A STUDY OF QUASI-GORENSTEIN RINGS II: DEFORMATION OF QUASI-GORENSTEIN PROPERTY

KAZUMA SHIMOMOTO, NAOKI TANIGUCHI, AND EHSAN TAVANFAR

Abstract. In the present article, we investigate the following deformation problem. Let $(R, \mathfrak{m})$ be a local (graded local) Noetherian ring with a (homogeneous) regular element $y \in \mathfrak{m}$ and assume that R/yR is quasi-Gorenstein. Then is R quasi-Gorenstein? We give positive answers to this problem under various assumptions, while we present a counter-example in general. We emphasize that absence of the Cohen-Macaulay condition requires delicate and subtle studies. Recently, the third-named author used the quasi-Gorenstein property to deduce some interesting results on the absolute integral closure of a complete local domain in [29]. Quasi-Gorenstein rings also appear in Du Bois singularities, which form a major class of singularities in birational geometry.

Contents

1. Introduction

In this article, we study the deformation problem of the quasi-Gorenstein property on local Noetherian rings and construct some examples of non-Cohen-Macaulay, quasi-Gorenstein and normal domains. Recall that a local ring $(R, \mathfrak{m})$ is quasi-Gorenstein, if it has a canonical module ω_R such that $\omega_R \cong R$. For completeness, we state the general deformation problem as follows:

Problem 1. *Let $(R, \mathfrak{m})$ be a local (graded local) Noetherian ring and M be a nonzero finitely generated R-module with a (homogeneous) M-regular element $y \in \mathfrak{m}$. Assume that M/yM has* $\mathbf{P}$. *Then does M possess* $\mathbf{P}$*?*

By specializing $\mathbf{P}$=quasi-Gorenstein, we prove the following result by constructing an explicit example using Macaulay2 (see Theorem 4.2):

Main Theorem 1. *There exists an example of a local Noetherian ring $(R, \mathfrak{m})$, together with a regular element $y \in \mathfrak{m}$ such that the following property holds: R/yR is quasi-Gorenstein and R is not quasi-Gorenstein.*

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We notice that if a local ring $(R, \mathfrak{m})$ is Cohen-Macaulay admitting a canonical module ω_R satisfying $\omega_R \cong R$, then it is Gorenstein. Thus, the local ring R that appears in Main Theorem 1 is not Cohen-Macaulay. In the absence of Cohen-Macaulay condition, various aspects have been studied around the deformation problem in a recent paper [30]. Our second main result is to provide some conditions under which the quasi-Gorenstein condition is preserved under deformation (see Theorem 3.2).

Main Theorem 2. *Let $(R, \mathfrak{m})$ be a local Noetherian ring with a regular element $y \in \mathfrak{m}$, such that R/yR is quasi-Gorenstein. If one of the following conditions holds, then R is also quasi-Gorenstein.*

(1) R is of equal-characteristic $p > 0$ that is F-finite and the Frobenius action on the local cohomology $H^{\dim R-1}_{\mathfrak{m}}(R/yR)$ is injective.
(2) R is essentially of finite type over $\mathbb{C}$ and R/yR has Du Bois singularities.
(3) $\operatorname{Ext}^1_{\widehat{R}}(\omega_{\widehat{R}}, \omega_{\widehat{R}}) = 0$ and $0 :_{\operatorname{Ext}^2_{\widehat{R}}(\omega_{\widehat{R}}, \omega_{\widehat{R}})} y = 0$, where $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R.
(4) Both R/yR and all of the formal fibers of R satisfy Serre's S_3.
(5) All of the formal fibers of R are Gorenstein, R is quasi-Gorenstein on $\operatorname{Spec}^\circ(R/yR)$ and $\operatorname{depth}(R) \geq 4$.
(6) All of the formal fibers of R are Gorenstein, R/yR is Gorenstein on its punctured spectrum and $\operatorname{depth}(R) \geq 4$.
(7) R is an excellent normal domain of equal-characteristic zero such that $R[\frac{1}{y}]$ is also quasi-Gorenstein.

While Main Theorem 2 is concerned about local rings, we establish the following result for the graded local rings using algebraic geometry, including Lefschetz condition and vanishing of sheaf cohomology (see Theorem 3.6).

Main Theorem 3. *Let $R = \bigoplus_{n \geq 0} R_n$ be a Noetherian standard graded ring such that $y \in R$ is a regular element which is homogeneous of positive degree, $R_0 = k$ is a field of characteristic zero. Suppose that R/yR is a quasi-Gorenstein graded ring such that $X := \operatorname{Proj}(R)$ is an integral normal variety and $X_1 := \operatorname{Proj}(R/yR)$ is nonsingular. Then R is a quasi-Gorenstein graded ring.*

At the time of writing, the following problem remains open, because the example given in Theorem 4.2 is not normal.

Problem 2. *Suppose that $(R, \mathfrak{m})$ be a local (or graded local) ring with a regular element $y \in \mathfrak{m}$ such that R/yR is a quasi-Gorenstein normal local (or graded local) domain. Is R quasi-Gorenstein?*

In the final section, we construct three non-trivial examples of quasi-Gorenstein normal local domains of depth equal to 2 that are not Cohen-Macaulay (the final one being with arbitrary admissible dimension at least 3) in Example 5.1. It will be interesting to ask the reader if any of these examples admits a non quasi-Gorenstein deformation. In the light of the above theorem, it is noteworthy to point out that any homogeneous deformation of the (standard) quasi-Gorenstein ring of Example 5.1(1) is again quasi-Gorenstein, provided that the deformation is standard of equal-characteristic zero. Let us end with a remark on the ubiquity of quasi-Gorenstein rings.

(Algebraic side): One can easily construct a local ring $(R, \mathfrak{m})$ with a regular element $y \in \mathfrak{m}$ such that R/yR is quasi-Gorenstein but not Gorenstein, which deforms to a quasi-Gorenstein ring R. For instance, take $(S, \mathfrak{n})$ to be any non-Gorenstein and quasi-Gorenstein local ring. Then the trivial extension $R := S[[y]]$ provides such an example. More interestingly, let R be a complete local domain of arbitrary characteristic. Then it is shown in [29] that R is dominated by a module-finite extension domain over R that is quasi-Gorenstein and complete intersection at codimension ≤ 1.

(Geometric side): The class of quasi-Gorenstein rings appears in Du Bois singularities. Indeed, we learn from Main Theorem 2(2) together with the main result in [19] that if R/yR has quasi-Gorenstein Du Bois singularity, then R enjoys the same properties. This type of result will be essential for moduli problems as explained in [19]. We also recall from [18] that normal quasi-Gorenstein Du Bois singularities are log canonical. This was previously known as a conjecture of Kollár. On the other hand, it is known from [17] that log canonical singularities are Du Bois.

2. Notation and auxiliary lemmas

Let $(R, \mathfrak{m})$ be a local Noetherian ring with Krull dimension $d := \dim R$ and let M be a finitely generated module. We say that M is a *canonical module* for R, if there is an isomorphism $M \otimes_R \widehat{R} \cong H^d_{\mathfrak{m}}(R)^\vee$, where $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R. In general, assume that R is a Noetherian ring and M is a finitely generated R-module. Then M is a *canonical module* for R, if for any $\mathfrak{p} \in \operatorname{Spec}(R)$, $M_{\mathfrak{p}}$ is a canonical module for the local ring $R_{\mathfrak{p}}$. We will write a canonical module as ω_R in what follows. A local Noetherian ring $(R, \mathfrak{m})$ is *quasi-Gorenstein*, if there is an isomorphism $H^d_{\mathfrak{m}}(R)^\vee \cong \widehat{R}$. Equivalently, R is quasi-Gorenstein, if R admits a canonical module such that $\omega_R \cong R$ (see [1]). Let R be a Noetherian ring admitting a canonical module ω_R. Then R is *(locally) quasi-Gorenstein*, if the localization $R_{\mathfrak{p}}$ for $\mathfrak{p} \in \operatorname{Spec}(R)$ is quasi-Gorenstein in the sense above, or equivalently, ω_R is a projective module of constant rank 1. Let $R = \bigoplus_{n \geq 0} R_n$ be a graded Noetherian ring such that $R_0 = k$ is a field. Then R is *quasi-Gorenstein*, if $\omega_R \cong R(a)$ for some $a \in \mathbb{Z}$ as graded R-modules. For a local ring $(R, \mathfrak{m})$, we write the punctured spectrum $\operatorname{Spec}^\circ(R) := \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$. Let I be an ideal of a ring R. Then let $V(I)$ denote the set of all prime ideals of R that contain I. We also use some basic facts on *attached primes*. For an Artinian R-module M, we denote by $\operatorname{Att}_R(M)$ the set of attached primes of M (see [4] for a brief summary).

We start by proving the following two auxiliary lemmas. The first lemma is a restatement of [6, Lemma] and we reprove it only for the convenience of the reader.

