QUASI-GORENSTEIN EXTENDED REES ALGEBRAS ASSOCIATED WITH FILTRATIONS

NAOKI ENDO

ABSTRACT. This paper investigates the quasi-Gorenstein property of extended Rees algebras associated with the Hilbert filtrations on a Noetherian local ring. We provide necessary and sufficient conditions for the deformation of the quasi-Gorenstein property, characterized by the Cohen-Macaulayness of the Matlis dual of local cohomology modules. As a consequence, we offer a characterization of the quasi-Gorenstein property of extended Rees algebras in terms of conditions on the length of local cohomology.

1. INTRODUCTION

In this paper, we study the quasi-Gorenstein property for extended Rees algebras associated with the Hilbert filtrations of ideals in a Noetherian local ring. As introduced by Platte and Storch in 1977 [13], a Noetherian local ring $(R,\mathfrak{m})$ is defined to be *quasi-Gorenstein* if it possesses a canonical module K_R such that $R \cong \mathrm{K}_R$ as an R-module. Setting $d = \dim R$, this condition is equivalent to $\mathrm{H}^d_{\mathfrak{m}}(R) \cong \mathrm{E}_R(R/\mathfrak{m})$, where $\mathrm{H}^d_{\mathfrak{m}}(-)$ denotes the d-th local cohomology functor with respect to $\mathfrak{m}$, and $\mathrm{E}_R(R/\mathfrak{m})$ represents the injective envelope of $R/\mathfrak{m}$. Now, let $R = \bigoplus_{n\in\mathbb{Z}} R_n$ be a Noetherian $\mathbb{Z}$-graded ring with unique graded maximal ideal $\mathfrak{M}$. We define R to be *quasi-Gorenstein* if it has a graded canonical module and its localization $R_{\mathfrak{M}}$ is quasi-Gorenstein. Equivalently, R admits a graded canonical module K_R such that $\mathrm{K}_R \cong R(a)$ for some $a \in \mathbb{Z}$. Here, for a graded R-module M and for an integer ℓ, let $M(\ell)$ denote the graded R-module whose underlying R-module is the same as that of the R-module M and the grading is given by $[M(\ell)]_m = M_{\ell+m}$ for all $m \in \mathbb{Z}$, where $[-]_m$ denotes the m-th homogeneous component.

The quasi-Gorenstein property of the extended Rees algebra has been studied in previous works, such as [4, 5, 11]. This paper focuses on investigating the conditions under which quasi-Gorensteinness deforms, specifically examining its inheritance from the associated graded ring to the extended Rees algebra. The question of whether the quasi-Gorenstein property deforms – that is, whether R is quasi-Gorenstein if a Noetherian local ring $(R,\mathfrak{m})$ and its non-zerodivisor $x \in \mathfrak{m}$ satisfy R/xR being quasi-Gorenstein – has been a fundamental question in commutative ring theory. In 2020, Shimomoto, Taniguchi, and Tavanfar provided a counterexample, using Macaulay2, demonstrating that quasi-Gorensteinness does not deform in general ([18, Theorem 4.2]). Thus, characterizing the circumstances under which quasi-Gorensteinness deforms has become a central problem in this area. In [18], the authors presented various sufficient conditions under which the deformation of quasi-Gorensteinness holds. Moreover, they discussed conditions for the deformation of quasi-Gorensteinness in graded rings, but their results were restricted to $\mathbb{N}$-graded rings, particularly standard graded rings. In contrast, this paper examines

2020 *Mathematics Subject Classification.* 13A30, 13H10, 13C14.
Key words and phrases. Quasi-Gorenstein ring, extended Rees algebra, FLC ring.
The author was partially supported by JSPS Grant-in-Aid for Scientific Research (C) 23K03058.

the quasi-Gorenstein property of extended Rees algebras, which are $\mathbb{Z}$-graded rings associated with filtrations of ideals, in relation to the quasi-Gorensteinness of their associated graded rings. In particular, this study characterizes the deformation of quasi-Gorensteinness from the perspective of the Cohen-Macaulayness of the Matlis dual of local cohomology modules.

Let $(R,\mathfrak{m})$ be a Noetherian local ring with $d = \dim R \geq 3$ which is a homomorphic image of a Gorenstein ring. Let $\mathscr{F} = \{F_n\}_{n\in\mathbb{Z}}$ be the Hilbert filtration of ideals in R, i.e., it is a filtration of ideals such that F_1 is $\mathfrak{m}$-primary and $F_{n+1} = F_1F_n$ for all $n \gg 0$. Denote by

$$\mathscr{R}'(\mathscr{F}) = \sum_{n\in\mathbb{Z}} F_nt^n \subseteq R[t,t^{-1}] \ \text{ and } \ G(\mathscr{F}) = \mathscr{R}'(\mathscr{F})/t^{-1}\mathscr{R}'(\mathscr{F}) \cong \bigoplus_{n\geq 0} F_n/F_{n+1}$$

the *extended Rees algebra of* $\mathscr{F}$ and the *associated graded ring of* $\mathscr{F}$, respectively, where t is an indeterminate over R.

With this notation, the main result of this paper is stated as follows.

Theorem 1.1. *Suppose that $G(\mathscr{F})$ is a quasi-Gorenstein graded ring,* $\operatorname{depth}\mathscr{R}'(\mathscr{F}) \geq d$*, and $\mathrm{H}^{d-1}_{\mathfrak{M}}(G(\mathscr{F}))$ is finitely generated as an $\mathscr{R}'(\mathscr{F})$-module, where $\mathrm{H}^i_{\mathfrak{M}}(-)$ denotes the i-th graded local cohomology functor with respect to the unique graded maximal ideal $\mathfrak{M}$ of $\mathscr{R}'(\mathscr{F})$. Then $\mathscr{R}'(\mathscr{F})$ is quasi-Gorenstein if and only if the length of $\mathrm{H}^{d-1}_{\mathfrak{m}}(R)$ as an R-module coincides with the length of $\mathrm{H}^{d-1}_{\mathfrak{M}}(G(\mathscr{F}))$ as a $G(\mathscr{F})$-module.*

This paper is organized as follows. In Section 2, we review fundamental concepts and results pertaining to canonical modules, modules with finite local cohomology (FLC), and blow-up algebras associated with filtrations of ideals. Additionally, we present a refined version of the well-known result [21, Corollary 2.7] concerning regular sequences on associated graded rings. Section 3 focuses on the conditions under which the quasi-Gorenstein property is preserved under deformations, utilizing the Cohen-Macaulay property of the Matlis dual of local cohomology modules. Finally, in Section 4, we provide a proof of Theorem 1.1.

Throughout this paper, unless otherwise specified, we use the following terminology and notation. For a commutative ring R and an R-module N, let $\ell_R(N)$ denote the length of N. When $(R,\mathfrak{m})$ is a Noetherian local ring, we denote by $\widehat{R}$ the $\mathfrak{m}$-adic completion of R. The Matlis dual functor is denoted by $(-)^\vee = \mathrm{Hom}_R(-,\mathrm{E}_R(R/\mathfrak{m}))$, where $\mathrm{E}_R(R/\mathfrak{m})$ is the injective envelope of $R/\mathfrak{m}$. Let $\mathrm{H}^i_{\mathfrak{m}}(-)$ be the i-th local cohomology functor with respect to $\mathfrak{m}$. Furthermore, for an $\mathfrak{m}$-primary ideal I in R and a finitely generated R-module M with $s = \dim_R M$, there exist integers $\mathrm{e}_i(I,M)$, called the *Hilbert coefficients* of M with respect to I, satisfying the equality

$$\ell_R(M/I^{n+1}M) = \mathrm{e}_0(I,M)\binom{n+s}{s} - \mathrm{e}_1(I,M)\binom{n+s-1}{s-1} + \cdots + (-1)^s\mathrm{e}_s(I,M)$$

for all $n \gg 0$. When $R = \bigoplus_{n\in\mathbb{Z}} R_n$ is a Noetherian $\mathbb{Z}$-graded ring with unique graded maximal ideal $\mathfrak{M}$, we denote by $\mathrm{H}^i_{\mathfrak{M}}(-)$ the i-th graded local cohomology functor with respect to $\mathfrak{M}$.

2. Preliminaries

In this section, we provide an overview of the preliminaries that will be utilized throughout this paper. Let $(R,\mathfrak{m})$ be a Noetherian local ring with $d = \dim R$. For a finitely generated R-module M, we define $\mathrm{Assh}_R M = \{\mathfrak{p} \in \mathrm{Supp}_R M \mid \dim R/\mathfrak{p} = \dim_R M\}$. For an ideal I of R, let $V(I)$ denote the set of all prime ideals of R containing I.

2.1. **Canonical modules.** Recall that a *canonical module* K of R is a finitely generated R-module satisfying the isomorphism

$$\widehat{R}\otimes_R K \cong \mathrm{Hom}_{\widehat{R}}(\mathrm{H}^d_{\widehat{\mathfrak{m}}}(\widehat{R}), \mathrm{E}_{\widehat{R}}(\widehat{R}/\widehat{\mathfrak{m}}))$$

([7, Definition 5.6]). The canonical module is uniquely determined up to isomorphism ([1, (1.5)]; see also [7, Lemma 5.8]), provided it exists. We denote the canonical module by K_R. A canonical module exists for the ring R if R is a homomorphic image of a Gorenstein ring, and the converse also holds when R is Cohen-Macaulay ([15, 17]).

