BOUNDS FOR THE FIRST HILBERT COEFFICIENTS OF $\mathfrak{m}$-PRIMARY IDEALS

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Abstract. This paper purposes to characterize Noetherian local rings $(A, \mathfrak{m})$ of positive dimension such that the first Hilbert coefficients of $\mathfrak{m}$-primary ideals in A range among only finitely many values. Examples are explored to illustrate our theorems.

1. Introduction

Let A be a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. For each $\mathfrak{m}$-primary ideal I in A we set

$$\mathrm{H}_I(n) = \ell_A(A/I^{n+1}) \quad \text{for} \quad n \geq 0$$

and call it the Hilbert function of A with respect to I, where $\ell_A(A/I^{n+1})$ denotes the length of the A-module A/I^{n+1}. Then there exist integers $\{\mathrm{e}_i(I)\}_{0 \leq i \leq d}$ such that

$$\mathrm{H}_I(n) = \mathrm{e}_0(I)\binom{n+d}{d} - \mathrm{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_d(I) \quad \text{for all} \quad n \gg 0.$$

The integers $\mathrm{e}_i(I)$'s are called the Hilbert coefficients of A with respect to I. These integers describe the complexity of given local rings, and there are a huge number of preceding papers about them, e.g., [1, 2, 3, 4, 5]. In particular, the integer $\mathrm{e}_0(I) > 0$ is called the multiplicity of A with respect to I and has been explored very intensively. One of the most spectacular results on the multiplicity theory says that *A is a regular local ring if and only if* $\mathrm{e}_0(\mathfrak{m}) = 1$, provided A is unmixed. This was proven by P. Samuel [9] in the case where A contains a field of characteristic 0 and then by M. Nagata [7] in the above form. Recall that a local ring A is *unmixed*, if $\dim \widehat{A} = \dim \widehat{A}/\mathfrak{p}$ for every associated prime ideal $\mathfrak{p}$ of the $\mathfrak{m}$-adic completion $\widehat{A}$ of A. The Cohen-Macaulayness in A is characterized in terms of $\mathrm{e}_0(Q)$ of parameter ideals Q of A. On the other hand, L. Ghezzi and other authors [1] analyzed the boundness of the values $\mathrm{e}_1(Q)$ for parameter ideals Q of A and deduced that the local cohomology modules $\{\mathrm{H}^i_{\mathfrak{m}}(A)\}_{i \neq d}$ are finitely generated, once A is unmixed and the set $\Lambda(A) = \{\mathrm{e}_1(Q) \mid Q \text{ is a parameter ideal of } A\}$ is finite.

In the present paper we focus on the first Hilbert coefficients $\mathrm{e}_1(I)$ for $\mathfrak{m}$-primary ideals I of A. Our study dates back to the paper of M. Narita [8], who showed that if A is a Cohen-Macaulay local ring, then $\mathrm{e}_1(I) \geq 0$, and also $\mathrm{e}_2(I) \geq 0$ when $d = \dim A \geq 2$. We consider the set

$$\Delta(A) = \{\mathrm{e}_1(I) \mid I \text{ is an } \mathfrak{m}\text{-primary ideal in } A\}$$

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and are interested in the problem of when $\Delta(A)$ is finite. Under the light of Narita's theorem, if A is a Cohen-Macaulay local ring of positive dimension, our problem is equivalent to the question of when the values $\mathrm{e}_1(I)$ has a finite upper bound, and Theorem 1.1 below settles the question, showing that such Cohen-Macaulay local rings are exactly of dimension one and analytically unramified, where $\mathrm{H}^0_{\mathfrak{m}}(A)$ denotes the 0-th local cohomology module of A with respect to $\mathfrak{m}$.

Theorem 1.1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with $d = \dim A > 0$. Then the following conditions are equivalent.*

(1) *$\Delta(A)$ is a finite set.*
(2) *$d = 1$ and $A/\mathrm{H}^0_{\mathfrak{m}}(A)$ is analytically unramified.*

We prove Theorem 1.1 in Section 3. Section 2 is devoted to preliminaries for the proof. Let $\overline{A}$ denote the integral closure of A in the total ring of fractions of A. The key is the following, which we shall prove in Section 2.

Theorem 1.2. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim A = 1$. Then*

$$\sup \Delta(A) = \ell_A(\overline{A}/A).$$

Hence $\Delta(A)$ is a finite set if and only if A is analytically unramified.