Lemma 2.1. *Suppose that $(R, \mathfrak{m})$ is a local Noetherian ring with $\operatorname{depth}(R) \geq 2$. Let $\mathfrak{a}$ be an ideal of R such that $\mathfrak{m}$ is not associated to $\mathfrak{a}$, the ideal $\mathfrak{a}$ is not contained in any associated prime of R and $\mathfrak{a}R_{\mathfrak{p}}$ is principal for $\mathfrak{p} \in \operatorname{Spec}^\circ(R)$. Then $\mathfrak{a}$ defines an element of $\operatorname{Pic}\big(\operatorname{Spec}^\circ(R)\big)$. Moreover if the line bundle attached to $\mathfrak{a}$ is a trivial element of $\operatorname{Pic}\big(\operatorname{Spec}^\circ(R)\big)$, then $\mathfrak{a}$ is a principal ideal.*

Proof. For each $\mathfrak{p} \in \operatorname{Spec}^\circ(R)$, we have $\mathfrak{a}R_{\mathfrak{p}} = (s)$ for some $s \in R_{\mathfrak{p}}$ by assumption. We need to show that we can choose s as a regular element. Since $\mathfrak{a}$ is not contained in any associated prime of R, we have $\mathfrak{a} \nsubseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass}(R)} \mathfrak{p}$ by Prime Avoidance Lemma. So the $\mathcal{O}_{\operatorname{Spec}^\circ(R)}$-module $\widetilde{\mathfrak{a}}$ is invertible on $\operatorname{Spec}^\circ(R)$, which defines an element

$$[\tilde{\mathfrak{a}}] \in \operatorname{Pic}\big(\operatorname{Spec}^\circ(R)\big).$$

There are two exact sequences: $0 \to \mathfrak{a}/\Gamma_{\mathfrak{m}}(\mathfrak{a}) \to H^0(\operatorname{Spec}^\circ(R), \tilde{\mathfrak{a}}) \to H^1_{\mathfrak{m}}(\mathfrak{a}) \to 0$ and $\Gamma_{\mathfrak{m}}(R/\mathfrak{a}) \to H^1_{\mathfrak{m}}(\mathfrak{a}) \to H^1_{\mathfrak{m}}(R)$, where the first exact sequence is due to [14, III, Exercise 2.3.(e)] and [14, III, Exercise 3.3.(b)]. We have $H^1_{\mathfrak{m}}(R) = 0$, because of $\operatorname{depth}(R) \geq 2$. We also have $\Gamma_{\mathfrak{m}}(R/\mathfrak{a}) = 0$, because $\mathfrak{m}$ is not associated to $\mathfrak{a}$. Hence we get $H^0(\operatorname{Spec}^\circ(R), \tilde{\mathfrak{a}}) = \mathfrak{a}$ ($\Gamma_{\mathfrak{m}}(\mathfrak{a}) \subseteq \Gamma_{\mathfrak{m}}(R) = 0$). Now suppose that $\tilde{\mathfrak{a}}$ is the trivial element in $\operatorname{Pic}(\operatorname{Spec}^\circ(R))$. Then we have $\tilde{\mathfrak{a}} = \mathcal{O}_{\operatorname{Spec}^\circ(R)}$ and hence

$$\mathfrak{a} = H^0(\operatorname{Spec}^\circ(R), \tilde{\mathfrak{a}}) \cong H^0(\operatorname{Spec}^\circ(R), \mathcal{O}_{\operatorname{Spec}^\circ(R)}) = R,$$

where the last equality follows from the exact sequence

$$0 \to R/\Gamma_{\mathfrak{m}}(R) \to H^0(\operatorname{Spec}^\circ(R), \mathcal{O}_{\operatorname{Spec}^\circ(R)}) \to H^1_{\mathfrak{m}}(R) \to 0.$$

□

Definition 2.2. Let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of a local ring $(R, \mathfrak{m})$. We say that R is *formally unmixed*, if $\dim(\widehat{R}/\mathfrak{p}) = \dim(\widehat{R})$ for all $\mathfrak{p} \in \operatorname{Ass}(\widehat{R})$.

Lemma 2.3. *Let $(R, \mathfrak{m})$ be local Noetherian ring and suppose that $y \in \mathfrak{m}$ is a regular element such that R/yR is quasi-Gorenstein. Then R is formally unmixed.*

Proof. First of all, recall that a quasi-Gorenstein local ring is unmixed by [1, (1.8), page 87]. By definition of formal unmixedness, we can assume that R is complete and we proceed by induction on the Krull dimension $d := \dim(R)$. If $d \leq 3$, then R/yR is a quasi-Gorenstein ring of dimension at most 2, which implies that R/yR and R are Gorenstein rings, hence R is an unmixed ring. So suppose that $d \geq 4$ and the statement has been proved for smaller values than d. Pick $\mathfrak{q} \in \operatorname{Ass}(R)$. Then we have $\dim(R/\mathfrak{q}) \geq 2$, because if otherwise, $\operatorname{depth}(R) \leq \dim(R/\mathfrak{q}) \leq 1$ by [5, Proposition 1.2.13], violating $\operatorname{depth}(R) \geq 3$. Thus, $\dim(R/\mathfrak{q} + yR) \geq 1$ (note that $y \notin \mathfrak{q}$, as y is a regular element). So we can choose $\mathfrak{p}/yR \in V(\mathfrak{q}+yR/yR)\backslash\{\mathfrak{m}/yR\} \subset \operatorname{Spec}(R/yR)$ such that $\dim(R/\mathfrak{p}) = 1$. Since $R_\mathfrak{p}/yR_\mathfrak{p}$ is quasi-Gorenstein, the inductive hypothesis implies that $R_\mathfrak{p}$ is formally unmixed. Hence we have that $R_\mathfrak{p}$ is unmixed and $\dim(R/\mathfrak{q}) - 1 \geq \operatorname{ht}(\mathfrak{p}/\mathfrak{q}) = \dim(R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}) = \operatorname{ht}(\mathfrak{p})$. On the other hand, since R/yR is a complete and quasi-Gorenstein local ring, it is catenary and equi-dimensional. Therefore, we have $\operatorname{ht}(\mathfrak{m}/yR) = \operatorname{ht}(\mathfrak{p}/yR) + 1$ and $\dim(R/\mathfrak{q}) = \dim(R)$, as required. □

Let us recall that the quasi-Gorenstein property admits a nice variant of deformation in [30, Theorem 2.9]:

Theorem 2.4 (Tavanfar-Tousi)**.** *Let $(R, \mathfrak{m})$ be a local Noetherian ring with a regular element $y \in \mathfrak{m}$. If R/y^nR is quasi-Gorenstein for infinitely many $n \in \mathbb{N}$, then R is quasi-Gorenstein.*

3. Deformation of quasi-Gorensteinness

The aim of this section is to present some cases where the quasi-Gorenstein property deforms. We recall the notion of surjective elements which is given in [15].

Definition 3.1. Let $(R, \mathfrak{m})$ be a local Noetherian ring. A regular element $y \in \mathfrak{m}$ is called a *surjective element*, if the natural map of local cohomology modules $H^i_\mathfrak{m}(R/y^nR) \to H^i_\mathfrak{m}(R/yR)$, which is induced by the natural surjection $R/y^nR \to R/yR$, is surjective for all $n > 0$ and $i \geq 0$.

In the parts (1) and (2) of the following theorem, the surjective elements will play a role. In (2), a precise understanding of Du Bois singularities is not necessary, as we only need to use some established facts that follow from the definition.

Theorem 3.2. *Let $(R, \mathfrak{m})$ be a local Noetherian ring with a regular element $y \in \mathfrak{m}$, such that R/yR is quasi-Gorenstein. If one of the following conditions holds, then R is also quasi-Gorenstein.*

(1) R is of equal-characteristic $p > 0$ that is F-finite and the Frobenius action on the local cohomology $H^{\dim R - 1}_\mathfrak{m}(R/yR)$ is injective.
(2) R is essentially of finite type over $\mathbb{C}$ and R/yR has Du Bois singularities.
(3) $\operatorname{Ext}^1_{\widehat{R}}(\omega_{\widehat{R}}, \omega_{\widehat{R}}) = 0$ and $0 :_{\operatorname{Ext}^2_{\widehat{R}}(\omega_{\widehat{R}}, \omega_{\widehat{R}})} y = 0$, where $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R.
(4) Both R/yR and all of the formal fibers of R satisfy Serre's S_3.
(5) All of the formal fibers of R are Gorenstein, R is quasi-Gorenstein on $\operatorname{Spec}^\circ(R/yR)$ and $\operatorname{depth}(R) \geq 4$.

(6) All of the formal fibers of R are Gorenstein, R/yR is Gorenstein on its punctured spectrum[1] *and* $\operatorname{depth}(R) \geq 4$.

(7) R is an excellent normal domain of equal-characteristic zero such that $R[\frac{1}{y}]$ is also quasi-Gorenstein.

Proof. In each of the cases (4), (5) and (6), we can suppose that R is complete without loss of generality. More precisely, we apply the assumption on the formal fibers and [2, Theorem 4.1] is needed in addition for part (5) and (6). By Lemma 2.3, R is unmixed and in view of [1, (1.8), page 87], R is quasi-Gorenstein if and only it has a cyclic canonical module.