Now assume that the canonical module K_R exists. For every $\mathfrak{p} \in \mathrm{Supp}_R \mathrm{K}_R$, the localization $(\mathrm{K}_R)_\mathfrak{p}$ serves as the canonical module of $R_\mathfrak{p}$ ([1, (1.6)]). Moreover, it holds that

$$\mathrm{Supp}_R \mathrm{K}_R = \{\mathfrak{p} \in \mathrm{Spec}\, R \mid \dim R_\mathfrak{p} + \dim R/\mathfrak{p} = d\}$$

([1, (1.9)]). Additionally, for every $\mathfrak{p} \in \mathrm{Supp}_R \mathrm{K}_R$, any subsystem of parameters for $R_\mathfrak{p}$ of length at most 2 forms a K_R-regular sequence. Thus, K_R satisfies Serre's (S_2)-condition ([1, (1.10)]). Here, a finitely generated R-module M satisfies *Serre's (S_n)-condition*, if $\mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \geq \inf\{n, \dim_{R_\mathfrak{p}} M_\mathfrak{p}\}$ for every $\mathfrak{p} \in \mathrm{Spec}\, R$.

2.2. **FLC modules.** Following [16], a finitely generated R-module M is said to have *finite local cohomology* (abbr. FLC) if $\mathrm{H}^i_\mathfrak{m}(M)$ is finitely generated (or equivalently, of finite length) for all $i \neq \dim_R M$. For instance, if either $\dim_R M \leq 1$ or M is a Buchsbaum module (e.g., Cohen-Macaulay), then M possesses FLC. Thus, modules with FLC can be regarded as a generalization of Buchsbaum modules, and they are occasionally referred to as *generalized Buchsbaum* modules or *generalized Cohen-Macaulay* modules.

Note that M has FLC if and only if the $\widehat{R}$-module $\widehat{M}$ does as well. When $s = \dim_R M \geq 1$, the condition for the R-module M to have FLC is equivalent to the condition that the supremum $\mathbb{I}(M) = \sup_\mathfrak{q}(\ell_R(M/\mathfrak{q}M) - \mathrm{e}_0(\mathfrak{q}, M))$, taken over all parameter ideals $\mathfrak{q}$ of M, is finite ([16, (3.3) Satz], [20, Lemma 1.1]). Furthermore, it is also equivalent to the condition that there exists an integer $\ell \gg 0$ such that every system of parameters contained in $\mathfrak{m}^\ell$ acts as a d-sequence on M ([3, Theorem]). In this case, the equality

$$\mathbb{I}(M) = \sum_{i=0}^{s-1} \binom{s-1}{i} \ell_R(\mathrm{H}^i_\mathfrak{m}(M))$$

holds ([16, (3.7) Satz], [20, Lemma 1.5]). If M has FLC with $s = \dim_R M \geq 1$, the localization $M_\mathfrak{p}$ at $\mathfrak{p} \in \mathrm{Supp}_R M \setminus \{\mathfrak{m}\}$ is necessarily a Cohen-Macaulay $R_\mathfrak{p}$-module satisfying the equality

$$\dim_R M = \dim_{R_\mathfrak{p}} M_\mathfrak{p} + \dim R/\mathfrak{p}.$$

The converse holds if R is a homomorphic image of a Cohen-Macaulay ring ([16, (2.5) Satz], [2, (1.17)], see also [10, Corollary 1.2]). Consequently, all normal isolated singularities that appear as homomorphic images of Cohen-Macaulay rings possess FLC. For more details on FLC modules, the reader may refer to [2, 3, 6, 16, 19, 20], and others.

2.3. **Blow-up algebras associated with filtrations.** Let $\mathscr{F} = \{F_n\}_{n\in\mathbb{Z}}$ be a filtration of ideals in R, meaning that F_n is an ideal of R, $F_n \supseteq F_{n+1}$, $F_m F_n \subseteq F_{m+n}$ for all $m, n \in \mathbb{Z}$, and $F_0 = R$. Define

$$\mathscr{R}'(\mathscr{F}) = \sum_{n\in\mathbb{Z}} F_n t^n \subseteq R[t, t^{-1}] \text{ and } G(\mathscr{F}) = \mathscr{R}'(\mathscr{F})/t^{-1}\mathscr{R}'(\mathscr{F}) \cong \bigoplus_{n\geq 0} F_n/F_{n+1},$$

as the *extended Rees algebra of* $\mathscr{F}$ and the *associated graded ring of* $\mathscr{F}$, respectively, where t is an indeterminate over R. Recall that the filtration $\mathscr{F} = \{F_n\}_{n\in\mathbb{Z}}$ is *Noetherian*, if the ring $\mathscr{R}'(\mathscr{F})$ is Noetherian. We say that the filtration $\mathscr{F} = \{F_n\}_{n\in\mathbb{Z}}$ is *Hilbert*, if F_1 is $\mathfrak{m}$-primary and $F_{n+1} = F_1F_n$ for all $n \gg 0$. In addition, for an ideal I of R, we denote by $\mathscr{F}/I$ the filtration $\{(F_n+I)/I\}_{n\in\mathbb{Z}}$ of ideals in R/I.

Examples of Hilbert filtrations are plentiful. Beyond the classical instance of the ideal-adic filtration, when R is analytically unramified, the filtration $\{\overline{I^n}\}_{n\in\mathbb{Z}}$, consisting of the integral closures of powers of an $\mathfrak{m}$-primary ideal I, constitutes the Hilbert filtration ([14]). Furthermore, the filtration $\{\widetilde{(I^n)}\}_{n\in\mathbb{Z}}$, formed by the Ratliff-Rush closures of the powers of I, also qualifies as the Hilbert filtration.

The following constitutes a generalization of the works of [8, Proposition 6] and [21, Corollary 2.7], while also serving as a partial generalization of [9, Proposition 3.5]. Although this may be familiar to experts in the field, we include a proof for the sake of completeness.

Proposition 2.1. *Let* $(R,\mathfrak{m})$ *be a Noetherian local ring and* $\mathscr{F} = \{F_n\}_{n\in\mathbb{Z}}$ *the Hilbert filtration of ideals in R. Let* $a_1,a_2,\dots,a_r \in R$ $(r>0)$ *such that* $a_i \in F_{n_i}$ *with* $n_i \geq 0$. *Then the following conditions are equivalent.*

(1) $a_1t^{n_1}, a_2t^{n_2}, \dots, a_rt^{n_r} \in \mathscr{R}'(\mathscr{F})$ *forms a regular sequence on* $G(\mathscr{F})$.
(2) $a_1,a_2,\dots,a_r$ *forms a regular sequence on R and the equality* $(a_1,a_2,\dots,a_r)\cap F_n = \sum_{j=1}^{r} a_jF_{n-n_j}$ *holds for all* $n\in\mathbb{Z}$.

When this is the case, one has an isomorphism

$$G(\mathscr{F})/(a_1t^{n_1},a_2t^{n_2},\dots,a_rt^{n_r})G(\mathscr{F}) \cong G(\mathscr{F}/(a_1,a_2,\dots,a_r))$$

of rings.

Proof. (1) $\Rightarrow$ (2) By induction on r. Suppose $r=1$. Let $x\in R$ with $a_1x=0$. Since the Hilbert filtration is separated, if $x\neq 0$, we can choose $n\geq 0$ such that $x\in F_n\setminus F_{n+1}$. Then

$$a_1t^{n_1}\cdot\overline{xt^n} = \overline{(a_1x)t^{n_1+n}} = 0$$

where $\overline{(-)}$ denotes the image in $G(\mathscr{F})$. Since $a_1t^{n_1}$ is $G(\mathscr{F})$-regular, it follows that $\overline{xt^n}=0$ in $G(\mathscr{F})$. This implies $x\in F_{n+1}$, which contradicts $x\in F_n\setminus F_{n+1}$. Hence $x=0$. If $a_1\notin\mathfrak{m}$, then $n_1=0$ because $a\in F_{n_1}$. Thus, $a_1t^{n_1}=a_1$ is a unit in $\mathscr{R}'(\mathscr{F})$, a contradiction. Therefore $a\in\mathfrak{m}$ is R-regular. Next, we will show that $(a_1)\cap F_n = a_1F_{n-n_1}$ holds for all $n\in\mathbb{Z}$. Indeed, let $x\in(a_1)\cap F_n$, and write $x=a_1y$ for some $y\in R$. Suppose $y\notin F_{n-n_1}$. Choose $\ell\geq 0$ such that $y\in F_\ell\setminus F_{\ell+1}$. Then $n-n_1\geq \ell+1$. Consequently, $a_1y=x\in F_n\subseteq F_{n_1+\ell+1}$. Since $a_1t^{n_1}$ is $G(\mathscr{F})$-regular, we have $y\in F_{\ell+1}$, a contradiction. Thus, $y\in F_{n-n_1}$, and hence $(a_1)\cap F_n\subseteq a_1F_{n-n_1}$. The converse inclusion follows from $a_1\in F_{n_1}$. Therefore, $(a_1)\cap F_n = a_1F_{n-n_1}$. The canonical surjection $\varepsilon: R\to R/(a_1)$ induces a surjective graded ring homomorphism

$$\varphi: G(\mathscr{F})\to G(\mathscr{F}/(a_1))$$

defined by $\varphi(\overline{xt^n}) = \overline{(x+F_{n+1})t^n}$ for each $x\in F_n$. For $x\in F_n$, we have $xt^n\in\operatorname{Ker}\varphi$ if and only if $x\in(F_{n+1}+(a_1))\cap F_n$, or equivalently $x\in F_{n+1}+a_1F_{n-n_1}$, because $(a_1)\cap F_n = a_1F_{n-n_1}$. Hence $\operatorname{Ker}\varphi = (a_1t^{n_1})$, and therefore $G(\mathscr{F})/(a_1t^{n_1})G(\mathscr{F})\cong G(\mathscr{F}/(a_1))$. Thus, (2) holds for $r=1$.