For the proof we need particular calculation of $\mathrm{e}_1(I)$ in one-dimensional Cohen-Macaulay local rings, which we explain also in Section 2.

When $A = k[[t^{a_1}, t^{a_2}, \ldots, t^{a_\ell}]]$ is the semigroup ring of a numerical semigroup $H = \{\sum_{i=1}^{\ell} c_i a_i \mid c_i \in \mathbb{N}\}$ over a field k (here t is the indeterminate over k and $0 < a_1 < a_2 < \ldots < a_\ell$ are integers such that $\gcd(a_1, a_2, \ldots, a_\ell) = 1$), the set $\Delta(A)$ is finite and $\Delta(A) = \{0, 1, \ldots, \sharp(\mathbb{N} \setminus H)\}$, where $\mathbb{N}$ denotes the set of non-negative integers (Example 4.1). However, despite this result and the fact $\sup \Delta(A) = \ell_A(\overline{A}/A)$ in Theorem 1.2, the equality

$$\Delta(A) = \{n \in \mathbb{Z} \mid 0 \le n \le \ell_A(\overline{A}/A)\}$$

does not necessarily hold true in general. In Section 4 we will explore several concrete examples, including an example for which the equality is not true (Example 4.7).

Unless otherwise specified, throughout this paper let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let $\mathrm{Q}(A)$ denote the total ring of fractions of A. For each finitely generated A-module M, let $\ell_A(M)$ and $\mu_A(M)$ denote respectively the length of and the number of elements in a minimal system of generators of M.

2. Proof of Theorem 1.2

In this section let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim A = 1$. Let I be an $\mathfrak{m}$-primary ideal of A and assume that I contains a parameter ideal $Q = (a)$ as a reduction. Hence there exists an integer $r \ge 0$ such that $I^{r+1} = QI^r$. This assumption is automatically satisfied, when the residue class field $A/\mathfrak{m}$ of A is infinite. We set

$$\frac{I^n}{a^n} = \left\{ \frac{x}{a^n} \;\middle|\; x \in I^n \right\} \subseteq \mathrm{Q}(A) \quad \text{for} \quad n \ge 0$$

and let

$$B = A\left[\frac{I}{a}\right] \subseteq \mathrm{Q}(A),$$

where $\mathrm{Q}(A)$ denotes the total ring of fractions of A. Then

$$B = \bigcup_{n \geq 0} \frac{I^n}{a^n} = \frac{I^r}{a^r} \cong I^r$$

as an A-module, because $\frac{I^n}{a^n} = \frac{I^r}{a^r}$ if $n \geq r$ as $\frac{I^n}{a^n} \subseteq \frac{I^{n+1}}{a^{n+1}}$ for all $n \geq 0$. Therefore B is a finitely generated A-module, whence $A \subseteq B \subseteq \overline{A}$, where $\overline{A}$ denotes the integral closure of A in $\mathrm{Q}(A)$. We furthermore have the following.

Lemma 2.1 ([3, Lemma 2.1])**.**
(1) $\mathrm{e}_0(I) = \ell_A(A/Q)$.
(2) $\mathrm{e}_1(I) = \ell_A(I^r/Q^r) = \ell_A(B/A) \leq \ell_A(\overline{A}/A)$.

Conversely, let $A \subseteq B \subseteq \overline{A}$ be an arbitrary intermediate ring and assume that B is a finitely generated A-algebra. We choose a non-zerodivisor $a \in \mathfrak{m}$ of A so that $aB \subsetneq A$ and set $I = aB$. Then I is an $\mathfrak{m}$-primary ideal of A and $I^2 = a^2B = aI$. Hence $B = A\left[\frac{I}{a}\right] = \frac{I}{a}$, so that we get the following.

Corollary 2.2. $\ell_A(B/A) = \mathrm{e}_1(I) \in \Delta(A)$.

Let us note the following.