We prove the assertions (1) and (2) simultaneously. Then we prove y is a surjective element for all $n > 0$ and $i \geq 0$. When R/yR has Du Bois singularities, then it follows from [21, Lemma 3.3] that $y \in \mathfrak{m}$ is a surjective element. So assume that R satisfies the condition (1). Without loss of generality, we may assume that R is complete. In this case, the Matlis dual of the Frobenius action $H^{\dim R-1}_{\mathfrak{m}}(R/yR) \hookrightarrow H^{\dim R-1}_{\mathfrak{m}}(F_*(R/yR))$ yields a surjection $\phi : F_*(R/yR) \twoheadrightarrow R/yR$ in view of the assumption that $R/yR \cong \omega_{R/yR}$. Then there is an element $F_*a \in F_*(R/yR)$ such that $\phi(F_*a) = 1 \in R/yR$. Define a surjective R-module map $\Phi : F_*(R/yR) \to R/yR$ by letting $\Phi(F_*t) := \phi(F_*(at))$. Then the map Φ splits the Frobenius $R/yR \to F_*(R/yR)$. Hence R/yR is F-split. As F-pure (split) rings are F-anti-nilpotent by [20, Theorem 1.1 and Theorem 2.3], it follows that $H^i_{\mathfrak{m}}(R/y^nR) \to H^i_{\mathfrak{m}}(R/yR)$ is surjective by [22, Proposition 3.5].

We have proved that y is a surjective element in (1) and (2). It follows from [22, Proposition 3.3] that the multiplication map $H^i_{\mathfrak{m}}(R) \xrightarrow{\cdot y} H^i_{\mathfrak{m}}(R)$ is surjective for all $i \geq 0$. Letting $d = \dim R$, the short exact sequence $0 \to R \xrightarrow{\cdot y} R \to R/yR \to 0$ induces a short exact sequence

$$0 \to H^{d-1}_{\mathfrak{m}}(R/yR) \to H^d_{\mathfrak{m}}(R) \xrightarrow{\cdot y} H^d_{\mathfrak{m}}(R) \to 0.$$

Taking the Matlis dual of this exact sequence, we obtain the exact sequence:

$$0 \to \omega_{\widehat{R}} \xrightarrow{\cdot y} \omega_{\widehat{R}} \to \omega_{\widehat{R}/y\widehat{R}} \to 0.$$

Hence we have $\omega_{\widehat{R}/y\widehat{R}} \simeq \omega_{\widehat{R}}/y\omega_{\widehat{R}}$. Nakayama's lemma allows us to write $\omega_{\widehat{R}} \simeq \widehat{R}/J$ for some ideal $J \subset \widehat{R}$. By Lemma 2.3, R is formally unmixed. Then we conclude that $J = 0$ in view of [1, (1.8)]. Hence $\omega_{\widehat{R}} \simeq \widehat{R}$.

We prove (3) and argue by induction on dimension d. We may assume that R is complete, $d \geq 4$ and that the statement is true in the case $d < 4$. Let us prove that $\operatorname{Hom}_{R/yR}(\omega_R/y\omega_R, \omega_R/y\omega_R) \cong R/yR$. By dualizing the exact sequence $H^{d-1}_{\mathfrak{m}}(R/yR) \to H^d_{\mathfrak{m}}(R) \xrightarrow{\cdot y} H^d_{\mathfrak{m}}(R) \to 0$, we have an exact sequence:

$$0 \to \omega_R/y\omega_R \xrightarrow{g} \omega_{R/yR} \to C \to 0. \tag{3.1}$$

Consider the commutative diagram:

$$\begin{array}{ccc} R/yR & \xrightarrow{\alpha} & \operatorname{Hom}_{R/yR}(\omega_R/y\omega_R, \omega_R/y\omega_R) \\ \cong\Big\downarrow R/yR \text{ is } S_2 & & \operatorname{Hom}(\mathrm{id},g)\Big\downarrow \text{injective} \\ \operatorname{Hom}_{R/yR}(\omega_{R/yR}, \omega_{R/yR}) & \xrightarrow{\operatorname{Hom}(g,\mathrm{id})} & \operatorname{Hom}_{R/yR}(\omega_R/y\omega_R, \omega_{R/yR}) \end{array} \tag{3.2}$$

[1] According to [4, 9.5.7 Exercise], that a local ring $(R, \mathfrak{m})$ is generalized Cohen-Macaulay is equivalent to the condition that R is Cohen-Macaulay over the punctured spectrum, provided that R admits the dualizing complex. Moreover, recall that a quasi-Gorenstein Cohen-Macaulay ring is Gorenstein and vice versa.

where α is the natural map $\overline{r} \mapsto \{t \mapsto \overline{r}t\}$. Upon the localization at $\mathfrak{p} \in \operatorname{Spec}^{\circ}(R/yR)$, the exact sequence (3.1) becomes

$$0 \to \omega_{R_{\mathfrak{p}}}/y\omega_{R_{\mathfrak{p}}} \xrightarrow{g} \omega_{R_{\mathfrak{p}}/yR_{\mathfrak{p}}} \to C_{\mathfrak{p}} \to 0,$$

where $C_{\mathfrak{p}}$ is the Matlis dual to $H^{\dim(R_{\mathfrak{p}})-1}_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}})/(y/1)H^{\dim(R_{\mathfrak{p}})-1}_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}})$ (see [30, Remark 2.3.(b)]). But by our inductive hypothesis, $R_{\mathfrak{p}}$ is quasi-Gorenstein for each $\mathfrak{p} \in \operatorname{Spec}^{\circ}(R/yR)$ and so [30, Corollary 2.8] implies that $H^{\dim(R_{\mathfrak{p}})-1}_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}})/(y/1)H^{\dim(R_{\mathfrak{p}})-1}_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) = 0$ for each $\mathfrak{p} \in \operatorname{Spec}^{\circ}(R/yR)$. It follows that C is of finite length. In particular, $\operatorname{Ext}^i_{R/yR}(C, \omega_{R/yR}) = 0$ for $i = 0, 1$ in view of the fact that $\omega_{R/yR} \cong R/yR$ and [4, Theorem 6.2.2].

By applying $\operatorname{Hom}_{R/yR}(-, \omega_{R/yR})$ to the exact sequence (3.1), we find that $\operatorname{Hom}(g, \mathrm{id})$ is an isomorphism. Therefore, the commutative diagram (3.2) in conjunction with the injectivity of $\operatorname{Hom}(\mathrm{id}, g)$ implies that α is an isomorphism.

Since $\operatorname{depth}(R/yR) \geq 2$ and $\operatorname{Hom}_{R/yR}(\omega_R/y\omega_R, \omega_R/y\omega_R) \cong R/yR$, we get $\operatorname{depth}(\omega_R/y\omega_R) \geq 1$. Applying the hypothesis $\operatorname{Ext}^1_R(\omega_R, \omega_R) = 0$ and $0 :_{\operatorname{Ext}^2_R(\omega_R, \omega_R)} y = 0$ to the exact sequence $0 \to \omega_R \xrightarrow{\cdot y} \omega_R \to \omega_R/y\omega_R \to 0$, we get $\operatorname{Ext}^2_R(\omega_R/y\omega_R, \omega_R) = 0$. So it follows from [5, Lemma 3.1.16] that $\operatorname{Ext}^1_{R/yR}(\omega_R/y\omega_R, \omega_R/y\omega_R) = 0$. Set $N := \omega_R/y\omega_R$ and assume that $z \in R/yR$ is an N-regular element. This choice is possible due to $\operatorname{depth}(\omega_R/y\omega_R) \geq 1$. By applying $\operatorname{Hom}_{R/yR}(N, -)$ to the exact sequence $0 \to N \xrightarrow{\cdot z} N \to N/zN \to 0$, we get an exact sequence:

$$0 \to \operatorname{Hom}_{R/yR}(N, N)/z \operatorname{Hom}_{R/yR}(N, N) \to \operatorname{Hom}_{R/yR}(N, N/zN) \to \operatorname{Ext}^1_{R/yR}(N, N),$$

which gives

$$\operatorname{Hom}_{R/yR}(N, N)/z \operatorname{Hom}_{R/yR}(N, N) \cong \operatorname{Hom}_{R/yR}(N, N/zN).$$

So we have $\operatorname{depth}(N/zN) \geq 1$, because if otherwise, we would have $\operatorname{depth}(\operatorname{Hom}_{R/yR}(N, N)) \leq 1$, which contradicts $\operatorname{Hom}_{R/yR}(N, N) \cong R/yR$ and $\operatorname{depth}(R/yR) \geq 2$ as proved above. It follows that $\operatorname{depth}(\omega_R/y\omega_R) \geq 2$. Thus, we have $\operatorname{depth}(\omega_R) \geq 3$ and $\mathfrak{m} \notin \operatorname{Att}(H^{d-1}_{\mathfrak{m}}(R))$ in view of [2, Lemma 2.1 (2)(i)]. We claim that

$$y \notin \bigcup_{\mathfrak{p} \in \operatorname{Att}_R\left(H^{d-1}_{\mathfrak{m}}(R)\right)} \mathfrak{p}.$$

Indeed, this implies that the multiplication map $H^{d-1}_{\mathfrak{m}}(R) \xrightarrow{\cdot y} H^{d-1}_{\mathfrak{m}}(R)$ is surjective in view of [4, Proposition 7.2.11]. So suppose to the contrary that $y \in \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Att}_R\big(H^{d-1}_{\mathfrak{m}}(R)\big)$. Then by Shifted Localization Theorem, we have $y/1 \in \mathfrak{p}R_{\mathfrak{p}} \in \operatorname{Att}_{R_{\mathfrak{p}}}\big(H^{\operatorname{ht}(\mathfrak{p})-1}_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}})\big)$. As we already proved that $\mathfrak{p} \neq \mathfrak{m}$, the induction hypothesis implies that $R_{\mathfrak{p}}$ is quasi-Gorenstein and by [30, Corollary 2.8], we must get

$$y/1 \notin \bigcup_{\mathfrak{q}R_{\mathfrak{p}} \in \operatorname{Att}_{R_{\mathfrak{p}}}\left(H^{\operatorname{ht}(\mathfrak{p})-1}_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}})\right)} \mathfrak{q}R_{\mathfrak{p}},$$

a contradiction. By a similar argument as in part (1) or (2), we can establish $\omega_R \cong R$.