Now, assume $r \geq 2$ and that (2) holds for $r-1$. Since $G(\mathscr{F})/(a_1t^{n_1})G(\mathscr{F}) \cong G(\mathscr{F}/(a_1))$, the sequence $a_2t^{n_2}, a_3t^{n_3}, \dots, a_rt^{n_r}$ is $G(\mathscr{F}/(a_1))$-regular. Note that $\mathscr{F}/(a_1)$ is the Hilbert filtration of ideals in $R/(a_1)$. By the induction hypothesis, $a_2, a_3, \dots, a_r$ is an R-regular sequence, and the equality

$$(a_2, a_3, \dots, a_r)\overline{R} \cap F_n\overline{R} = \sum_{j=2}^{r} a_jF_{n-n_j}\overline{R}$$

holds, where $\overline{R} = R/(a_1)$. In particular, $a_1, a_2, \dots, a_r \in \mathfrak{m}$ forms an R-regular sequence. Finally, let $x \in (a_1, a_2, \dots, a_r) \cap F_n$. We can choose $y_i \in F_{n-n_i}$ such that $x - (a_2y_2 + a_3y_3 + \cdots + a_ry_r) \in (a_1)$. Since $a_iy_i \in F_n$ for all $2 \leq i \leq r$, it follows that $x - (a_2y_2 + a_3y_3 + \cdots + a_ry_r) \in (a_1) \cap F_n = a_1F_{n-n_1}$. Thus $x \in \sum_{j=1}^{r} a_jF_{n-n_j}$. Consequently, $(a_1, a_2, \dots, a_r) \cap F_n = \sum_{j=1}^{r} a_jF_{n-n_j}$. Furthermore, the induction hypothesis implies

$$\begin{aligned}
G(\mathscr{F})/(a_1t^{n_1}, a_2t^{n_2}, \dots, a_rt^{n_r})G(\mathscr{F}) &\cong G(\mathscr{F}/(a_1))/(a_2t^{n_2}, a_3t^{n_3}, \dots, a_rt^{n_r})G(\mathscr{F}/(a_1)) \\
&\cong G((\mathscr{F}/(a_1))/(a_2, a_3, \dots, a_r)(\mathscr{F}/(a_1))) \\
&\cong G(\mathscr{F}/(a_1, a_2, \dots, a_r)).
\end{aligned}$$

Thus, the result follows.

(2) $\Rightarrow$ (1) Let $J_i = (a_1, a_2, \dots, a_i)$ for each $1 \leq i \leq r$. We prove by descending induction that

$$J_i \cap F_n = \sum_{j=1}^{r} a_jF_{n-n_j}$$

holds for all $1 \leq i \leq r$ and $n \in \mathbb{Z}$. Assuming the assertion holds for $i+1$, we proceed by induction on i. Without loss of generality, we assume $1 \leq i < r$ and that the assertion holds for $i+1$. We now further proceed by induction on n. Notice that the desired equality holds trivially for $n \leq 0$. Let $n > 0$, and assume that the equality holds for $n-1$. By defining $L = \sum_{j=1}^{i} a_jF_{n-n_j}$, we obtain

$$J_i \cap F_n \subseteq J_{i+1} \cap F_n = \sum_{j=1}^{i+1} a_jF_{n-n_j} = L + a_{i+1}F_{n-n_{i+1}}.$$

Thus, $J_i \cap F_n \subseteq \left[L + (a_{i+1}F_{n-n_{i+1}})\right] \cap J_i = L + \left[J_i \cap (a_{i+1}F_{n-n_{i+1}})\right]$. It remains to verify that

$$J_i \cap (a_{i+1}F_{n-n_{i+1}}) \subseteq L.$$

Let $x \in J_i \cap (a_{i+1}F_{n-n_{i+1}})$, and write $x = a_{i+1}y$ with $y \in F_{n-n_{i+1}}$. As $a_{i+1}y = x \in F_i$ and a_{i+1} is a non-zerodivisor modulo J_i, we have $y \in J_i \cap F_{n-n_{i+1}}$.

Case 1. $n_{i+1} > 0$

In this case, $n - n_{i+1} < n$. By the induction hypothesis on n, we have $y \in J_i \cap F_{n-n_{i+1}} = \sum_{j=1}^{i} a_jF_{(n-n_{i+1})-n_j}$. Since $a_{i+1} \in F_{n_{i+1}}$, it follows that $x = a_{i+1}y \in \sum_{j=1}^{i} a_jF_{(n-n_{i+1})-n_j} \cdot F_{n_{i+1}} \subseteq \sum_{j=1}^{i} a_jF_{n-n_j} = L$, as desired.

Case 2. $n_{i+1} = 0$

We first show that $J_i \cap (a_{i+1}F_n) \subseteq (a_{i+1}^mF_n) + L$ for all $m > 0$. If $m = 1$, this inclusion is immediate. Assume $m > 1$, and that the inclusion holds for $m-1$. Let $x \in J_i \cap (a_{i+1}F_n)$. Then $x = a_{i+1}^{m-1}y + \ell$ for some $y \in F_n$ and $\ell \in L$. Thus, $a_{i+1}^{m-1}y = x - \ell \in J_i$, and since a_{i+1} is R/J_i-regular, it follows that $y \in J_i$. Therefore $y \in J_i \cap F_n \subseteq J_{i+1} \cap F_n \subseteq L + a_{i+1}F_n$. Hence

$x = a_{i+1}^{m-1}y + \ell \in a_{i+1}^{m}F_n + L$, that is, $J_i \cap (a_{i+1}F_n) \subseteq (a_{i+1}^{m}F_n) + L$. Taking the intersection over all $m > 0$, we obtain

$$J_i \cap (a_{i+1}F_n) \subseteq \bigcap_{m>0} \left[(a_{i+1}^{m}F_n) + L\right] \subseteq \bigcap_{m>0} [(\mathfrak{m}^m F_n) + L] \subseteq L.$$

Consequently, $J_i \cap (a_{i+1}F_{n-n_{i+1}}) \subseteq L$.

In conclusion, for both cases, we have $J_i \cap (a_{i+1}F_{n-n_{i+1}}) \subseteq L$, which implies $J_i \cap F_n = L = \sum_{j=1}^{i} a_j F_{n-n_j}$. Next, it is straightforward to verify that the equality

$$(a_1t^{n_1}, a_2t^{n_2}, \dots, a_it^{n_i})G(\mathscr{F}) :_{G(\mathscr{F})} a_{i+1}t^{n_{i+1}} = (a_1t^{n_1}, a_2t^{n_2}, \dots, a_it^{n_i})G(\mathscr{F})$$

holds for all $0 \le i < r$. Consider a surjective graded ring homomorphism

$$\varphi : G(\mathscr{F}) \to G(\mathscr{F}/J_r)$$

defined by $\varphi(\overline{xt^n}) = \overline{(x+F_{n+1})t^n}$ for each $x \in F_n$. Since $\operatorname{Ker}\varphi = (a_1t^{n_1}, a_2t^{n_2}, \dots, a_rt^{n_r})G(\mathscr{F})$, we obtain the isomorphism $G(\mathscr{F})/(a_1t^{n_1}, a_2t^{n_2}, \dots, a_rt^{n_r})G(\mathscr{F}) \cong G(\mathscr{F}/J_r)$. Finally, because $a_1, a_2, \dots, a_r \in \mathfrak{m}$ and F_1 is $\mathfrak{m}$-primary, we have $(F_1 + J_r)/J_r \neq R/J_r$. This implies $G(\mathscr{F}/J_r) \neq (0)$. Therefore, $a_1t^{n_1}, a_2t^{n_2}, \dots, a_rt^{n_r} \in \mathscr{R}'(\mathscr{F})$ forms a regular sequence on $G(\mathscr{F})$. ◻