Lemma 2.3. *Let* $(A, \mathfrak{m})$ *be a Cohen-Macaulay local ring with* $\dim A = 1$*. Then*

$$\sup \Delta(A) \geq \ell_A(\overline{A}/A).$$

Proof. We set $s = \sup \Delta(A)$. Assume $s < \ell_A(\overline{A}/A)$ and choose elements $y_1, y_2, \ldots, y_\ell$ of $\overline{A}$ so that $\ell_A(\left[\sum_{i=1}^{\ell} Ay_i\right]/A) > s$. We consider the ring $B = A[y_1, y_2, \ldots, y_\ell]$. Then $A \subseteq B \subseteq \overline{A}$ and

$$s < \ell_A(\left[\sum_{i=1}^{\ell} Ay_i\right]/A) \leq \ell_A(B/A),$$

which is impossible, as $\ell_A(B/A) \in \Delta(A)$ by Corollary 2.2. Hence $s \geq \ell_A(\overline{A}/A)$. ☐

The assumption in the following Corollary 2.4 that the field $A/\mathfrak{m}$ is infinite is necessary to assure a given $\mathfrak{m}$-primary ideal I of A the existence of a reduction generated by a single element. We notice that even if the field $A/\mathfrak{m}$ is finite, the existence is guaranteed when $\overline{A}$ is a discrete valuation ring (see Section 4).

Corollary 2.4. *Let* $(A, \mathfrak{m})$ *be a Cohen-Macaulay local ring with* $\dim A = 1$*. Suppose that the field* $A/\mathfrak{m}$ *is infinite. We then have*

$$\Delta(A) = \{\ell_A(B/A) \mid A \subseteq B \subseteq \overline{A} \text{ is an intermediate ring which is a module-finite extension of } A\}$$

Proof. Let $\Gamma(A)$ denote the set of the right hand side. Let I be an $\mathfrak{m}$-primary ideal of A and choose a reduction $Q = (a)$ of I. We put $B = A\left[\frac{I}{a}\right]$. Then B is a module-finite extension of A and Lemma 2.1 (2) shows $\mathrm{e}_1(I) = \ell_A(B/A)$. Hence $\Delta(A) \subseteq \Gamma(A)$. The reverse inclusion follows from Corollary 2.2. □

We finish the proof of Theorem 1.2.

Proof of Theorem 1.2. By Lemma 2.3 it suffices to show $\sup \Delta(A) \leq \ell_A(\overline{A}/A)$. Enlarging the residue class field $A/\mathfrak{m}$ of A, we may assume that the field $A/\mathfrak{m}$ is infinite. Let I be an $\mathfrak{m}$-primary ideal of A and choose $a \in I$ so that aA is a reduction of I. Then

$$\mathrm{e}_1(I) \leq \ell_A(\overline{A}/A)$$

by Lemma 2.1 (2). Hence the result. □

3. Proof of Theorem 1.1

Let us prove Theorem 1.1. Let $(A, \mathfrak{m})$ be a Noetherian local ring with $d = \dim A > 0$. We begin with the following.

Lemma 3.1. *Suppose that $\Delta(A)$ is a finite set. Then $d = 1$.*

Proof. Let I be an $\mathfrak{m}$-primary ideal of A. Then for all $k \geq 1$

$$\mathrm{e}_0(I^k) = k^d{\cdot}\mathrm{e}_0(I) \quad \text{and} \quad \mathrm{e}_1(I^k) = \frac{d-1}{2}{\cdot}\mathrm{e}_0(I){\cdot}k^d + \frac{2\mathrm{e}_1(I) - \mathrm{e}_0(I){\cdot}(d-1)}{2}{\cdot}k^{d-1}.$$

In fact, we have

$$(1) \quad \ell_A(A/(I^k)^{n+1}) = \mathrm{e}_0(I^k)\binom{n+d}{d} - \mathrm{e}_1(I^k)\binom{n+d-1}{d-1} + \cdots + (-1)^d\mathrm{e}_d(I^k)$$

for $n \gg 0$, while

$$\begin{aligned}(2) \quad \ell_A(A/(I^k)^{n+1}) &= \ell_A(A/I^{(kn+k-1)+1}) \\ &= \mathrm{e}_0(I)\binom{(kn+k-1)+d}{d} - \mathrm{e}_1(I)\binom{(kn+k-1)+d-1}{d-1} \\ &\quad + \cdots + (-1)^d\mathrm{e}_d(I),\end{aligned}$$

$$\binom{kn+k+d-1}{d} = k^d\binom{n+d}{d} + a\binom{n+d-1}{d-1} + (\text{lower terms})$$

