp}_R M$.*

From now on, let us explore the localization property for the graded rings. Let $R = \sum_{n\in\mathbb{Z}} R_n$ be a Noetherian $\mathbb{Z}$-graded ring and assume that R is an H-local ring with an H-maximal ideal P of R in the sense of S. Goto and K.-i. Watanabe (see [GW, Definition 1.1.5, Definition 1.1.6]). Let $M \neq (0)$ be a finitely generated graded R-module of dimension d. Let $\{D_i\}_{0\le i\le \ell}$ be the dimension filtration of M. We put $q = \dim R/P$. For an arbitrary ideal I of a graded ring, I^* stands for the ideal generated by every homogeneous elements in I.

Lemma 2.3. $\dim_R M = \dim_{R_P} M_P + q$. *Therefore* $\dim_R M = \dim_{R_\mathfrak{m}} M_\mathfrak{m}$ *for every maximal ideal $\mathfrak{m}$ of R with $\mathfrak{m} \supseteq P$.*

Proof. We may assume $q > 0$ (thus $q = 1$). Let $\mathfrak{m}$ be a maximal ideal of R such that $\mathfrak{m} \supseteq P$ and $\dim R_\mathfrak{m}/PR_\mathfrak{m} = 1$. Let $\mathfrak{p} \in \mathrm{Ass}_R M$ such that $P \supseteq \mathfrak{p}$ and $\dim R_P/\mathfrak{p}R_P = \dim_{R_P} M_P$. Then we have $\dim_R M \ge \dim_{R_P} M_P + 1$. Conversely, we choose $\mathfrak{p} \in \mathrm{Ass}_R M$ and a maximal ideal $\mathfrak{m}$ of R such that $\mathfrak{p} \subseteq \mathfrak{m}$, $\dim R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m} = d$. Since $\mathfrak{p}$ is a graded ideal of R, $\mathfrak{p} \subseteq \mathfrak{m}^*$. Notice that $\mathfrak{m}$ is not a graded ideal of R because $q = 1$. Hence we get $\mathfrak{m}^* \subseteq P$ and

$$d = \dim R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m} = \dim R_{\mathfrak{m}^*}/\mathfrak{p}R_{\mathfrak{m}^*} + 1 \le \dim R_P/\mathfrak{p}R_P + 1 \le \dim_{R_P} M_P + 1.$$

$\square$

Apply [GHS, Theorem 2.3] and Lemma 2.3, we get the following.

Corollary 2.4. *The following assertions hold true.*

(1) *$[D_0]_\mathfrak{m} = (0) \subsetneq [D_1]_\mathfrak{m} \subsetneq \ldots \subsetneq [D_\ell]_\mathfrak{m} = M_\mathfrak{m}$ is the dimension filtration of $M_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of R such that $\mathfrak{m} \supseteq P$.*
(2) *$[D_0]_P = (0) \subsetneq [D_1]_P \subsetneq \ldots \subsetneq [D_\ell]_P = M_P$ is the dimension filtration of M_P, so that M is a sequentially Cohen-Macaulay R-module if and only if M_P is a sequentially Cohen-Macaulay R_P-module.*

Finally we reach the goal of this section.

Theorem 2.5. *The following conditions are equivalent.*

(1) *M is a sequentially Cohen-Macaulay R-module.*
(2) *M_P is a sequentially Cohen-Macaulay R_P-module.*

When this is the case, $M_\mathfrak{p}$ is a sequentially Cohen-Macaulay $R_\mathfrak{p}$-module for every $\mathfrak{p} \in \mathrm{Supp}_R M$.