Proposition 2.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $\mathscr{F} = \{F_n\}_{n\in\mathbb{Z}}$ the Hilbert filtration of ideals in R. Suppose that* $\operatorname{depth} G(\mathscr{F}) = r > 0$. *Then there exists homogeneous elements $f_1, f_2, \dots, f_r$ in $\mathscr{R}'(\mathscr{F})$ of non-negative degree such that $f_1, f_2, \dots, f_r$ forms a regular sequence on $G(\mathscr{F})$.*

Proof. Let $\mathfrak{N}$ denote the graded maximal ideal of $G(\mathscr{F})$. We aim to prove that $\operatorname{Ass} G(\mathscr{F}) \subseteq \operatorname{Proj} G(\mathscr{F})$. Indeed, note that $\mathfrak{N} \notin \operatorname{Ass} G(\mathscr{F})$. For each $P \in \operatorname{Ass} G(\mathscr{F})$, the ideal P is graded and satisfies $P \subsetneq \mathfrak{N}$. Assume, for the sake of contradiction, that $P \supseteq G(\mathscr{F})_+$, where $G(\mathscr{F})_+ = \bigoplus_{n>0} F_n/F_{n+1}$. Since F_1 is $\mathfrak{m}$-primary and $\mathfrak{m}G(\mathscr{F})$ is contained in the integral closure of $G(\mathscr{F})_+$, it follows that $G(\mathscr{F})_+$ is a reduction of $\mathfrak{N}$. Consequently, we would have $P = \mathfrak{N}$, which is a contradiction. Hence, $P \in \operatorname{Proj} G(\mathscr{F})$. Therefore, we conclude that $\operatorname{Ass} G(\mathscr{F}) \subseteq \operatorname{Proj} G(\mathscr{F})$, as desired.

By employing the same technique as in the proof of [12, Lemma 3.1], for any subset $\mathscr{X} \subseteq \operatorname{Ass} G(\mathscr{F})$ and any graded ideal I of $G(\mathscr{F})$ with $I \not\subseteq \bigcup_{P\in\mathscr{X}} P$, there exists a homogeneous element $g \in I$ such that $g \notin \bigcup_{P\in\mathscr{X}} P$. In particular, we can choose a homogeneous $g \in \mathfrak{N}$ such that $g \notin \bigcup_{P\in \operatorname{Ass} G(\mathscr{F})} P$. Write $g = \overline{a_1t^{n_1}}$, where $a_1 \in F_{n_1}$ and $n_1 \ge 0$, with $\overline{(-)}$ denoting the image in $G(\mathscr{F})$. By Proposition 2.1, it follows that $a_1 \in \mathfrak{m}$ is R-regular and that

$$G(\mathscr{F})/(a_1t^{n_1})G(\mathscr{F}) \cong G(\mathscr{F}/(a_1)).$$

Since the filtration $\mathscr{F}/(a_1)$ is Hilbert and $\operatorname{depth} G(\mathscr{F}/(a_1)) = r - 1$, the induction hypothesis ensures the existence of homogeneous elements $g_2, \dots, g_r \in G(\mathscr{F}/(a_1))$ of non-negative degree such that these elements form a regular sequence on $G(\mathscr{F}/(a_1))$. Consequently, there exist homogeneous elements $f_1, f_2, \dots, f_r$ in $\mathscr{R}'(\mathscr{F})$ of non-negative degree such that $f_1, f_2, \dots, f_r$ is $G(\mathscr{F})$-regular. ◻

3. When does the quasi-Gorenstein property deform?

In this section, we explore the conditions under which the quasi-Gorenstein property deforms. Let $(R,\mathfrak{m})$ be a Noetherian local ring with $d=\dim R>0$ that admits a canonical module K_R. Note that R is Gorenstein if $d\le 3$ and R/xR is quasi-Gorenstein for some non-zerodivisor $x\in\mathfrak{m}$. Thus, when addressing the deformation problem of the quasi-Gorenstein property, it is sufficient to focus on cases where $d\ge 4$ and R is not Cohen-Macaulay. In such cases, one has $\operatorname{depth} R\ge 3$.

We begin with the following, which plays a key in our argument.

Theorem 3.1. *Let $(R,\mathfrak{m})$ be a Noetherian local ring with $d=\dim R\ge 4$ and $\operatorname{depth} R=d-1$, admitting the canonical module K_R. Suppose that there exists a non-zerodivisor $x\in\mathfrak{m}$ on R such that R/xR is quasi-Gorenstein and $\mathrm{H}^{d-2}_{\mathfrak{m}}(R/xR)$ is finitely generated as an R-module. Then the following assertions hold true, where M denotes the Matlis dual of $\mathrm{H}^{d-1}_{\mathfrak{m}}(R)$.*

1. *R is quasi-Gorenstein if and only if M is a Cohen-Macaulay R-module with $\dim_R M=1$.*
2. *$\operatorname{Supp}_R M$ is the non-Cohen-Macaulay locus of R, i.e., the set of prime ideals $\mathfrak{p}$ of R for which the local ring $R_{\mathfrak{p}}$ is not Cohen-Macaulay.*
3. *If $\dim_R M=1$, then M is a Cohen-Macaulay R-module if and only if the equality*
$$\ell_R(\mathrm{H}^{d-2}_{\mathfrak{m}}(R/xR))=\sum_{\mathfrak{p}\in\operatorname{Assh}_R M}\ell_{R_{\mathfrak{p}}}(\mathrm{H}^{d-2}_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}))\cdot \mathrm{e}_0(x,R/\mathfrak{p})$$
holds.

Proof. Without loss of generality, we may assume that R is $\mathfrak{m}$-adically complete. Let $\overline{R}=R/xR$. Note that $M\ne(0)$. By applying the functor $\mathrm{H}^i_{\mathfrak{m}}(-)$ to the sequence $0\to R\xrightarrow{x}R\to\overline{R}\to 0$, we obtain the long exact sequence of R-modules
$$0\to\mathrm{H}^{d-2}_{\mathfrak{m}}(\overline{R})\to\mathrm{H}^{d-1}_{\mathfrak{m}}(R)\xrightarrow{x}\mathrm{H}^{d-1}_{\mathfrak{m}}(R)\to\mathrm{H}^{d-1}_{\mathfrak{m}}(\overline{R})\to\mathrm{H}^{d}_{\mathfrak{m}}(R)\xrightarrow{x}\mathrm{H}^{d}_{\mathfrak{m}}(R)\to 0.$$
Taking Matlis duality yields the exact sequence
$$0\to\mathrm{K}_R\xrightarrow{x}\mathrm{K}_R\to\mathrm{K}_{\overline{R}}\to M\xrightarrow{x}M\to[\mathrm{H}^{d-2}_{\mathfrak{m}}(\overline{R})]^{\vee}\to 0$$
which can be divided into the following three parts
$$0\to\mathrm{K}_R\xrightarrow{x}\mathrm{K}_R\to\mathrm{K}_R/x\mathrm{K}_R\to 0,$$
$$0\to\mathrm{K}_R/x\mathrm{K}_R\to\mathrm{K}_{\overline{R}}\to C\to 0,\ \text{ and}$$
$$0\to C\to M\xrightarrow{x}M\to[\mathrm{H}^{d-2}_{\mathfrak{m}}(\overline{R})]^{\vee}\to 0$$
where $C=(0):_M x$. Since $[\mathrm{H}^{d-2}_{\mathfrak{m}}(\overline{R})]^{\vee}$ has finite length, it follows that $\dim_R M\le 1$.

(1) We first assume that R is quasi-Gorenstein. If $\dim_R M=0$, the ring R has FLC, because $\ell_R(\mathrm{H}^{d-1}_{\mathfrak{m}}(R))<\infty$. By [2, (1.16)], we get the isomorphisms
$$M=[\mathrm{H}^{d-1}_{\mathfrak{m}}(R)]^{\vee}\cong\mathrm{H}^{d-(d-1)+1}_{\mathfrak{m}}(\mathrm{K}_R)\cong\mathrm{H}^{2}_{\mathfrak{m}}(R)=(0)$$
which leads to a contradiction. Thus $\dim_R M=1$, so $x\in\mathfrak{m}$ forms a system of parameters for M. Therefore, it suffices to prove that $C=(0)$. Assume, to the contrary, that $C\ne(0)$, and seek a contradiction. As $\ell_R(M/xM)<\infty$, for each $\mathfrak{p}\in\operatorname{Spec}R\setminus\{\mathfrak{m}\}$, we have the exact sequence
$$0\to C_{\mathfrak{p}}\to M_{\mathfrak{p}}\xrightarrow{x}M_{\mathfrak{p}}\to 0$$

of $R_\mathfrak{p}$-modules. This implies that $C_\mathfrak{p} = (0)$, and hence $\ell_R(C) < \infty$. Next, applying the depth lemma to the exact sequence

$$0 \to \mathrm{K}_R/x\mathrm{K}_R \to \mathrm{K}_{\overline{R}} \cong \overline{R} \to C \to 0,$$

we have $\operatorname{depth} \mathrm{K}_R = 2$. However, this contradicts the facts that $R \cong \mathrm{K}_R$ and $\operatorname{depth} R \geq 3$. Consequently $C = (0)$, as desired.