and

$$\binom{kn+k+d-2}{d-1} = k^{d-1}\binom{n+d-1}{d-1} + (\text{lower terms}),$$

where

$$a = k^{d-1}\left(k + \frac{d-1}{2}\right) - \frac{k^d}{2}(d+1).$$

Comparing the coefficients of n^d in equations (1) and (2), we see

$$\mathrm{e}_0(I^k) = k^d{\cdot}\mathrm{e}_0(I).$$

We similarly have

$$\begin{aligned}
\mathrm{e}_1(I^k) &= -\mathrm{e}_0(I)a + \mathrm{e}_1(I)k^{d-1} \\
&= -\mathrm{e}_0(I)\Big(k^d + \frac{d-1}{2}k^{d-1} - \frac{d+1}{2}k^d\Big) + \mathrm{e}_1(I)k^{d-1} \quad \text{and} \\
&= \frac{d-1}{2}\cdot\mathrm{e}_0(I)\cdot k^d + \frac{2\mathrm{e}_1(I) - \mathrm{e}_0(I)\cdot(d-1)}{2}\cdot k^{d-1},
\end{aligned}$$

considering n^{d-1}. Hence $d = 1$, if the set $\{\mathrm{e}_1(I^k) \mid k \geq 1\}$ is finite. □

Lemma 3.1 and the following estimations finish the proof of Theorem 1.1. Remember that $\overline{A}$ is a finitely generated A-module if and only if the $\mathfrak{m}$-adic completion $\widehat{A}$ of A is a reduced ring, provided A is a Cohen-Macaulay local ring with $\dim A = 1$.

Theorem 3.2. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with $\dim A = 1$ and set $B = A/\mathrm{H}^0_{\mathfrak{m}}(A)$. Then*

$$\begin{aligned}
\sup \Delta(A) &= \ell_B(\overline{B}/B) - \ell_A(\mathrm{H}^0_{\mathfrak{m}}(A)) \quad \textit{and} \\
\inf \Delta(A) &= -\ell_A(\mathrm{H}^0_{\mathfrak{m}}(A)).
\end{aligned}$$

Proof. We set $W = \mathrm{H}^0_{\mathfrak{m}}(A)$. Then $B = A/W$ is a Cohen-Macaulay local ring with $\dim B = 1$. Let I be an $\mathfrak{m}$-primary ideal of A. We consider the exact sequence

$$0 \to W/[I^{n+1} \cap W] \to A/I^{n+1} \to B/I^{n+1}B \to 0$$

of A-modules. Then since $I^{n+1} \cap W = (0)$ for all $n \gg 0$,

$$\begin{aligned}
\ell_A(A/I^{n+1}) &= \ell_A(B/I^{n+1}B) + \ell_A(W) \\
&= \mathrm{e}_0(IB)\binom{n+1}{1} - \mathrm{e}_1(IB) + \ell_A(W).
\end{aligned}$$

Hence

$$\mathrm{e}_0(I) = \mathrm{e}_0(IB) \quad \text{and} \quad \mathrm{e}_1(I) = \mathrm{e}_1(IB) - \ell_A(W) \geq -\ell_A(W),$$

because $\mathrm{e}_1(IB) \geq 0$ by Lemma 2.1 (2). If I is a parameter ideal of A, then IB is a parameter ideal of B and

$$\mathrm{e}_1(I) = \mathrm{e}_1(IB) - \ell_A(W) = -\ell_A(W).$$

Thus from Theorem 1.2 the estimations

$$\begin{aligned}
\sup \Delta(A) &= \sup \Delta(B) - \ell_A(W) \\
&= \ell_B(\overline{B}/B) - \ell_A(W) \quad \text{and} \\
\inf \Delta(A) &= -\ell_A(W)
\end{aligned}$$

follow, since every $\mathfrak{m}B$-primary ideal J of B has the form $J = IB$ for some $\mathfrak{m}$-primary ideal I of A. □

4. Examples

We explore concrete examples. Let $0 < a_1 < a_2 < \ldots < a_\ell$ ($\ell \geq 1$) be integers such that $\gcd(a_1, a_2, \ldots, a_\ell) = 1$. Let $V = k[[t]]$ be the formal power series ring over a field k. We set $A = k[[t^{a_1}, t^{a_2}, \ldots, t^{a_\ell}]]$ and $H = \left\langle \sum_{i=1}^{\ell} c_i a_i \mid c_i \in \mathbb{N} \right\rangle$. Hence A is the semigroup ring of the numerical semigroup H. We have $V = \overline{A}$ and $\ell_A(V/A) = \sharp(\mathbb{N} \setminus H)$. Let $c = \mathrm{c}(H)$ be the conductor of H.