We prove (4). This can be reduced to the situation of part (5), using the Noetherian induction. However, we will deduce it via a simpler proof than the proof of part (5). Since both R/yR and the formal fibers of R/yR have S_3, the $\mathfrak{m}$-adic completion of R/yR satisfies the same hypothesis. So let us assume that R is complete. Now R/yR is a quasi-Gorenstein complete local with S_3, so we have $H^{d-2}_{\mathfrak{m}}(R/yR) = 0$ in view of [26, Corollary 1.15]. It follows that the multiplication map

$H^{d-1}_{\mathfrak{m}}(R) \xrightarrow{\cdot y} H^{d-1}_{\mathfrak{m}}(R)$ is injective on the $\mathfrak{m}$-torsion module $H^{d-1}_{\mathfrak{m}}(R)$, which yields $H^{d-1}_{\mathfrak{m}}(R) = 0$. We conclude that $R/yR \cong \omega_{R/yR} \cong \omega_R/y\omega_R$, showing that ω_R is cyclic, as required.

We prove (5). Notice that by [30, Corollary 2.8] together with Theorem 2.4, we easily deduce that R is quasi-Gorenstein on $\mathrm{Spec}^\circ(R/yR)$ if and only if R/y^nR is quasi-Gorenstein on $\mathrm{Spec}^\circ(R/y^nR)$ for each $n \geq 2$. Suppose that R satisfies these equivalent conditions and $\mathrm{depth}(R) \geq 4$. Moreover, since R has Gorenstein formal fibers, we can suppose that R is a complete local ring without loss of generality. Then both $\omega_R/y^n\omega_R$ and ω_{R/y^nR} define line bundles on $\mathrm{Spec}^\circ(R/y^nR)$. We claim that these line bundles are identical on $\mathrm{Spec}^\circ(R/y^nR)$. By [30, Remark 2.3], there exists a natural embedding: $\omega_R/y^n\omega_R \hookrightarrow \omega_{R/y^nR}$ whose cokernel C is locally (by Matlis duality) dual to $H^{\dim(R_\mathfrak{p})-1}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})/y^nH^{\dim(R_\mathfrak{p})-1}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})$ for each $\mathfrak{p} \in \mathrm{Spec}(R/yR)$. Since both $R_\mathfrak{p}$ and $R_\mathfrak{p}/y^nR_\mathfrak{p}$ are quasi-Gorenstein for each $\mathfrak{p} \in \mathrm{Spec}^\circ(R/yR)$, we have $C_\mathfrak{p} = 0$ for $\mathfrak{p} \in \mathrm{Spec}^\circ(R/y^nR)$ in view of [30, Corollary 2.8] and hence our claim follows. There is a group homomorphism:

$$\pi_n : \mathrm{Pic}\left(\mathrm{Spec}^\circ(R/y^nR)\right) \to \mathrm{Pic}\left(\mathrm{Spec}^\circ(R/y^{n-1}R)\right),$$

which is induced by the natural surjection $M \mapsto M/y^{n-1}M$ for each $n \geq 2$. Since R/yR is quasi-Gorenstein, we have

$$0 = [\omega_{R/yR}] = [\omega_R/y\omega_R] = [\pi_2(\omega_R/y^2\omega_R)] = [\pi_2(\omega_{R/y^2R})],$$

that is to say, we have $[\omega_{R/y^2R}] \in \mathrm{Ker}(\pi_2)$. Since $\mathrm{depth}(R) \geq 4$, arguing as in [14, III, Exercise 4.6], we can apply [14, III, Exercise 2.3(e)], [14, III, Exercise 3.3(b)] and [14, III, Theorem 3.7] to see that π_2 is injective and thus, $[\omega_{R/y^2R}]$ is trivial in $\mathrm{Pic}\big(\mathrm{Spec}^\circ(R/y^2R)\big)$. By considering the maps π_n inductively and using a different but similar exact sequence as in [14, III, Exercise 4.6],[2] we can deduce that $[\omega_{R/y^nR}] = 0$ as an element of $\mathrm{Pic}\left(\mathrm{Spec}^\circ(R/y^nR)\right)$ for each $n \geq 1$.

Suppose to the contrary that R is not quasi-Gorenstein. Then according to Theorem 2.4, there exists an integer $n \geq 2$ such that R/y^nR is not quasi-Gorenstein. For each $n \geq 2$, R/y^nR satisfies Serre's S_2-condition, we have $H^{d-1}_{\mathfrak{m}}(\omega_{R/y^nR}) \cong E_{R/y^nR}(R/\mathfrak{m})$ in view of [2, Remark 1.4], because it is quasi-Gorenstein on $\mathrm{Spec}^\circ(R/y^nR)$ and $\mathrm{depth}(R/y^nR) \geq 3$ by assumption. Since R/y^nR is generically Gorenstein, $\omega_{R/y^nR} \cong \mathfrak{a}$ for an ideal $\mathfrak{a} \subseteq R/y^nR$ by applying [5, Lemma 1.4.4] and [5, 1.4.18]. Since R/y^nR has S_2, but is not quasi-Gorenstein, after applying the functor $\Gamma_\mathfrak{m}(-)$ to the exact sequence $0 \to \mathfrak{a} \to R/y^nR \to (R/y^nR)/\mathfrak{a} \to 0$, we conclude that $\mathrm{ht}(\mathfrak{a}) \leq 1$; otherwise we would get $H^{d-1}_{\mathfrak{m}}(R/y^nR) \cong H^{d-1}_{\mathfrak{m}}(\omega_{R/y^nR}) \cong E_{R/y^nR}(R/\mathfrak{m})$, contradicting to our hypothesis that R/y^nR is not quasi-Gorenstein. On the other hand, $\mathfrak{a}$ has trivial annihilator, because R/y^nR is unmixed by [1, (1.8), page 87] and [2, Lemma 1.1]. So it follows that $\mathrm{ht}(\mathfrak{a}) = 1$. Since $\mathfrak{a}$ satisfies S_2, we get $\Gamma_\mathfrak{m}\big((R/y^nR)/\mathfrak{a}\big) \cong H^1_\mathfrak{m}(\mathfrak{a}) = 0$. Therefore, $\mathfrak{a}$ satisfies the hypothesis of Lemma 2.1 and hence it is principal, i.e. R/y^nR is quasi-Gorenstein. But this is a contradiction and we must get that R/y^nR is quasi-Gorenstein for all $n > 0$. That is, R is quasi-Gorenstein.

The assertion (6) is a special case of part (5).

Finally, we prove the assertion (7). Suppose the contrary. Then using the Noetherian induction, we may assume that $R_\mathfrak{p}$ is quasi-Gorenstein for all $\mathfrak{p} \in \mathrm{Spec}^\circ(R/yR)$. Since $R[\frac{1}{y}]$ is quasi-Gorenstein by assumption, ω_R defines an element of $\mathrm{Pic}\left(\mathrm{Spec}^\circ(R)\right)$ which, in view of our hypothesis, belongs to

$$\mathrm{Ker}\Big(\mathrm{Pic}\left(\mathrm{Spec}^\circ(R)\right) \to \mathrm{Pic}\left(\mathrm{Spec}^\circ(R/yR)\right)\Big).$$

[2] More precisely, consider the exact sequence $0 \to \mathcal{O}_1 \xrightarrow{g} \mathcal{O}^*_{n+1} \to \mathcal{O}^*_n \to 0$, where $\mathcal{O}^*_n$ denotes the sheaf of the group of invertible elements on $\mathrm{Spec}^\circ(R/y^nR)$ and g is defined by $t \mapsto 1 + ty^n$.

Then by virtue of a theorem of Bhatt and de Jong [3, Theorem 0.1], ω_R is the trivial element in $\mathrm{Pic}(\mathrm{Spec}^\circ(R))$. Then the desired conclusion follows by applying Lemma 2.1 to R. □

Let us prove a positive result in the graded normal case. First, we prepare a few lemmas.

Lemma 3.3. *Suppose that $R = \bigoplus_{n \geq 0} R_n$ is a Noetherian standard graded ring with $\mathfrak{m} := \bigoplus_{n>0} R_n$ and that M is a finitely generated graded R-module with $\mathrm{grade}_{\mathfrak{m}}(M) \geq 2$. Then*

$$M \cong \bigoplus_{n \in \mathbb{Z}} H^0\big(X, \widetilde{M}(n)\big),$$

where we put $X := \mathrm{Proj}(R)$.

Proof. According to [12, (2.1.5)], there is an exact sequence

$$0 \to H^0_{\mathfrak{m}}(M) \to M \to \bigoplus_{n \in \mathbb{Z}} H^0\big(X, \widetilde{M}(n)\big) \to H^1_{\mathfrak{m}}(M) \to 0$$

under the stated hypothesis on $(R, \mathfrak{m})$. Since $\mathrm{grade}_{\mathfrak{m}}(M) \geq 2$ by assumption, we have the claimed isomorphism. □

We need some tools from algebraic geometry.