Conversely, we assume M is Cohen-Macaulay and of dimension one. Since $\ell_R(M/xM) < \infty$, note that $x \in \mathfrak{m}$ is a non-zerodivisor on M. Thus $C = (0)$ and $\mathrm{K}_R/x\mathrm{K}_R \cong \overline{R}$. It is straightforward to check that the local ring R is quasi-Gorenstein.

(2) Let $\mathfrak{p} \in \operatorname{Supp}_R M$. To show that $R_\mathfrak{p}$ is not Cohen-Macaulay, we may assume $\mathfrak{p} \neq \mathfrak{m}$. Then $\dim_R M = 1$ and $\mathfrak{p} \in \operatorname{Assh}_R M$. In particular, $\operatorname{Supp}_R M \setminus \{\mathfrak{m}\} \subseteq \operatorname{Assh}_R M$. Choose a Gorenstein complete local ring S with $\dim S = d$ such that R is a homomorphic image of S. Let $P = \mathfrak{p} \cap S \in \operatorname{Spec} S$. Then $S/P \cong R/\mathfrak{p}$, which implies $\dim S_P = d-1$. As S is Gorenstein and complete, we get the isomorphism

$$M = [\mathrm{H}^{d-1}_\mathfrak{m}(R)]^\vee \cong \operatorname{Ext}^1_S(R, S)$$

which yields $M_\mathfrak{p} \cong \operatorname{Ext}^1_{S_P}(R_\mathfrak{p}, S_P)$. Hence

$$(0) \neq \widehat{R_\mathfrak{p}} \otimes_{R_\mathfrak{p}} M_\mathfrak{p} \cong \widehat{S_P} \otimes_{S_P} \operatorname{Ext}^1_{S_P}(R_\mathfrak{p}, S_P) \cong [\mathrm{H}^{d-2}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})]^\vee.$$

This shows $0 < \ell_{R_\mathfrak{p}}(\mathrm{H}^{d-2}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})) < \infty$, because $\mathfrak{p} \in \operatorname{Assh}_R M$. Then $R_\mathfrak{p}$ is not Cohen-Macaulay. Indeed, we assume the contrary, i.e., $R_\mathfrak{p}$ is Cohen-Macaulay. As $\operatorname{depth} R_\mathfrak{p} = d-2 \geq 2$, we can choose a non-zerodivisor $\alpha \in \mathfrak{p}R_\mathfrak{p}$ on $R_\mathfrak{p}$. The sequence $0 \to R_\mathfrak{p} \to R_\mathfrak{p} \to R_\mathfrak{p}/\alpha R_\mathfrak{p} \to 0$ induces the exact sequence

$$0 \to \mathrm{H}^{d-3}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p}/\alpha R_\mathfrak{p}) \to \mathrm{H}^{d-2}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p}) \xrightarrow{\alpha} \mathrm{H}^{d-2}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p}) \to 0$$

of $R_\mathfrak{p}$-modules. Since $\mathrm{H}^{d-2}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})$ is finitely generated, it follows that $\mathrm{H}^{d-3}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p}/\alpha R_\mathfrak{p}) = (0)$. This contradicts the fact that $R_\mathfrak{p}/\alpha R_\mathfrak{p}$ is Cohen-Macaulay and of dimension $d-3$. Therefore, $R_\mathfrak{p}$ cannot be Cohen-Macaulay, as required.

On the other hand, pick $\mathfrak{p} \in \operatorname{Spec} R$ such that $R_\mathfrak{p}$ is not Cohen-Macaulay. Let S be a Gorenstein complete local ring with $\dim S = d$ such that R is a homomorphic image of S. Setting $P = \mathfrak{p} \cap S$, $n = \dim S_P$, and $t = \operatorname{depth} R_\mathfrak{p}$, we have

$$n = \dim S_P \geq \dim R_\mathfrak{p} > t = \operatorname{depth} R_\mathfrak{p},$$

which implies $\operatorname{Ext}^{n-t}_S(R, S) \neq (0)$. If $n - t \geq 2$, then

$$\operatorname{Ext}^{n-t}_S(R, S) \cong [\mathrm{H}^{d-(n-t)}_\mathfrak{m}(R)]^\vee = (0),$$

a contradiction. Hence, $n - t = 1$, and we conclude that

$$M_\mathfrak{p} \cong \operatorname{Ext}^1_{S_P}(R_\mathfrak{p}, S_P) \neq (0).$$

Thus, $\operatorname{Supp}_R M$ is precisely the non-Cohen-Macaulay locus of R.

(3) Suppose $\dim_R M = 1$. Since $M/xM \cong [\mathrm{H}^{d-2}_\mathfrak{m}(\overline{R})]^\vee$, we have $\ell_R(M/xM) = \ell_R(\mathrm{H}^{d-2}_\mathfrak{m}(\overline{R}))$. First, assume that M is Cohen-Macaulay. Because $x \in \mathfrak{m}$ is a system of parameters for M, we get $\mathrm{e}_0(x, M) = \ell_R(M/xM)$. Using the associativity formula for multiplicity, we derive the equalities

$$\mathrm{e}_0(x, M) = \sum_{\mathfrak{p} \in \operatorname{Assh}_R M} \ell_{R_\mathfrak{p}}(M_\mathfrak{p}) \cdot \mathrm{e}_0(x, R/\mathfrak{p}) = \sum_{\mathfrak{p} \in \operatorname{Assh}_R M} \ell_{R_\mathfrak{p}}(\mathrm{H}^{d-2}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})) \cdot \mathrm{e}_0(x, R/\mathfrak{p})$$

where the second equality follows from the fact that

$$\ell_{R_\mathfrak{p}}(M_\mathfrak{p}) = \ell_{R_\mathfrak{p}}(\widehat{R_\mathfrak{p}} \otimes_{R_\mathfrak{p}} M_\mathfrak{p}) = \ell_{R_\mathfrak{p}}[(\mathrm{H}^{d-2}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p}))^\vee) = \ell_{R_\mathfrak{p}}(\mathrm{H}^{d-2}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p}))$$

for all $\mathfrak{p} \in \mathrm{Assh}_R M$. This establishes the desired equality. Conversely, assume that

$$\ell_R(\mathrm{H}^{d-2}_\mathfrak{m}(R/xR)) = \sum_{\mathfrak{p} \in \mathrm{Assh}_R M} \ell_{R_\mathfrak{p}}(\mathrm{H}^{d-2}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})) \cdot \mathrm{e}_0(x, R/\mathfrak{p}).$$

Since $\ell_{R_\mathfrak{p}}(M_\mathfrak{p}) = \ell_{R_\mathfrak{p}}(\mathrm{H}^{d-2}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p}))$, it follows that $\ell_R(M/xM) = \mathrm{e}_0(x, M)$. Hence, M is Cohen-Macaulay. □

Remark 3.2. Under the assumption of Theorem 3.1, the condition that $\mathrm{H}^{d-2}_\mathfrak{m}(R/xR)$ is finitely generated as an R-module is equivalent to stating that the ring R/xR has FLC. The latter condition holds if either R/xR is locally Cohen-Macaulay on the punctured spectrum, or $\dim R = 4$.

Corollary 3.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with $\dim R = 4$ admitting the canonical module K_R. Suppose that R is not Cohen-Macaulay and there exists a non-zerodivisor $x \in \mathfrak{m}$ on R such that R/xR is quasi-Gorenstein. Then the following assertions hold true, where M denotes the Matlis dual of $\mathrm{H}^3_\mathfrak{m}(R)$.*

(1) *R is quasi-Gorenstein if and only if M is a Cohen-Macaulay R-module with $\dim_R M = 1$.*
(2) *$\mathrm{Supp}_R M$ is the non-Cohen-Macaulay locus of R.*
(3) *If $\dim_R M = 1$, then M is a Cohen-Macaulay R-module if and only if the equality*

$$\ell_R(\mathrm{H}^2_\mathfrak{m}(R/xR)) = \sum_{\mathfrak{p} \in \mathrm{Assh}_R M} \ell_{R_\mathfrak{p}}(\mathrm{H}^2_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})) \cdot \mathrm{e}_0(x, R/\mathfrak{p})$$

holds.