Example 4.1. Let $q = \sharp(\mathbb{N} \setminus H)$. Then $\Delta(A) = \{0, 1, \ldots, q\}$.

Proof. We may assume $q \geq 1$, whence $c \geq 2$. We write $\mathbb{N} \setminus H = \{c_1, c_2, \ldots, c_q\}$ with $1 = c_1 < c_2 < \cdots < c_q = c - 1$ and set $B_i = A[t^{c_i}, t^{c_{i+1}}, \ldots, t^{c_q}]$ for each $1 \leq i \leq q$. Then the descending chain $V = B_1 \supsetneq B_2 \supsetneq \cdots \supsetneq B_q \supsetneq B_{q+1} := A$ of A-algebras gives rise to a composition series of the A-module V/A, since $\ell_A(V/A) = q$. Therefore $\ell_A(B_i/A) = q + 1 - i$ for all $1 \leq i \leq q+1$ and hence, setting $a = t^c$ and $I_i = aB_i$ ($\subsetneq A$), by Corollary 2.2 we have $\mathrm{e}_1(I_i) = q + 1 - i$. Thus $\Delta(A) = \{0, 1, \ldots, q\}$ as asserted. $\square$

Because $q = \mathrm{c}(H)/2$ if H is symmetric (that is $A = k[[t^{a_1}, t^{a_2}, \ldots, t^{a_\ell}]]$ is a Gorenstein ring), we readily have the following.

Corollary 4.2. *Suppose that H is symmetric. Then $\Delta(A) = \{0, 1, \ldots, \mathrm{c}(H)/2\}$.*

Corollary 4.3. *Let $A = k[[t^a, t^{a+1}, \ldots, t^{2a-1}]]$ ($a \geq 2$). Then $\Delta(A) = \{0, 1, \ldots, a-1\}$. For the ideal $I = (t^a, t^{a+1}, \ldots, t^{2a-2})$ of A, one has*

$$\mathrm{e}_1(I) = \begin{cases} \mathrm{r}(A) - 1 & (a = 2), \\ \mathrm{r}(A) & (a \geq 3) \end{cases}$$

where $\mathrm{r}(A) = \ell_A(\mathrm{Ext}_A^1(A/\mathfrak{m}, A))$ denotes the Cohen-Macaulay type of A.

Proof. See Example 4.1 for the first assertion. Let us check the second one. If $a = 2$, then A is a Gorenstein ring and I is a parameter ideal of A, so that $\mathrm{e}_1(I) = \mathrm{r}(A) - 1$ ($=0$). Let $a \geq 3$ and put $Q = (t^a)$. Then Q is a reduction of I, since $IV = QV$. Because $A\left[\frac{I}{t^a}\right] = k[[t]]$ and $\mathfrak{m} = t^a V$, we get $A :_{\mathrm{Q}(A)} \mathfrak{m} = k[[t]]$. Thus $\mathrm{e}_1(I) = \ell_A(k[[t]]/A) = \ell_A([A :_{\mathrm{Q}(A)} \mathfrak{m}]/A) = \mathrm{r}(A)$ ([6, Bemerkung 1.21]). $\square$

Remark 4.4. In Example 4.3 I is a canonical ideal of A ([6]). Therefore the equality $\mathrm{e}_1(I) = \mathrm{r}(A)$ shows that if $a \geq 3$, A is not a Gorenstein ring but an almost Gorenstein ring in the sense of [3, Corollary 3.12].

Let us consider local rings which are not analytically irreducible.

Example 4.5. Let $(R, \mathfrak{n})$ be a regular local ring with $n = \dim R \geq 2$. Let $X_1, X_2, \ldots, X_n$ be a regular system of parameters of S and set $P_i = (X_j \mid 1 \leq j \leq n,\ j \neq i)$ for each $1 \leq i \leq n$. We consider the ring $A = R/\bigcap_{i=1}^{n} P_i$. Then A is a one-dimensional Cohen-Macaulay local ring with $\Delta(A) = \{0, 1, \ldots, n-1\}$.