Definition 3.4 (Lefschetz condition)**.** Let X be a Noetherian scheme and let $Y \subset X$ be a closed subscheme. Denote by $\widehat{(\ )}$ the formal completion along Y. Then we say that the pair (X, Y) satisfies the *Lefschetz condition*, written as $\mathrm{Lef}(X, Y)$, if for every open neighborhood U of Y in X and a locally free sheaf $\mathcal{F}$ on U, there exists an open subset U' of X with $Y \subset U' \subset U$ such that the natural map

$$H^0(U', \mathcal{F}|_{U'}) \to H^0(\widehat{X}, \widehat{\mathcal{F}})$$

is an isomorphism.

The Lefschetz condition has been used to study the behavior of Picard groups or algebraic fundamental groups under the restriction maps. We refer the reader to [13, Chapter IV] for these topics.

Lemma 3.5. *Let X be an integral projective variety of dimension ≥ 2 over a field of characteristic zero and let $D \subset X$ be a nonsingular effective ample divisor. Then the pair (X, D) satisfies the Lefschetz property $\mathrm{Lef}(X, D)$.*

Proof. Since D is locally principal and nonsingular, there exists an open neighborhood $D \subset V$ in X such that V is nonsingular and dense in X. By Hironaka's theorem of desingularization, there exists a nonsingular integral variety Y and a proper birational morphism $\pi : Y \to X$ such that $\pi^{-1}(V) \cong V$. By [25, Lemma 3.4],[3] there exists an effective Cartier divisor $E \subset Y$ such that either $E = 0$ or $\dim \pi(\mathrm{Supp}(E)) = 0$ and

$$H^0\big(Y, \mathcal{F} \otimes \mathcal{O}_Y(E)\big) \cong H^0(\widehat{Y}, \widehat{\mathcal{F}}) \tag{3.3}$$

for a fixed coherent reflexive sheaf $\mathcal{F}$ on Y, where $\mathcal{F}$ is locally free around some neighborhood of $D \cong \pi^{-1}(D)$. Here, $\widehat{(\ )}$ is the completion along $\pi^{-1}(D) \subset Y$. For any open neighborhood $\pi^{-1}(D) \subset U$ such that $U \cap \mathrm{Supp}(E) = \emptyset$, the map (3.3) factors as

$$H^0\big(Y, \mathcal{F} \otimes \mathcal{O}_Y(E)\big) \to H^0\big(U, \mathcal{F} \otimes \mathcal{O}_Y(E)\big) \to H^0(\widehat{Y}, \widehat{\mathcal{F}})$$

[3]To apply the lemma, we need that $X \setminus D$ is affine and the cohomological dimension of $Y \setminus \pi^{-1}(D)$ is at most $\dim Y - 1$; these are satisfied in our case in view of [13, Corollary 3.5 at page 98].

and we have an isomorphism $H^0(U, \mathcal{F} \otimes \mathcal{O}_Y(E)) \cong H^0(U, \mathcal{F})$. Therefore,

$$H^0(U, \mathcal{F}) \to H^0(\widehat{Y}, \widehat{\mathcal{F}}) \tag{3.4}$$

is surjective. Let us prove that (3.4) is injective. Let $\mathcal{I}$ be the ideal sheaf of $D' := \pi^{-1}(D)$ (as a closed subscheme of U). Then we have a short exact sequence: $0 \to \mathcal{I}^n \to \mathcal{O}_U \to \mathcal{O}_{D'_n} \to 0$, where D_n is the n-th infinitesimal thickening of D'. Now we get a short exact sequence

$$0 \to \mathcal{I}^n\mathcal{F} \to \mathcal{F} \to \mathcal{F}/\mathcal{I}^n\mathcal{F} \to 0.$$

Taking cohomology, we get an exact sequence $0 \to H^0(U, \mathcal{I}^n\mathcal{F}) \to H^0(U, \mathcal{F}) \to H^0(D'_n, \mathcal{F}/\mathcal{I}^n\mathcal{F})$. Using [14, Chapter II, Proposition 9.2],[4] one gets an exact sequence

$$0 \to \varprojlim_n H^0(U, \mathcal{I}^n\mathcal{F}) \to H^0(U, \mathcal{F}) \to H^0(\widehat{Y}, \widehat{\mathcal{F}}),$$

where the latter map coincides with (3.4). So it suffices to prove that $\varprojlim_n H^0(U, \mathcal{I}^n\mathcal{F}) = 0$. In view of [14, Chapter II, Proposition 9.2], one is reduced to proving that $\varprojlim_n \mathcal{I}^n\mathcal{F} = 0$. Since this question is local, we may assume that $U = \operatorname{Spec}(R)$ for a Noetherian ring R. Since Y is an integral variety, its open subset U is also integral. Therefore, R is a Noetherian domain. We have

$$\widetilde{I^nF} \cong \mathcal{I}^n\mathcal{F}$$

for an ideal $I \subset R$ and a projective R-module F of finite rank. However, R is a Noetherian domain, it follows from Krull's intersection theorem that $\bigcap_{n>0} I^nF = 0$ and thus

$$\varprojlim_n \mathcal{I}^n\mathcal{F} = 0,$$

as desired.

For any locally free sheaf $\mathcal{G}$ over an open subset $W \subset X$ such that $D \subset W \subset V$ with V as in the beginning of the proof, since $\pi^{-1}(W) \cong W$, we have the commutative diagram:

$$\begin{array}{ccc}
H^0\big(\pi^{-1}(W), \pi^*\mathcal{G}|_{\pi^{-1}(W)}\big) & \xrightarrow{\simeq} & H^0\big(\widehat{Y}, \pi^*\widehat{\mathcal{G}|_{\pi^{-1}(W)}}\big) \\
\Big\| & & \uparrow \\
H^0(W, \mathcal{G}) & \longrightarrow & H^0(\widehat{X}, \widehat{\mathcal{G}})
\end{array}$$

where the vertical map on the right is induced by the map π and the horizontal map on the top is an isomorphism, due to (3.4). On the other hand, letting $\mathcal{J}$ be the ideal sheaf of $D \subset W$, we have isomorphisms $\pi^{-1}(D)_n \cong D_n$ and

$$H^0(\widehat{X}, \widehat{\mathcal{G}}) \cong \varprojlim_{n>0} H^0(D_n, \mathcal{G}/\mathcal{J}^n\mathcal{G}) \cong \varprojlim_{n>0} H^0\big(\pi^{-1}(D)_n, \pi^*\mathcal{G}/\pi^{-1}(\mathcal{J})^n\pi^*\mathcal{G}\big) \cong H^0\big(\widehat{Y}, \pi^*\widehat{\mathcal{G}|_{\pi^{-1}(W)}}\big).$$

In summary, $H^0(W, \mathcal{G}) \to H^0(\widehat{X}, \widehat{\mathcal{G}})$ is an isomorphism, which shows that the pair (X, D) satisfies $\operatorname{Lef}(X, D)$, as desired.

□

Let us prove the following result.

Theorem 3.6. *Let $R = \bigoplus_{n \geq 0} R_n$ be a Noetherian standard graded ring such that $y \in R$ is a regular element which is homogeneous of positive degree, $R_0 = k$ is a field of characteristic zero. Suppose that R/yR is a quasi-Gorenstein graded ring such that $X := \operatorname{Proj}(R)$ is an integral normal variety and $X_1 := \operatorname{Proj}(R/yR)$ is nonsingular. Then R is a quasi-Gorenstein graded ring.*

[4] There is a result asserting that the cohomology functor commutes with inverse limit functor under the Mittag-Leffler condition; see [16, Proposition 8.2.5.3].

Proof. Let us fix notation: $R_{(n)} := R/y^nR$, $\mathfrak{m} := \bigoplus_{n\geq 1} R_n$ and $X_n := \mathrm{Proj}(R/y^nR)$ for each $n > 0$. Since $R_{(n)}$ is a standard graded ring over the field k, the sheaves $\mathcal{O}_{X_n}(m)$ are invertible for $m \in \mathbb{Z}$ and $n > 0$.[5] Assume that R/yR is quasi-Gorenstein. Then:

$$\operatorname{depth} R \geq 3,\ \mathcal{O}_X(n) \text{ is invertible and } \widetilde{\omega_R}(n) \text{ is an } S_2\text{-sheaf.} \tag{3.5}$$

Now let us prove that R is quasi-Gorenstein. First, assume that $\dim X \leq 2$, or equivalently $\dim R \leq 3$. Since R/yR is quasi-Gorenstein, it has $\dim R/yR = \operatorname{depth} R/yR \geq 2$, in which case it is immediate to see that R is a Gorenstein graded ring. In what follows, let us assume that $\dim X \geq 3$ and set $d := \deg(y)$. Then we have a short exact sequence: $0 \to y^nR/y^{n+1}R \to R/y^{n+1}R \to R/y^nR \to 0$. Put $\mathcal{O}_{X_1}(-dn) := \widetilde{R/yR}(-dn)$. Then there is an isomorphism

$$\mathcal{O}_{X_1}(-dn) \xrightarrow{\cdot y^n} \big(y^n \widetilde{R/y^{n+1}}R\big) \text{ as } \mathcal{O}_{X_1}\text{-modules.}$$