Proof. We may assume that R is complete. Set $\overline{R} = R/xR$. Note that $\operatorname{depth} R = 3$. For every $P \in \operatorname{Spec} \overline{R} \setminus \{\overline{\mathfrak{m}}\}$, we have $\dim \overline{R}_P \le 2$; hence $\overline{R}_P$ is Cohen-Macaulay because $\overline{R}$ satisfies Serre's (S_2) condition. Since $(\mathrm{K}_{\overline{R}})_P \cong \overline{R}_P \neq (0)$, it follows that $P \in \mathrm{Supp}_{\overline{R}} \mathrm{K}_{\overline{R}}$. Thus, $3 = \dim \overline{R}_P + \dim \overline{R}/P$. As $\overline{R}$ is a homomorphic image of a Cohen-Macaulay ring, $\overline{R}$ has FLC. Therefore, $\mathrm{H}^2_\mathfrak{m}(\overline{R})$ is finitely generated as an R-module. Hence, the assertions follow from Theorem 3.1. □

Corollary 3.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with $d = \dim R > 0$ admitting the canonical module K_R. Suppose that R has FLC and there exists a non-zerodivisor $x \in \mathfrak{m}$ on R such that R/xR is quasi-Gorenstein. Then R is quasi-Gorenstein if and only if $\mathrm{H}^{d-1}_\mathfrak{m}(R) = (0)$. In particular, R is quasi-Gorenstein if and only if R is Gorenstein, provided $d \le 4$.*

Proof. Let $\overline{R} = R/xR$. For $d \ge 3$, the ring R is Gorenstein, so we may assume $d \ge 4$. Hence, $\operatorname{depth} R \ge 3$. Suppose that $\mathrm{H}^{d-1}_\mathfrak{m}(R) = (0)$. Applying the functor $\mathrm{H}^i_\mathfrak{m}(-)$ to the exact sequence $0 \to \mathrm{K}_R \xrightarrow{x} \mathrm{K}_R \to \overline{R} \to 0$, we obtain the sequence $0 \to \mathrm{H}^{d-1}_\mathfrak{m}(\overline{R}) \to \mathrm{H}^d_\mathfrak{m}(R) \xrightarrow{x} \mathrm{H}^d_\mathfrak{m}(R) \to 0$. This implies the exact sequence

$$0 \to \mathrm{K}_R \xrightarrow{x} \mathrm{K}_R \to \overline{R} \to 0$$

of R-modules. Therefore, we have $R \cong \mathrm{K}_R$. Conversely, assume R is quasi-Gorenstein. Then we have the exact sequence

$$0 \to \overline{R} \to \overline{R} \to C \to 0$$

where M denotes the Matlis dual of $\mathrm{H}^{d-1}_\mathfrak{m}(R)$ and $C = (0) :_M x$. Suppose, for the sake of contradiction, that $\mathrm{H}^{d-1}_\mathfrak{m}(R) \neq (0)$. Then $M \neq (0)$, and $\ell_R(M) = \ell_R(\mathrm{H}^{d-1}_\mathfrak{m}(R))$ is finite because R has

FLC. Thus, $\dim_R M = 0$. Consequently, $C \neq (0)$, which implies $\dim_R C = 0$ and $\operatorname{depth} R = 2$. This is a contradiction. Hence, $\mathrm{H}^{d-1}_{\mathfrak{m}}(R) = 0$, as required. Finally, assume $d = 4$ and R is quasi-Gorenstein. Since $\mathrm{H}^3_{\mathfrak{m}}(R) = 0$ and $\operatorname{depth} R \geq 3$, it follows that $\operatorname{depth} R \geq 4$. Hence, R is Cohen-Macaulay, which means R is Gorenstein. $\square$

To conclude this section, we articulate the criteria under which a quasi-Gorenstein ring possessing the FLC property is guaranteed to be Gorenstein.

Remark 3.5. Let $(R, \mathfrak{m})$ be a Noetherian local ring with $d = \dim R \geq 2$ that admits the canonical module K_R. Suppose R is quasi-Gorenstein having FLC. Then R is Gorenstein if and only if $\operatorname{depth} R \geq \frac{d}{2} + 1$.

Proof. We may first assume that R is complete. Suppose R is Gorenstein. Since $d \geq 2$, we have $d \geq \frac{d}{2} + 1$; hence, $\operatorname{depth} R \geq \frac{d}{2} + 1$. Conversely, when $d = 2$, the ring R is Cohen-Macaulay because $\operatorname{depth} R \geq \frac{d}{2} + 1 = 2$. This implies that R is Gorenstein. Therefore, we may assume $d \geq 3$. Since R is complete and has FLC, we have the isomorphisms

$$\mathrm{H}^i_{\mathfrak{m}}(R) \cong [\mathrm{H}^{d-i+1}_{\mathfrak{m}}(\mathrm{K}_R)]^{\vee} \cong [\mathrm{H}^{d-i+1}_{\mathfrak{m}}(R)]^{\vee}$$

for every $2 \leq i \leq d-1$. Set $t = \operatorname{depth} R$. Then $t \geq \frac{d}{2} + 1$, which implies that $t \geq 3$. Assume R is not Cohen-Macaulay. Then $t \leq d-1$. This leads us to $(0) \neq \mathrm{H}^t_{\mathfrak{m}}(R) \cong [\mathrm{H}^{d-t+1}_{\mathfrak{m}}(R)]^{\vee}$. Thus, $\mathrm{H}^{d-t+1}_{\mathfrak{m}}(R) \neq (0)$. Consequently, $t \leq d-t+1$. This yields $\frac{d}{2} + 1 \leq t \leq \frac{d+1}{2}$, which is a contradiction. Therefore, R must be Cohen-Macaulay, and hence R is Gorenstein. $\square$

4. Proof of main theorem

First, we establish the notation and assumptions that form the basis of all the results in this section.

Setup 4.1. Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let $\mathscr{F} = \{F_n\}_{n \in \mathbb{Z}}$ denote a Noetherian filtration of ideals of R with $F_1 \neq R$. Let t be an indeterminate over R. The *extended Rees algebra of* $\mathscr{F}$ and the *associated graded ring of* $\mathscr{F}$ are defined as

$$\mathscr{R}'(\mathscr{F}) = \sum_{n \in \mathbb{Z}} F_n t^n \subseteq R[t, t^{-1}], \quad G(\mathscr{F}) = \mathscr{R}'(\mathscr{F})/t^{-1}\mathscr{R}'(\mathscr{F}) \cong \bigoplus_{n \geq 0} F_n/F_{n+1},$$

respectively. Let $\mathfrak{M}$ and $\mathfrak{N}$ denote the unique graded maximal ideals of $\mathscr{R}'(\mathscr{F})$ and $G(\mathscr{F})$, respectively. For each $\mathfrak{p} \in \operatorname{Spec} R$, we define $\mathfrak{p}' = \mathfrak{p} \cdot R[t, t^{-1}] \cap \mathscr{R}'(\mathscr{F})$.

Note that $\mathfrak{p}'$ is a graded prime ideal of $\mathscr{R}'(\mathscr{F})$ with $t^{-1} \notin \mathfrak{p}'$ and $\mathfrak{p}' \subsetneq \mathfrak{M}$. There exists a one-to-one correspondence between $\operatorname{Spec} R$ and ${}^*D(t^{-1})$, the set of all graded prime ideals P of $\mathscr{R}'(\mathscr{F})$ such that $t^{-1} \notin P$, where $\mathfrak{p} \in \operatorname{Spec} R$ corresponds to $\mathfrak{p}' \in {}^*D(t^{-1})$. Furthermore, we have the isomorphisms

$$\mathscr{R}'(\mathscr{F})_{\mathfrak{p}'} \cong R[t, t^{-1}]_{\mathfrak{p}R[t,t^{-1}]} \cong R[t]_{\mathfrak{p}R[t]} \quad \text{and} \quad \mathscr{R}'(\mathscr{F})/\mathfrak{m}' \cong (R/\mathfrak{m})[t^{-1}].$$

Lemma 4.2. *Suppose the following conditions hold.*

(1) *R is not a Cohen-Macaulay ring but possesses FLC.*
(2) *F_1 is an $\mathfrak{m}$-primary ideal of R.*
(3) *$G(\mathscr{F})_Q$ is Cohen-Macaulay for every graded prime ideal Q of $G(\mathscr{F})$ with $Q \neq \mathfrak{N}$.*

Then, for every $P \in \operatorname{Spec} \mathscr{R}'(\mathscr{F})$ such that P is graded and $P \neq \mathfrak{M}$, the local ring $\mathscr{R}'(\mathscr{F})_P$ is not Cohen-Macaulay if and only if $P = \mathfrak{m}'$. In particular, $V(\mathfrak{m}')$ coincides with the non-Cohen-Macaulay locus of $\mathscr{R}'(\mathscr{F})$.