Proof. The equivalence of conditions (1) and (2) follows from Corollary 2.4. Let us make sure of the last assertion. Note that $\mathfrak{p}^* \subseteq P$ for any $\mathfrak{p} \in \mathrm{Supp}_R M$. Thanks to [CGT, Proposition 2.6], $M_{\mathfrak{p}^*}$ is a sequentially Cohen-Macaulay $R_{\mathfrak{p}^*}$-module. Since $\mathfrak{p}^* R_{(\mathfrak{p})}$ is an H-maximal ideal of the homogeneous localization $R_{(\mathfrak{p})}$ of R, $M_{(\mathfrak{p})}$ is a sequentially Cohen-Macaulay $R_{(\mathfrak{p})}$-module. We may assume that $\mathfrak{p}$ is not a graded ideal of R, so that $\mathfrak{p}R_{(\mathfrak{p})}$ is a maximal ideal of $R_{(\mathfrak{p})}$. Therefore $M_\mathfrak{p}$ is a sequentially Cohen-Macaulay $R_\mathfrak{p}$-module. $\square$

3. Module-finite extension of sequentially Cohen-Macaulay modules

In this section, we assume that R is a local ring with maximal ideal $\mathfrak{m}$. Remember that M_n stands for the largest R-submodule of M with $\dim_R M_n \le n$ for each $n \in \mathbb{Z}$.

We note the following.

Lemma 3.1. *Let M and N be finitely generated R-modules. Then $[M \oplus N]_n = M_n \oplus N_n$ for every $n \in \mathbb{Z}$.*

Proof. We have $[M \oplus N]_n \supseteq M_n \oplus N_n$, since $\dim_R(M_n \oplus N_n) = \max\{\dim_R M_n, \dim_R N_n\} \le n$. Let $p : L = M \oplus N \to M, (x, y) \mapsto x$ be the first projection. Then $p(L_n) \subseteq M_n$, since $\dim_R p(L_n) \le \dim_R L_n \le n$. We similarly have $q(L_n) \subseteq N_n$, where $q : M \oplus N \to N, (x, y) \mapsto y$ denotes the second projection. Hence $[M \oplus N]_n \subseteq M_n \oplus N_n$ as claimed. □

The following proposition includes the result [CN, Proposition 4.5].

Proposition 3.2. *Let M and N $(M, N \neq (0))$ be finitely generated R-modules. Then $M \oplus N$ is a sequentially Cohen-Macaulay R-module if and only if both M and N are sequentially Cohen-Macaulay R-modules.*

Proof. We set $L = M \oplus N$ and $\ell = \sharp\mathcal{S}(L)$. Then $\mathcal{S}(L) = \mathcal{S}(M) \cup \mathcal{S}(N)$, as $\mathrm{Ass}_R L = \mathrm{Ass}_R M \cup \mathrm{Ass}_R N$. Hence if $\ell = 1$, then $\mathcal{S}(L) = \mathcal{S}(M) = \mathcal{S}(N)$ and $\dim_R L = \dim_R M = \dim_R N$. Therefore when $\ell = 1$, L is a sequentially Cohen-Macaulay R-module if and only if L is a Cohen-Macaulay R-module, and the second condition is equivalent to saying that the R-modules M and N are Cohen-Macaulay, that is M and N are sequentially Cohen-Macaulay R-modules. Suppose that $\ell > 1$ and that our assertion holds true for $\ell - 1$. Let

$$D_0 = (0) \subsetneq D_1 \subsetneq D_2 \subsetneq \cdots \subsetneq D_\ell = L$$

be the dimension filtration of $L = M \oplus N$, where $\mathcal{S}(L) = \{d_1 < d_2 < \cdots < d_\ell\}$. Then $\{D_i/D_1\}_{1 \le i \le \ell}$ is the dimension filtration of L/D_1 and hence L is a sequentially Cohen-Macaulay R-module if and only if D_1 is a Cohen-Macaulay R-module and L/D_1 is a sequentially Cohen-Macaulay R-module. Because

$$D_1 = \begin{cases} M_{d_1} \oplus (0) & (d_1 \in \mathcal{S}(M) \setminus \mathcal{S}(N)), \\ M_{d_1} \oplus N_{d_1} & (d_1 \in \mathcal{S}(M) \cap \mathcal{S}(N)), \\ (0) \oplus N_{d_1} & (d_1 \in \mathcal{S}(N) \setminus \mathcal{S}(M)) \end{cases}$$

by Lemma 3.1, the hypothesis on ℓ readily shows the second condition is equivalent to saying that the R-modules M and N are sequentially Cohen-Macaulay. □

In what follows, let A be a Noetherian local ring and assume that A is a module-finite algebra over R.