Then we get an exact sequence of abelian sheaves:

$$0 \to \mathcal{O}_{X_1}(-dn) \xrightarrow{\alpha} \mathcal{O}^*_{X_{n+1}} \to \mathcal{O}^*_{X_n} \to 0 \tag{3.6}$$

on the topological space X_1, where $\alpha(t) := 1 + ty^n$. Since $\mathcal{O}_{X_1}(-dn)$ is the dual of an ample divisor for $n > 0$, we have $H^1\big(X_1, \mathcal{O}_{X_1}(-dn)\big) = 0$ for $n > 0$ by Kodaira's vanishing theorem. Hence the map between Picard groups induced by (3.6)

$$\pi_{n+1} : \mathrm{Pic}(X_{n+1}) \to \mathrm{Pic}(X_n) \tag{3.7}$$

is injective in view of [14, III, Exercise 4.6]. Denote by $a := a(R_{(1)})$ the a-invariant of $R_{(1)}$. Then we have $\omega_{R_{(1)}} \cong R_{(1)}(a)$ and thus by [11, Lemma (5.1.2)],

$$\widetilde{\omega_{R_{(1)}}}(-a) \cong \widetilde{\omega_{R_{(1)}}} \otimes \mathcal{O}_{X_1}(-a) \cong \mathcal{O}_{X_1}(a) \otimes \mathcal{O}_{X_1}(-a) \cong \mathcal{O}_{X_1}.$$

Since $y \in R$ is regular and $X_1 \subset X$ is a nonsingular divisor, X is nonsingular in a neighborhood of X_1 and $X_1 = X_2 = \cdots$ as topological spaces. In particular, X_n is a Gorenstein scheme for $n \geq 1$. By [31, Theorem (A.3.9)], we have $\big[\widetilde{\omega_{R_{(n)}}}\big] \in \mathrm{Pic}(X_n)$ for $n \geq 1$. Consider the short exact sequence $0 \to R(-dn) \xrightarrow{\cdot y^n} R \to R_{(n)} \to 0$. By [11, Proposition (2.2.9)], we get an injection:

$$\big(\omega_R/y^n\omega_R\big)(dn) \hookrightarrow \omega_{R_{(n)}}.$$

Then an inspection of the proof of [11, Proposition (2.2.10)], together with the fact that X_n is Gorenstein, yields that

$$\big(\widetilde{\omega_R/y^n\omega_R}\big)(dn) \cong \widetilde{\omega_{R_{(n)}}} \text{ for } n > 0.$$

Hence we have $\big[\big(\widetilde{\omega_R/y^n\omega_R}\big)(dn)\big] \in \mathrm{Pic}(X_n)$ and $\big[\big(\widetilde{\omega_R/y^n\omega_R}\big)(m)\big] \in \mathrm{Pic}(X_n)$ for $m \in \mathbb{Z}$ and $n \geq 2$. Since $\big[\big(\widetilde{\omega_R/y\omega_R}\big)(d-a)\big] \in \mathrm{Pic}(X_1)$ is trivial, it follows from (3.7) that

$$\big(\widetilde{\omega_R/y^{n+1}\omega_R}\big)(2d-a) \cong \mathcal{O}_{X_{n+1}}(d)$$

for $n > 0$. Since $X_1 \subset X$ is a nonsingular divisor, there is an open neighborhood $X_1 \subset U$ such that U is nonsingular. In particular, it follows that $\widetilde{\omega_R}(2d-a)\big|_U$ is a line bundle. There are isomorphisms for all $n > 0$ and $m \in \mathbb{Z}$:

$$\mathcal{O}_{X_{n+1}}(d+m) \cong \big(\widetilde{\omega_R/y^{n+1}\omega_R}\big)(2d-a+m) \cong \widetilde{\omega_R}(2d-a+m)\Big/\widetilde{y^{n+1}\omega_R}(2d-a+m).$$

[5]The paper [11] considers a more generalized version of standard graded rings, known as "condition (#)" in [11, page 206].

Hence we get $\widehat{\mathcal{O}_X(d+m)} \cong \widehat{\widetilde{\omega_R}(2d-a+m)}$, where $\widehat{(\ )}$ is the formal completion along the closed subscheme $X_1 \subset X$. Therefore,

$$\big[\widetilde{\omega_R}(2d-a+m)\big|_U\big] - \big[\mathcal{O}_X(d+m)\big|_U\big] \in \operatorname{Ker}\big(\operatorname{Pic}(U) \to \operatorname{Pic}(\widehat{X})\big).$$

Notice that $X_1 \subset X$ is a nonsingular Cartier divisor and the pair (X, X_1) satisfies the property $\operatorname{Lef}(X, X_1)$ in view of Lemma 3.5. So after possibly shrinking U more, it follows that
(3.8)

$$H^0\big(U, \widetilde{\omega_R}(2d-a+m)\big) \cong H^0\big(\widehat{X}, \widehat{\widetilde{\omega_R}(2d-a+m)}\big) \cong H^0\big(\widehat{X}, \widehat{\mathcal{O}_X(d+m)}\big) \cong H^0\big(U, \mathcal{O}_X(d+m)\big).$$

We claim that $Z := X \setminus U$ is zero-dimensional. Indeed, the complement $X \setminus X_1$ is affine. On the other hand, Z is a proper scheme over k that is contained in $X \setminus X_1$, so Z must be a zero-dimensional closed set in X. Using these facts together with the hypothesis $\dim X \geq 3$ and (3.5), we have an exact sequence:

$$0 = H^0_Z\big(X, \widetilde{\omega_R}(m)\big) \to H^0\big(X, \widetilde{\omega_R}(m)\big) \to H^0\big(U, \widetilde{\omega_R}(m)\big) \to H^1_Z\big(X, \widetilde{\omega_R}(m)\big) = 0$$

in view of [14, III, Exercise 2.3 (e) and (f)], and so an isomorphism $H^0\big(X, \widetilde{\omega_R}(m)\big) \cong H^0\big(U, \widetilde{\omega_R}(m)\big)$. Likewise, we have $H^0\big(X, \mathcal{O}_X(m)\big) \cong H^0\big(U, \mathcal{O}_X(m)\big)$. So it follows from (3.8) and Lemma 3.3 that

$$\omega_R \cong \bigoplus_{m \in \mathbb{Z}} H^0\big(X, \widetilde{\omega_R}(m)\big) \cong \bigoplus_{m \in \mathbb{Z}} H^0\big(X, \mathcal{O}_X(-d+a+m)\big) \cong R(-d+a),$$

and R is quasi-Gorenstein, as desired. □

Remark 3.7. One could try to prove results similar to Theorem 3.6 for non standard graded rings. It is worth pointing out that examples of non Cohen-Macaulay quasi-Gorenstein, non standard graded rings constructed by using ample invertible sheaves are given in [7] and examples constructed by using non-integral $\mathbb{Q}$-divisors are given in Example 5.1(2), while examples that are standard graded are easily constructed as in Example 5.1(1).

The following proposition shows ubiquity of quasi-Gorestein graded rings, which is an unpublished result due to K-i.Watanabe.

Proposition 3.8 (K-i.Watanabe)**.** *Let X be an integral normal projective variety of dimension at least 2 defined over an algebraically closed field k. Then there exists a quasi-Gorenstein, Noetherian normal graded domain $R = \bigoplus_{n \geq 0} R_n$ with $R_0 = k$ such that $X \simeq \operatorname{Proj}(R)$.*

Proof. The proof cited in [27, Proposition 5.9] applies directly to our case after dropping the assumption that $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$. □

4. Failure of deformation of quasi-Gorensteinness

In view of Theorem 3.2 (7) together with [30, Theorem 2.9], it seems to be promising that the quasi-Gorenstein property deforms (at least in equal-characteristic zero). However, counterexamples exist in both of prime characteristic and equal-characteristic zero cases.

Counterexample 4.1. Suppose that k is a field of either characteristic 2 or of characteristic zero. Let us define S to be the Segre product:

$$S := k[x, y, z]/(x^3) \# k[a, b, c]/(a^3),$$

i.e. S is the graded direct summand ring of the complete intersection ring $k[x, y, z, a, b, c]/(x^3, a^3)$ generated by the set of monomials $G := \{xa, xb, xc, ya, yb, yc, za, zb, zc\}$. By [11, Theorem (4.3.1)], S is quasi-Gorenstein. By [11, Proposition (4.2.2)], S has dimension 3 and it has depth 2 by [11, Proposition (4.1.5)]. We define the homomorphism $\varphi : k[Z_1, \ldots, Z_9] \to S$ by setting $Z_i \mapsto G_i$.

Then the ideal $\mathfrak{b} := \ker \varphi$ of $k[Z_1, \ldots, Z_9]$ is generated by the 2-sized minors of the matrix $M := \begin{pmatrix} Z_1 & Z_2 & Z_3 \\ Z_4 & Z_5 & Z_6 \\ Z_7 & Z_8 & Z_9 \end{pmatrix}$ as well as the elements

$$\begin{aligned}
&Z_1^3, Z_2^3, Z_3^3, Z_4^3, Z_7^3, \\
&Z_1^2Z_2, Z_1^2Z_3, Z_1Z_2^2, Z_1Z_3^2, Z_2^2Z_3, Z_2Z_3^2, Z_1Z_2Z_3, \\
&Z_1^2Z_4, Z_1^2Z_7, Z_1Z_4^2, Z_1Z_7^2, Z_4^2Z_7, Z_4Z_7^2, Z_1Z_4Z_7.
\end{aligned} \tag{4.1}$$

So we have $S = k[Z_1, \ldots, Z_9]/\mathfrak{b}$. Now set $A := k[Z_1, \ldots, Z_9, Y]$ and let $\mathfrak{a}$ be the ideal of A generated by the equations (4.1) as well as the 2-sized minors of the matrix M with two exceptions: $Z_4Z_7Y - Z_6Z_8 + Z_5Z_9$ instead of the determinant of $\begin{pmatrix} Z_5 & Z_6 \\ Z_8 & Z_9 \end{pmatrix}$ and $Z_1Z_7Y - Z_3Z_8 + Z_2Z_9$ instead of the determinant of $\begin{pmatrix} Z_2 & Z_3 \\ Z_8 & Z_9 \end{pmatrix}$. Let us set $R := A/\mathfrak{a}$ and suppose that y is the image of Y in R. Thus, we have $S = R/yR$. With the aid of the following Macaulay2 commands, one can verify that $y \in R$ is a regular element and R is not quasi-Gorenstein.