Proof. Let P be a graded prime ideal of $\mathscr{R}'(\mathscr{F})$ with $P \neq \mathfrak{M}$. Suppose that $\mathscr{R}'(\mathscr{F})_P$ is not Cohen-Macaulay. If $t^{-1} \in P$, then P contains $F_n = F_n t^n \cdot t^{-n}$ for all $n > 0$, which implies $F_\ell t^\ell \nsubseteq P$ for some $\ell > 0$. By our assumption, $G(I)_Q$ is Cohen-Macaulay, where $Q = P \cdot G(I) \subsetneq \mathfrak{N}$. Hence, $\mathscr{R}'(I)_P$ must be a Cohen-Macaulay local ring, which leads to a contradiction. Therefore, $t^{-1} \notin P$. Thus, $P \in {}^*D(t^{-1})$, and there exists $\mathfrak{p} \in \operatorname{Spec} R$ such that $P = \mathfrak{p}'$. This shows that

$$\mathscr{R}'(\mathscr{F})_P \cong \mathscr{R}'(\mathscr{F})_{\mathfrak{p}'} \cong R[t,t^{-1}]_{\mathfrak{p}R[t,t^{-1}]} \cong R[t]_{\mathfrak{p}R[t]}$$

is not Cohen-Macaulay. Consequently, $R_\mathfrak{p}$ is not Cohen-Macaulay either. Since R has FLC, it follows that $\mathfrak{p} = \mathfrak{m}$. Hence, we conclude that $P = \mathfrak{m}'$. Conversely, if we assume $P = \mathfrak{m}'$, then the isomorphism $\mathscr{R}'(\mathscr{F})_P \cong R[t]_{\mathfrak{m}R[t]}$ guarantees that $\mathscr{R}'(\mathscr{F})_P$ is not Cohen-Macaulay. Let us verify the last assertion. For each $P \in V(\mathfrak{m}')$, there exists an isomorphism $(\mathscr{R}'(\mathscr{F})_P)_{\mathfrak{m}'\mathscr{R}'(\mathscr{F})_P} \cong \mathscr{R}'(\mathscr{F})'_\mathfrak{m}$. In particular, $\mathscr{R}'(\mathscr{F})_P$ is not Cohen-Macaulay. Conversely, suppose $P \in \operatorname{Spec} \mathscr{R}'(\mathscr{F})$ such that $\mathscr{R}'(\mathscr{F})_P$ is not Cohen-Macaulay. Let P^* denote the ideal of $\mathscr{R}'(\mathscr{F})$ generated by the homogeneous elements of P. Observe that $P^* \subseteq \mathfrak{M}$. If $P^* = \mathfrak{M}$, then $P = \mathfrak{M}$. Suppose $P^* \neq \mathfrak{M}$. Then $P^* = \mathfrak{m}'$, because $\mathscr{R}'(\mathscr{F})_{P^*}$ is not Cohen-Macaulay. In either case, we have $P \subseteq \mathfrak{m}'$, and thus $P \in V(\mathfrak{m}')$. □

We now present the main result of this paper.

Theorem 4.3. *Let $(R,\mathfrak{m})$ be a Noetherian local ring with $d = \dim R \geq 3$ which is a homomorphic image of a Gorenstein ring. Let $\mathscr{F} = \{F_n\}_{n\in\mathbb{Z}}$ be the Hilbert filtration of ideals in R. Suppose that $G(\mathscr{F})$ is a quasi-Gorenstein graded ring,* $\operatorname{depth} \mathscr{R}'(\mathscr{F}) \geq d$*, and $\mathrm{H}^{d-1}_{\mathfrak{M}}(G(\mathscr{F}))$ is finitely generated as an $\mathscr{R}'(\mathscr{F})$-module. Then the following conditions are equivalent.*

1. *$\mathscr{R}'(\mathscr{F})$ is a quasi-Gorenstein graded ring.*
2. *The equality $\ell_R(\mathrm{H}^{d-1}_\mathfrak{m}(R)) = \ell_{G(\mathscr{F})}(\mathrm{H}^{d-1}_\mathfrak{M}(G(\mathscr{F})))$ holds.*

Proof. Set $\mathscr{R}' = \mathscr{R}'(\mathscr{F})$ and $G = G(\mathscr{F})$. The ring $\mathscr{R}'$ admits the graded canonical module $\mathrm{K}_{\mathscr{R}'}$.

$(2) \Rightarrow (1)$ We may assume that R is not Cohen-Macaulay. Indeed, if R were Cohen-Macaulay, then G would also be Cohen-Macaulay due to the equality $\ell_R(\mathrm{H}^{d-1}_\mathfrak{m}(R)) = \ell_G(\mathrm{H}^{d-1}_\mathfrak{M}(G))$ and the fact that $\operatorname{depth} G \geq d-1$. Consequently, G would be Gorenstein, which in turn implies that $\mathscr{R}'$ is Gorenstein as well. Now, assume for contradiction that $\mathscr{R}'$ is Cohen-Macaulay. The Cohen-Macaulayness of G would imply that the localization $R_\mathfrak{p}$ is Cohen-Macaulay for every $\mathfrak{p} \in V(F_1)$. However, this leads to a contradiction, as it conflicts with the facts that F_1 is $\mathfrak{m}$-primary and R is not Cohen-Macaulay. Therefore, $\mathscr{R}'$ cannot be Cohen-Macaulay. We conclude, in particular, that $\operatorname{depth} \mathscr{R}' = d$.

We set $A = \mathscr{R}'_\mathfrak{M}$. Then $\dim A = d+1 \geq 4$ and $\operatorname{depth} A = d$. Note that $G_\mathfrak{N} \cong G_\mathfrak{M} \cong A/t^{-1}A$. Let M denote the Matlis dual of $\mathrm{H}^d_{\mathfrak{M}A}(A)$. Observe that $M \neq (0)$. Applying the functor $\mathrm{H}^i_{\mathfrak{M}A}(-)$ to the exact sequence $0 \to A \xrightarrow{t^{-1}} A \to G_\mathfrak{M} \to 0$ of A-modules, we obtain the long exact sequence

$$0 \to \mathrm{H}^{d-1}_{\mathfrak{M}A}(G_\mathfrak{M}) \to \mathrm{H}^d_{\mathfrak{M}A}(A) \xrightarrow{t^{-1}} \mathrm{H}^d_{\mathfrak{M}A}(A) \to \mathrm{H}^d_{\mathfrak{M}A}(G_\mathfrak{M}) \to \mathrm{H}^{d+1}_{\mathfrak{M}A}(A) \xrightarrow{t^{-1}} \mathrm{H}^{d+1}_{\mathfrak{M}A}(A) \to 0.$$

Taking the Matlis dual of this sequence yields

$$0 \to \mathrm{K}_A \xrightarrow{t^{-1}} \mathrm{K}_A \to \mathrm{K}_{G_{\mathfrak{M}}} \to M \xrightarrow{t^{-1}} M \to [\mathrm{H}^{d-1}_{\mathfrak{M}A}(G_{\mathfrak{M}})]^{\vee} \to 0.$$

Since $\ell_A(M/t^{-1}M) = \ell_A(\mathrm{H}^{d-1}_{\mathfrak{M}A}(G_{\mathfrak{M}}))$ is finite, we conclude that $\dim_A M \le 1$. If $\dim_A M = 0$, the ring A has FLC. Consequently, A is locally Cohen-Macaulay on the punctured spectrum, and thus

$$A_{\mathfrak{m}'A} \cong (\mathscr{R}')_{\mathfrak{m}'} \cong R[t,t^{-1}]_{\mathfrak{m}R[t,t^{-1}]} \cong R[t]_{\mathfrak{m}R[t]}$$

is Cohen-Macaulay. By flat descent, this would imply that R is Cohen-Macaulay, which contradicts our assumption. Therefore, we must have $\dim_A M = 1$.

Observe that $\operatorname{depth} G = d-1 > 0$. By Proposition 2.2, it follows that $\operatorname{depth} R = d-1$. Since both $\mathrm{H}^{d-1}_{\mathfrak{m}}(R)$ and $\mathrm{H}^{d-1}_{\mathfrak{M}}(G)$ are finitely generated, the rings R and $G_{\mathfrak{N}}$ have FLC. We now claim that $\operatorname{Assh}_A M = \{\mathfrak{m}'A\}$. To establish this, let $P \in \operatorname{Assh}_A M$. By Theorem 3.1, the local ring A_P is not Cohen-Macaulay. Set $\mathfrak{p} = P \cap \mathscr{R}'$. Then $\mathfrak{p} \in \operatorname{Spec} \mathscr{R}'$ and $\mathfrak{p} \subseteq \mathfrak{M}$. Since $P \in \operatorname{Assh}_A M$ and $\dim_A M = 1$, we deduce that $\mathfrak{p} \neq \mathfrak{M}$. Note further that $A_P \cong \mathscr{R}'_{\mathfrak{p}}$. Let $\mathfrak{p}^*$ denote the ideal of $\mathscr{R}'$ generated by the homogeneous elements of $\mathfrak{p}$. It follows that $\mathscr{R}'_{\mathfrak{p}^*}$ is not Cohen-Macaulay. By Lemma 4.2, we conclude that $\mathfrak{p}^* = \mathfrak{m}'$. Hence $\mathfrak{m}' \subseteq \mathfrak{p} \subsetneq \mathfrak{M}$. Since $\dim \mathscr{R}'/\mathfrak{m}' = 1$, it must be that $\mathfrak{m}' = \mathfrak{p}$, and thus $P = \mathfrak{m}'A$. Consequently, $\operatorname{Assh}_A M = \{\mathfrak{m}'A\}$, as claimed.