Proof. Let x_i denote the image of X_i in A. We put $\mathfrak{p}_i = (x_j \mid 1 \leq j \leq n,\ j \neq i)$ and $B = \prod_{i=1}^{n}(A/\mathfrak{p}_i)$. Then the homomorphism $\varphi : A \to B,\ a \mapsto (\overline{a}, \overline{a}, \ldots, \overline{a})$ is injective and

$B = \overline{A}$. Since $\mathfrak{m}B = \mathfrak{m}$ and $\mu_A(B) = n$, $\ell_A(B/A) = n-1$. Let $\mathbf{e}_j = (0, \ldots, 0, \overset{j}{\check{1}}, 0, \ldots, 0)$ for $1 \le j \le n$ and $\mathbf{e} = \sum_{j=1}^{n} \mathbf{e}_j$. Then $B = A\mathbf{e} + \sum_{j=1}^{n-1} A\mathbf{e}_j$. We set $B_i = A\mathbf{e} + \sum_{j=1}^{i} A\mathbf{e}_j$ for each $1 \le i \le n-1$. Then since $B_i = A[\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_i]$, B_i is a finitely generated A-algebra and $B_i \subsetneq B_{i+1}$. Hence $B = B_{n-1} \supsetneq B_{n-2} \supsetneq \cdots \supsetneq B_1 \supsetneq B_0 := A$ gives rise to a composition series of the A-module B/A. Hence $\Delta(A) = \{0, 1, \ldots, n-1\}$, as $\ell_A(B_i/A) = i$ for all $0 \le i \le n-1$. □

Let A be a one-dimensional Cohen-Macaulay local ring. If A is not a reduced ring, then the set $\Delta(A)$ must be infinite. Let us note one concrete example.

Example 4.6. Let V be a discrete valuation ring and let $A = V \ltimes V$ denote the idealization of V over V itself. Then $\Delta(A) = \mathbb{N}$.

Proof. Let $K = \mathrm{Q}(V)$. Then $\mathrm{Q}(A) = K \ltimes K$ and $\overline{A} = V \ltimes K$. We set $B_n = V \ltimes \left(V \cdot \frac{1}{t^n}\right)$ for $n \ge 0$. Then $A \subseteq B_n \subseteq \overline{A}$ and

$$\begin{aligned} \ell_A(B_n/A) &= \ell_V(B_n/A) \\ &= \ell_V([V \oplus \left(V \cdot \frac{1}{t^n}\right)]/[V \oplus V]) \\ &= \ell_V(V \cdot \frac{1}{t^n}/V) \\ &= \ell_V(V/t^nV) \\ &= n. \end{aligned}$$

Hence $n \in \Delta(A)$ by Corollary 2.2, so that $\Delta(A) = \mathbb{N}$. □

Example 4.7. Let K/k ($K \neq k$) be a finite extension of fields and assume that there is no proper intermediate field between K and k. Let $n = [K : k]$ and choose a k-basis $\{\omega_i\}_{1 \le i \le n}$ of K. Let $K[[t]]$ be the formal power series ring over K and set $A = k[[\omega_1 t, \omega_2 t, \ldots, \omega_n t]] \subseteq K[[t]]$. Then $\Delta(A) = \{0, n-1\}$.

Proof. Let $V = K[[t]]$. Then $V = \displaystyle\sum_{i=1}^{n} A\omega_i$ and $V = \overline{A}$. Since $tV \subseteq A$, we have $\mathfrak{n} = tV = \mathfrak{m}$, where $\mathfrak{m}$ and $\mathfrak{n}$ stand for the maximal ideals of A and V, respectively. Therefore $\ell_A(V/A) = n-1$. Let $A \subseteq B \subseteq V$ be an intermediate ring. Then B is a local ring. Let $\mathfrak{m}_B$ denote the maximal ideal of B. We then have $\mathfrak{m} = \mathfrak{m}_B = \mathfrak{n}$ since $\mathfrak{m} = \mathfrak{n}$ and therefore, considering the extension of residue class fields $k = A/\mathfrak{n} \subseteq B/\mathfrak{n} \subseteq K = V/\mathfrak{n}$, we get $V = B$ or $B = A$. Since $V = \overline{A}$ is a discrete valuation ring, every $\mathfrak{m}$-primary ideal of A contains a reduction generated by a single element. Hence $\Delta(A) = \{0, n-1\}$ by Corollary 2.4. □

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