$i1 : A = \mathrm{QQ}[Z_1..Z_9, Y, \mathrm{Degrees} => \{9 : 1, 0\}]$
$o1 = A$
$o1$: PolynomialRing
$i2 : a = \mathrm{ideal}(Z_6 * Z_7 - Z_4 * Z_9, Z_5 * Z_7 - Z_4 * Z_8, Z_3 * Z_7 - Z_1 * Z_9, Z_2 * Z_7 - Z_1 * Z_8, Z_3 * Z_5 - Z_2 * Z_6, Z_3 * Z_4 - Z_1 * Z_6, Z_2 * Z_4 - Z_1 * Z_5, Z_4 * Z_7 * Y - Z_6 * Z_8 + Z_5 * Z_9, Z_1 * Z_7 * Y - Z_3 * Z_8 + Z_2 * Z_9, \quad Z_1^3, Z_2^3, Z_3^3, Z_4^3, Z_7^3, \quad Z_1^2 * Z_2, Z_1^2 * Z_3, Z_1 * Z_2^2, Z_1 * Z_3^2, Z_2^2 * Z_3, Z_2 * Z_3^2, Z_1 * Z_2 * Z_3, \quad Z_1^2 * Z_4, Z_1^2 * Z_7, Z_1 * Z_4^2, Z_1 * Z_7^2, Z_4^2 * Z_7, Z_4 * Z_7^2, Z_1 * Z_4 * Z_7);$
$o2$: Ideal of A
$i3 : c = \mathrm{ideal}(Z_1^3, Z_2^3, Z_3^3, Z_4^3, Z_7^3, Z_4 * Z_7 * Y - Z_6 * Z_8 + Z_5 * Z_9);$
$o3$: Ideal of A
$i4$: codim $c ==$ codim a
$o4 =$ true
$i5$: codim $c == 6$
$o5 =$ true
$i6 : d = c : a;$
$o6$: Ideal of A
$i7 : C = \mathrm{module}(d)/\mathrm{module}(c);$
$i8 : N = C/((\text{ideal gens ring } C) * C);$
$i9$: numgens source basis N
$o9 = 9$
$i10 : a : Y == a$
$o10 =$ true

Thus, the canonical module of R, which is the module C in the above Macaulay2 code, is generated minimally by 9 elements. Note that the last command shows that y is a regular element of R. We remark that the quasi-Gorenstein local ring $S = R/yR$ is Gorenstein on its punctured spectrum, which also shows that the depth condition of Theorem 3.2(6) is necessary and is sharp. Also we remark that, replacing QQ with $ZZ/\mathrm{ideal}(2)$ in the first command of the above Macaulay2 code, leads to the same conclusion.

Thus, we obtain the following result.

Theorem 4.2. *There exists an example of a local Noetherian ring* $(R, \mathfrak{m})$*, together with a regular element* $y \in \mathfrak{m}$ *such that the following property holds:* R/yR *is quasi-Gorenstein and* R *is not quasi-Gorenstein.*

Remark 4.3. In spite of Counterexample 4.1, the quasi-Gorenstein analogue of Ulrich's result [32, Proposition 1] holds: A quasi-Gorenstein ring which is a homomorphic image of a regular ring and which is a complete intersection at codimension ≤ 1 has a deformation to an excellent unique factorization domain in view of [28, Proposition 3.1].

The local ring $(R, \mathfrak{m})$ constructed in Counterexample 4.1 is not normal. At the time of preparation of this paper, we do not have any concrete counterexample for the deformation of quasi-Gorensteinness in the context of normal domains. For standard graded normal domains, we have Theorem 3.6.

5. Construction of quasi-Gorenstein rings which are not Cohen-Macaulay

In this section, we offer three different potential instances of quasi-Gorenstein normal domains and we are curious to know whether or not any of these instances of quasi-Gorenstein normal (local) domains admits a deformation to a quasi-Gorenstein ring.

Example 5.1.

(1) Let k be any field with $\mathrm{char}(k) \neq 3$ and suppose that S is the Segre product of the cubic Fermat hypersurface:

$$k[x, y, z]/(x^3 + y^3 + z^3)\#k[a, b, c]/(a^3 + b^3 + c^3).$$

Then in view of [11], S is a quasi-Gorenstein normal domain of dimension 3 and depth 2 such that $\mathrm{Proj}(S)$ is the product of two elliptic curves and so $\mathrm{Proj}(S)$ is an Abelian surface. In contrast to Counterexample 4.1, we expect that any deformation of S would be again quasi-Gorenstein. In view of Theorem 3.2(1), perhaps it is worth remarking that, when characteristic of k varies over the prime numbers distinct from 3, S can be either F-pure or non-F-pure. In the case when S is F-pure, any deformation of the local ring of the affine cone attached to $\mathrm{Proj}(S)$ is quasi-Gorenstein due to Theorem 3.2(1). On the other hand, if $\mathrm{char}(k) = 0$, then any standard homogeneous deformation of S is quasi-Gorenstein in view of Theorem 3.6.

(2) In contrast to the previous example, we hereby present an example of a non-Cohen-Macaulay quasi-Gorenstein normal graded domain S with $\mathrm{Proj}(S) = \mathbb{P}^1_k \times \mathbb{P}^1_k$, where k is a field either of characteristic zero or of prime characteristic $p > 0$ such that p varies over a Zariski-dense open (cofinite) subset of prime numbers. The construction of such a quasi-Gorenstein normal domain is much more complicated than the previous one, and the ring S has to be a non-standard graded ring. Thanks to Demazure's theorem [8], any (not necessarily quasi-Gorenstein) normal $\mathbb{N}_0$-graded ring $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ with $X := \mathrm{Proj}(R) = \mathbb{P}^1 \times \mathbb{P}^1$ is the generalized section ring:

$$R = R(X, D) = \bigoplus_{n \in \mathbb{N}_0} H^0\big(X, \mathcal{O}_X(\lfloor nD \rfloor)\big)$$

for some rational coefficient Weil divisor $D \in \mathrm{Div}(X, \mathbb{Q}) = \mathrm{Div}(X) \otimes \mathbb{Q}$ such that nD is an ample Cartier divisor for some $n \gg 0$ (see [33, Theorem, page 203] for the general statement of this fact and also [33, §1] for the definitions and the background). We shall give an example of a rational Weil divisor D on $\mathbb{P}^1 \times \mathbb{P}^1$ whose generalized section ring $R(X, D)$ is a non-Cohen-Macaulay quasi-Gorenstein normal domain with a-invariant 5, and we will also present $R(X,D)$ explicitly as the Segre product of two hypersurfaces.[6] On the genus zero smooth curve $\mathbb{P}^1 = \mathrm{Proj}(k[x,y])$ (respectively, with different coordinates, $\mathbb{P}^1 = \mathrm{Proj}(k[w,z])$), consider the $\mathbb{Q}$-divisor

$$D_1 := 2P_0 - \sum_{i=1}^{3} 5/8 P_i,$$

where P_i corresponds to the prime ideal, $x+iy$, for $i = 0, \ldots, 3$, respectively,

$$D_2 := 5Q_0 - \sum_{i=1}^{9} 1/2 Q_i,$$

where Q_i corresponds to the prime ideal $w+iz$ for $i = 0, \ldots, 9$. We follow the notation used in the end of the statement of [33, Theorem (2.8)], and then we have $D_1' = \sum_{i=1}^{3} 7/8 P_i$ and $D_2' = \sum_{i=1}^{9} 1/2 Q_i$. Using the fact that $K_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$ is the canonical divisor of $\mathbb{P}^1$, one can easily verify that both of the $\mathbb{Q}$-divisors $K_{\mathbb{P}^1} + D_1' - 5D_1$ and $K_{\mathbb{P}^1} + D_2' - 5D_2$ are principal (integral) divisors and hence by [33, Corollary (2.9)], one can conclude that the section rings $G := R(\mathbb{P}^1, D_1)$ and $G' := R(\mathbb{P}^1, D_2)$ are both Gorenstein rings with a-invariant 5 (see also [33, Example (2.5)(b)] and [33, Remark (2.10)]). It follows that the Segre product $S := G \# G'$ is a quasi-Gorenstein ring. In the sequel, we will give a presentation of S and we show that it is not Cohen-Macaulay.