By setting $R(t) = R[t]_{\mathfrak{m}R[t]}$, we obtain

$$\begin{aligned}\sum_{P \in \operatorname{Assh}_A M} \ell_{A_P}(\mathrm{H}^{d-1}_{PA_P}(A_P)) \cdot \mathrm{e}_0(t^{-1}, A/P) &= \ell_{A_{\mathfrak{m}'A}}(\mathrm{H}^{d-1}_{(\mathfrak{m}'A)A_{\mathfrak{m}'A}}(A_{\mathfrak{m}'A})) \cdot \mathrm{e}_0(t^{-1}, A/\mathfrak{m}'A)\\ &= \ell_{R(t)}(\mathrm{H}^{d-1}_{\mathfrak{m}R(t)}(R(t))) = \ell_R(\mathrm{H}^{d-1}_{\mathfrak{m}}(R))\\ &= \ell_G(\mathrm{H}^{d-1}_{\mathfrak{M}}(G)) = \ell_A(\mathrm{H}^{d-1}_{\mathfrak{M}A}(A/t^{-1}A)).\end{aligned}$$

Here, the second equality follows from the fact that

$$A/\mathfrak{m}'A \cong (\mathscr{R}'/\mathfrak{m}')_{\mathfrak{M}} \cong ((R/\mathfrak{m})[t^{-1}])_{(t^{-1})},$$

which is a regular local ring. By Theorem 3.1, we conclude that M is Cohen-Macaulay. Consequently, the local ring $A = \mathscr{R}'_{\mathfrak{M}}$ is quasi-Gorenstein, and therefore $\mathscr{R}'$ is also quasi-Gorenstein.

(1) $\Rightarrow$ (2) Suppose $\mathscr{R}'$ is Cohen-Macaulay. Then both G and R are Cohen-Macaulay, and hence $\ell_R(\mathrm{H}^{d-1}_{\mathfrak{m}}(R)) = 0 = \ell_G(\mathrm{H}^{d-1}_{\mathfrak{M}}(G))$. Therefore, we may assume that $\mathscr{R}'$ is not Cohen-Macaulay, in which case $\operatorname{depth} \mathscr{R}' = d$. Let $A = \mathscr{R}'_{\mathfrak{M}}$, and denote by M the Matlis dual of $\mathrm{H}^d_{\mathfrak{M}A}(A)$. Then $M \neq (0)$. Since A is quasi-Gorenstein, Theorem 3.1 ensures that M is Cohen-Macaulay of dimension one. Consequently, the equality

$$\ell_G(\mathrm{H}^{d-1}_{\mathfrak{M}}(G)) = \ell_A(\mathrm{H}^{d-1}_{\mathfrak{M}A}(A/t^{-1}A)) = \sum_{P \in \operatorname{Assh}_A M} \ell_{A_P}(\mathrm{H}^{d-1}_{PA_P}(A_P)) \cdot \mathrm{e}_0(t^{-1}, A/P)$$

holds. We claim that $\operatorname{Assh}_A M = \{\mathfrak{m}'A\}$. To verify this, let $P \in \operatorname{Assh}_A M$. By Theorem 3.1, the localization A_P is not Cohen-Macaulay. Let $\mathfrak{p} = P \cap \mathscr{R}'$. Then $\mathfrak{p} \subseteq \mathfrak{M}$. If $\mathfrak{p} = \mathfrak{M}$, then $P = \mathfrak{m}A$, which contradicts the assumption that $P \in \operatorname{Assh}_A M$ and $\dim_A M = 1$. Thus $\mathfrak{p} \subsetneq \mathfrak{M}$. Next, let $\mathfrak{p}^*$ denote the ideal of $\mathscr{R}'$ generated by the homogeneous elements of $\mathfrak{p}$. It follows that $\mathscr{R}'_{\mathfrak{p}^*}$ is not Cohen-Macaulay, and consequently $A_{\mathfrak{p}^*A}$ is not Cohen-Macaulay either. By Theorem 3.1 again, we have $\mathfrak{p}^*A \in \operatorname{Supp}_A M$. Since M is Cohen-Macaulay, it holds that $\dim_{A_{\mathfrak{p}^*A}} M_{\mathfrak{p}^*A} \le 1$.

If $\dim_{A_{\mathfrak{p}^*A}} M_{\mathfrak{p}^*A} = 1$, then $\dim A/\mathfrak{p}^*A = 0$, which is a contradiction because $\mathfrak{p}^*A \subseteq \mathfrak{p}A \subsetneq \mathfrak{M}A$. Thus, $\dim_{A_{\mathfrak{p}^*A}} M_{\mathfrak{p}^*A} = 0$, and so $\dim A/\mathfrak{p}^*A = 1$. Consider the exact sequence

$$0 \to \mathrm{K}_A \xrightarrow{t^{-1}} \mathrm{K}_A \to \mathrm{K}_{G_{\mathfrak{M}}} \to M \xrightarrow{t^{-1}} M \to [\mathrm{H}^{d-1}_{\mathfrak{M}A}(G_{\mathfrak{M}})]^\vee \to 0.$$

This implies that $\ell_A(M/t^{-1}M) < \infty$, so t^{-1} acts as a non-zerodivisor on M. In particular, $t^{-1} \notin \mathfrak{p}^*$. Thus $\mathfrak{p}^* \in {}^*D(t^{-1})$. There exists $\mathfrak{q} \in \operatorname{Spec} R$ such that $\mathfrak{q}' = \mathfrak{p}^*$. Therefore

$$\mathfrak{p}^*A = \mathfrak{q}'A \subseteq \mathfrak{m}'A \subsetneq \mathfrak{M}A,$$

and since $\dim A/\mathfrak{p}^*A = 1$, we deduce $\mathfrak{p}^*A = \mathfrak{q}'A = \mathfrak{m}'A$. As $\mathfrak{m}'A = \mathfrak{p}^*A \subseteq \mathfrak{p}A = P \subsetneq \mathfrak{M}A$, it follows that $\mathfrak{m}'A = \mathfrak{p}A = P$, because $\dim A/\mathfrak{m}'A = 1$. Consequently, $\operatorname{Assh}_A M = \{\mathfrak{m}'A\}$. Finally, we obtain

$$\ell_G(\mathrm{H}^{d-1}_{\mathfrak{M}}(G)) = \sum_{P \in \operatorname{Assh}_A M} \ell_{A_P}(\mathrm{H}^{d-1}_{PA_P}(A_P)) \cdot \mathrm{e}_0(t^{-1}, A/P) = \ell_R(\mathrm{H}^{d-1}_{\mathfrak{m}}(R)).$$

as desired. This completes the proof. □

A direct application of Theorem 1.1 yields the following result.

Corollary 4.4. *Let $(R,\mathfrak{m})$ be a Noetherian local ring with $\dim R = 3$ and $\mathscr{F} = \{F_n\}_{n\in\mathbb{Z}}$ the Hilbert filtration of ideals in R. Suppose that R is a homomorphic image of a Gorenstein ring and $G(\mathscr{F})$ is quasi-Gorenstein. Then the following conditions are equivalent.*

(1) *$\mathscr{R}'(\mathscr{F})$ is a quasi-Gorenstein graded ring.*
(2) *The equality $\ell_R(\mathrm{H}^2_{\mathfrak{m}}(R)) = \ell_{G(\mathscr{F})}(\mathrm{H}^2_{\mathfrak{M}}(G(\mathscr{F})))$ holds.*

Recall that a Noetherian graded k-algebra $R = \bigoplus_{n\geq 0} R_n$ over a field $R_0 = k$ is called *homogeneous*, if $R = k[R_1]$, i.e., R is generated by R_1 as a k-algebra.

Corollary 4.5. *Let $R = k[R_1]$ be a quasi-Gorenstein homogeneous ring over a field k with* $\dim R = 3$. *Then the extended Rees algebra $\mathscr{R}'(\mathfrak{m}R_{\mathfrak{m}})$ is quasi-Gorenstein, where $\mathfrak{m} = R_+$ denotes the graded maximal ideal of $\mathscr{R}'(\mathfrak{m})$.*

Proof. As R is homogeneous, we have $R \cong G(\mathfrak{m})$. Thus, $G(\mathfrak{m}R_{\mathfrak{m}}) \cong R_{\mathfrak{m}} \otimes_R G(\mathfrak{m}) \cong G(\mathfrak{m})_{\mathfrak{m}}$ is quasi-Gorenstein. The $\mathfrak{m}R_{\mathfrak{m}}$-adic filtration is Hilbert, and the local ring $R_{\mathfrak{m}}$ is a homomorphic image of a Gorenstein ring. By Corollary 4.4, the ring $\mathscr{R}'(\mathfrak{m}R_{\mathfrak{m}})$ is quasi-Gorenstein. □

Example 4.6. Let Δ be a two-dimensional finite abstract simplicial complex whose geometric realization is homeomorphic to an orientable manifold that is not a sphere. Then the Stanley-Reisner ring $R = k[\Delta]$, defined over a field k of characteristic 0, is quasi-Gorenstein but not Gorenstein ([22, Remark 4.6]). Denote by $\mathfrak{m} = R_+$ the graded maximal ideal of R. Consequently, $\mathscr{R}'(\mathfrak{m}R_{\mathfrak{m}})$ is a non-Gorenstein quasi-Gorenstein ring.

Acknowledgments. The author would like to thank K.-i. Watanabe for his valuable comments.

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SCHOOL OF POLITICAL SCIENCE AND ECONOMICS, MEIJI UNIVERSITY, 1-9-1 EIFUKU, SUGINAMI-KU, TOKYO 168-8555, JAPAN
Email address: endo@meiji.ac.jp
URL: https://www.isc.meiji.ac.jp/~endo/