The main result of this section is the following.

Theorem 3.3. *Let $M \neq (0)$ be a finitely generated A-module. Then the following assertions hold true.*

1. *M_n is the largest A-submodule of M with $\dim_A M_n \le n$ for every $n \in \mathbb{Z}$.*
2. *The dimension filtration of M as an A-module coincides with that of M as an R-module.*
3. *M is a sequentially Cohen-Macaulay A-module if and only if M is a sequentially Cohen-Macaulay R-module.*

Proof. Let $n \in \mathbb{Z}$ and X denote the largest A-submodule of M with $\dim_A X \le n$. Then $X \subseteq M_n$, since $\dim_R X = \dim_A X \le n$. Let $Y = AM_n$. Then $\dim_A Y \le n$. In fact, let $\mathfrak{p} \in \mathrm{Ass}_R Y$. Then since $[M_n]_\mathfrak{p} \subseteq Y_\mathfrak{p} = A_\mathfrak{p}{\cdot}[M_n]_\mathfrak{p} \subseteq M_\mathfrak{p}$, we see $[M_n]_\mathfrak{p} \ne (0)$, so that $\mathfrak{p} \in \mathrm{Supp}_R M_n$. Hence $\dim R/\mathfrak{p} \le \dim_R M_n \le n$. Thus $\dim_A Y = \dim_R Y \le n$, whence $M_n \subseteq Y \subseteq X$, which shows $X = M_n$. Therefore assertions (1) and (2) follows. Since $\dim_A L = \dim_R L$ and $\mathrm{depth}_A L = \mathrm{depth}_R L$ for every finitely generated A-module L, we get assertion (3). $\square$

We summarize some consequences.

Corollary 3.4. *A is a sequentially Cohen-Macaulay local ring if and only if A is a sequentially Cohen-Macaulay R-module.*

Corollary 3.5. *Let M be a finitely generated A-module. Suppose that R is a direct summand of M as an R-module. If M is a sequentially Cohen-Macaulay A-module, then R is a sequentially Cohen-Macaulay local ring.*

Proof. We write $M = R \oplus N$ where N is an R-submodule of M. Since M is a sequentially Cohen-Macaulay A-module, by Theorem 3.3 it is a sequentially Cohen-Macaulay R-module as well, so that by Proposition 3.2, R is a sequentially Cohen-Macaulay local ring. $\square$

Corollary 3.6. *Suppose that R is a direct summand of A as an R-module. If A is a sequentially Cohen-Macaulay local ring, then R is a sequentially Cohen-Macaulay local ring.*

We consider the invariant subring $R = A^G$.

Corollary 3.7. *Let A be a Noetherian local ring, G a finite subgroup of* $\mathrm{Aut}\, A$. *Suppose that the order of G is invertible in A. If A is a sequentially Cohen-Macaulay local ring, then the invariant subring $R = A^G$ of A is a sequentially Cohen-Macaulay local ring.*

Proof. Since the order of G is invertible in A, A is a module-finite extension of $R = A^G$ such that R is a direct summand of A (see [BR] and reduce to the case where A is a reduced ring). Hence the assertion follows from Corollary 3.6. $\square$

Remark 3.8. In the setting of Corollary 3.7, let $\{D_i\}_{0 \le i \le \ell}$ be the dimension filtration of A. Then every D_i is a G-stable ideal of A (compare with Theorem 3.3 (1)) and the dimension filtration of R is given by a refinement of $\{D_i^G\}_{0 \le i \le \ell}$.

The goal of this section is the following.

Corollary 3.9. *Let R be a Noetherian local ring, $M \ne (0)$ a finitely generated R-module. We put $A = R \ltimes M$ the idealization of M over R. Then the following conditions are equivalent.*

1. *$A = R \ltimes M$ is a sequentially Cohen-Macaulay local ring.*
2. *$A = R \ltimes M$ is a sequentially Cohen-Macaulay R-module.*
3. *R is a sequentially Cohen-Macaulay local ring and M is a sequentially Cohen-Macaulay R-module.*

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