- **Presentation of G':** Applying [14, Chapter IV, Theorem 1.3 (Riemann-Roch)] we have $H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\lfloor 2nD_2 \rfloor)\big) = H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)\big)$ is an $(n+1)$-dimensional vector space for each $n \geq 0$ (because $\lfloor 2nD_2 \rfloor$ has degree n, $K_{\mathbb{P}^1} - \lfloor 2nD_2 \rfloor = \mathcal{O}_{\mathbb{P}^1}(-n-2)$ is not generated by global sections and $\mathbb{P}^1$ has genus zero). More precisely, we have $\lfloor 2nD_2 \rfloor = 10nQ_0 - \sum_{i=1}^{9} nQ_i \sim \mathcal{O}_{\mathbb{P}^1}(n)$ which yields

$$\begin{aligned} H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\lfloor 2nD_2 \rfloor)\big) &= \Big\{ f/g \in k(w,z) \;\Big|\; \mathrm{div}(f/g) + 10nQ_0 - \sum_{i=1}^{9} nQ_i \geq 0 \Big\} \\ &= \Big\{ \Big(\big(\prod_{i=1}^{9} (w+iz)^n\big) f \Big) / w^{10n} \;\Big|\; f \in k[w,z]_{[n]} \Big\}. \end{aligned}$$

  Consequently, $G'_{[2n]}$ is generated by $G'_{[2]}$ for each $n \geq 2$ (as the elements of the ring G'). Similarly, we can see that $H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\lfloor 9D_2 \rfloor)\big)$ is the 1-dimensional k-vector space spanned by $\big(\prod_{i=1}^{9}(w+iz)^5\big)/w^{45}$ which clearly provides us with a new generator of our section ring G'. One can then observe that, for $n \neq 4$, $H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\lfloor (2n+1)D_2 \rfloor)\big)$ is either zero for $n \leq 3$ or it is an $(n-3)$-dimensional vector space generated by ${G'}_{[9]}$ and ${G'}_{[2n-8]}$. It follows that G' has three generators and since it has dimension 2, we get

$$G' = k[A', B', C']/(f)$$

  for some irreducible element $f \in k[A', B', C']$ of degree 18, such that A' and B' have degree 2 while C' has degree 9. Namely, $f = C'^2 - (\prod_{i=1}^{9}(A' + iB'))$.

[6] A non-Cohen-Macaulay section ring, whose projective scheme is $\mathbb{P}^1 \times \mathbb{P}^1$, is given in [33, Example (2.6)]. Here a non-Cohen-Macaulay quasi-Gorenstein normal domain will be explicitly given.

- **Presentation of G:** Similarly as in the previous part, for any $m \geq 0$ and $0 \leq k \leq 7$, setting $0 \neq n := 8m + k$, we can observe that

$$H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\lfloor nD_1 \rfloor)\big) = \begin{cases} (m+1)\text{-dimensional vector space } H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m)\big), & k \overset{3}{\equiv} 0 \\ m\text{-dimensional vector space } H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m-1)\big), & k \overset{3}{\equiv} 1 \\ \max\{0, (m-1)\}\text{-dimensional vector space } H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m-2)\big), & k \overset{3}{\equiv} 2 \end{cases}$$

that $G_{[n]} = H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\lfloor nD_1 \rfloor)\big)$ is generated by $G_{[n-8]}$ and $G_{[8]}$ in the case where $m \geq 2$ and $n \neq 18, 21$, that $G_{[6]}$, $G_{[9]}$, $G_{[12]}$ and $G_{[15]}$ are generated by $G_{[3]}$, that $G_{[11]}$ (respectively, $G_{[14]}$) is generated by $G_{[8]}$ and $G_{[3]}$ (respectively, $G_{[11]}$ and $G_{[3]}$), that $G_{[18]}$ (respectively, $G_{[21]}$) is generated by $G_{[3]}$ and $G_{[15]}$ (respectively, $G_{[3]}$ and $G_{[18]}$) and that G is zero in the remained unmentioned degrees. Consequently,

$$G = k[A, B, C]/(g)$$

such that $\deg(A) = 3$, B and C are of degree 8 and $g = A^8 - \prod_{i=1}^{3}(B + iC)$ (Note that A corresponds to the element $\prod_{i=1}^{3}(x+iy)^2/x^6$, B corresponds to $\prod_{i=1}^{3}\big((x+iy)^5 x\big)/x^{16}$ and $C = \prod_{i=1}^{3}\big((x+iy)^5 y\big)/x^{16}$).
- **Non-Cohen-Macaulayness of $S = G\#G'$:** Note that by Serre duality theorem,

$$\begin{aligned} H^1\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\lfloor 3D_2 \rfloor)\big) = \operatorname{Hom}\big(\mathcal{O}_{\mathbb{P}^1}(\lfloor 3D_2 \rfloor), K_{\mathbb{P}^1}\big) &= H^0\Big(\mathbb{P}^1, \mathscr{H}om\big(\mathcal{O}_{\mathbb{P}^1}(\lfloor 3D_2 \rfloor), \mathcal{O}_{\mathbb{P}^1}(-2)\big)\Big) \\ &= H^0\Big(\mathbb{P}^1, \mathscr{H}om\big(\mathcal{O}_{\mathbb{P}^1}(-3), \mathcal{O}_{\mathbb{P}^1}(-2)\big)\Big) \\ &= H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)\big) \end{aligned}$$

is a non-zero 2-dimensional vector space. Thus,

$$\begin{aligned} H^2_{S+}(S)_{[3]} &= H^1\big(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\lfloor 3D_1 \rfloor) \boxtimes \mathcal{O}_{\mathbb{P}^1}(\lfloor 3D_2 \rfloor)\big) \\ &= \Big(H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\lfloor 3D_1 \rfloor)\big) \otimes H^1\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\lfloor 3D_2 \rfloor)\big)\Big) \\ &\oplus \Big(H^1\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\lfloor 3D_1 \rfloor)\big) \otimes \underbrace{H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\lfloor 3D_2 \rfloor)\big)}_{=G'_{[3]}=0}\Big) \\ &= G_{[3]} \otimes H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)\big) \\ &\neq 0, \end{aligned}$$

which implies that S is not-Cohen-Macaulay as required.

(3) We give an explicit construction of a unique factorization domain (so, being quasi-Gorenstein in our case), not being Cohen-Macaulay of depth 2 with arbitrarily large dimension, as an invariant subring. Fix a prime number $p \geq 5$ and an algebraically closed field k of characteristic p. Consider the k-automorphism on the polynomial algebra $k[x_1, \ldots, x_{p-1}]$ defined

by

$$\sigma(x_1) = x_1,$$
$$\sigma(x_2) = x_2 + x_1,$$
$$\vdots$$
$$\sigma(x_{p-1}) = x_{p-1} + x_{p-2}.$$

Now we have $\sigma((x_1, \dots, x_{p-2})) = (x_1, \dots, x_{p-2})$ which is a prime ideal, so σ gives rise to an action on the localization $R := k[x_1, \dots, x_{p-1}]_{(x_1,\dots,x_{p-2})}$. Let $\mathfrak{m}$ be the unique maximal ideal of R. Let $\langle\sigma\rangle$ be the cyclic group generated by σ. Then the ring of invariants $R^{\langle\sigma\rangle}$ enjoys the following properties:

- $R^{\langle\sigma\rangle}$ is a local ring which is essentially of finite type over k, $R^{\langle\sigma\rangle}$ is a unique factorization domain with a non Cohen-Macaulay isolated singularity, $\dim R^{\langle\sigma\rangle} = p-2$ and $\operatorname{depth} R^{\langle\sigma\rangle} = 2$. In particular, $R^{\langle\sigma\rangle}$ is quasi-Gorenstein.

Since R has characteristic p, σ generates the p-cyclic action by construction. Then $R^{\langle\sigma\rangle} \hookrightarrow R$ is an integral extension and we thus have $\dim R^{\langle\sigma\rangle} = p-2$. Quite obviously,

$$\big(\sigma(x_1) - x_1, \sigma(x_2) - x_2, \dots, \sigma(x_{p-1}) - x_{p-1}\big) = \big(x_1, \dots, x_{p-2}\big)$$

is an $\mathfrak{m}$-primary ideal. By [24, Lemma 3.2] (see also [9] for related results), the map $R^{\langle\sigma\rangle} \to R$ ramifies only at the maximal ideal. Since R is regular, $R^{\langle\sigma\rangle}$ has only isolated singularity. By [24, Corollary 1.6], we have $\operatorname{depth} R^{\langle\sigma\rangle} = 2$. Since $R^{\langle\sigma\rangle}$ has dimension $p-2 \geq 3$, we see that $R^{\langle\sigma\rangle}$ is not Cohen-Macaulay. It remains to show that $R^{\langle\sigma\rangle}$ is a unique factorization domain. For this, let us look at the action of $\langle\sigma\rangle$ on $k[x_1, \dots, x_{p-1}]$. Then by [10, Proposition 16.4], $k[x_1, \dots, x_{p-1}]^{\langle\sigma\rangle}$ is a unique factorization domain and we have

$$R^{\langle\sigma\rangle} = \big(k[x_1, \dots, x_{p-1}]_{(x_1,\dots,x_{p-2})}\big)^{\langle\sigma\rangle} = \big(k[x_1, \dots, x_{p-1}]^{\langle\sigma\rangle}\big)_{k[x_1,\dots,x_{p-1}]^{\langle\sigma\rangle} \cap (x_1,\dots,x_{p-2})}.$$

Since being a unique factorization domain is preserved under localization, it follows that $R^{\langle\sigma\rangle}$ is a unique factorization domain, as desired. The paper [23] examines more examples of non Cohen-Macaulay domains that are unique factorization domains.

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Email address: shimomotokazuma@gmail.com
Email address: nendo@purdue.edu
Email address: tavanfar@ipm.ir