# HOW MANY IDEALS WHOSE QUOTIENT RINGS ARE GORENSTEIN EXIST?

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ABSTRACT. This paper investigates a naive question of how many non-principal ideals whose residue class rings are Gorenstein exist in a given Gorenstein ring. The main result provides that the number of such graded ideals in a symmetric numerical semigroup ring $R$ over a field coincides with the conductor of the semigroup. We furthermore provide a complete list of non-principal graded ideals $I$ in $R$ whose quotient rings $R/I$ are Gorenstein.

## 1. INTRODUCTION

Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim A \geq 0$. An $\mathfrak{m}$-primary ideal $I$ is called *Ulrich*, if the associated graded ring $\mathrm{gr}_I(A) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ is a Cohen-Macaulay ring with $\mathrm{a}(\mathrm{gr}_I(A)) = 1 - d$ and $I/I^2$ is free as an $A/I$-module, where $\mathrm{a}(\mathrm{gr}_I(A))$ stands for the a-invariant of $\mathrm{gr}_I(A)$. When $I$ contains a parameter ideal $Q$ of $A$ as a reduction, i.e., $I^{r+1} = QI^r$ for some $r \geq 0$, the ideal $I$ is Ulrich if and only if $I \neq Q$, $I^2 = QI$, and $I/Q$ is a free $A/I$-module ([4, Definition 1.1, Lemma 2.3]). The notion of Ulrich ideal is one of the modifications of that of *stable* maximal ideal introduced in 1971 by his monumental paper [11] of J. Lipman. The present modification was formulated by S. Goto, K. Ozeki, R. Takahashi, K.-i. Watanabe, and K.-i. Yoshida [4] in 2014, where the authors developed and consolidated the basic theory of Ulrich ideals. As an example, since $I/Q$ is free as an $A/I$-module, we have the inequality $(\mu_A(I) - d) \cdot \mathrm{r}(A/I) \leq \mathrm{r}(A)$ ([4, Corollary 2.6 (b)]), where $\mu_A(-)$ and $\mathrm{r}(-)$ denote the number of generators and the Cohen-Macaulay type, respectively. Subsequently, the authors of [5] studied the structure of the complex $\mathbf{R}\mathrm{Hom}_A(A/I, A)$ in the derived category of $A$, and proved that the equality $(\mu_A(I) - d) \cdot \mathrm{r}(A/I) = \mathrm{r}(A)$ holds ([5, Corollary 2.6]). Hence, $A$ is Gorenstein if and only if $A/I$ is Gorenstein and $\mu_A(I) = d + 1$, if an Ulrich ideal $I$ exists.

Motivated by this observation, in this paper we investigate the following question.

**Question 1.1.** Let $A$ be a Gorenstien ring with $d = \dim A > 0$. How many ideals $I$ of $A$ with $\mathrm{ht}_A I = 1$ which satisfy the ring $A/I$ is Gorenstein and $\mu_A(I) \geq 2$ exist?

Every Ulrich ideal in a one-dimensional Gorenstein local ring satisfies the conditions stated as in Queistion 1.1. Whereas, based on the past experience on analysis for Ulrich ideals ([1, 2, 3, 5]), it is rather difficult to make a list of all the Ulrich ideals even for one-dimensional Cohen-Macaulay local rings, especially for numerical semigroup rings; see e.g., [1, Theorem 3.9, Theorem 4.1]. In light of the result that there are only finitely many Ulrich ideals generated by monomials in numerical semigroup rings ([4, Theorem 6.1]), we start our investigation on

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Question 1.1 by going over graded ideals. Still, it remains unclear the question even if we restrict to graded ideals in numerical semigroup rings, which we will clarify in this paper.

Let $\mathbb{N}$ be the set of non-negative integers. A *numerical semigroup* is a non-empty subset $H$ of $\mathbb{N}$ which is closed under addition, contains the zero element, and whose complement in $\mathbb{N}$ is finite. Every numerical semigroup $H$ admits a finite minimal system of generators, i.e., there exist positive integers $a_1, a_2, \dots, a_\ell \in H$ ($\ell \geq 1$) such that

$$H = \langle a_1, a_2, \dots, a_\ell \rangle = \left\{ \sum_{i=1}^{\ell} c_i a_i \;\middle|\; c_i \in \mathbb{N} \text{ for all } 1 \leq i \leq \ell \right\}.$$

For a field $k$, the ring $k[H] = k[t^{a_1}, t^{a_2}, \dots, t^{a_\ell}]$ is called the *numerical semigroup ring* of $H$ over $k$, where $t$ denotes an indeterminate over $k$. Then $R = k[H]$ forms a one-dimensional Noetherian graded integral domain; moreover the ring $R$ enjoys a beautiful relation with its corresponding semigroup $H$. A typical example is that the maximum integer $\mathrm{f}(H)$ in the set $\mathbb{N} \setminus H$ coincides with the a-invariant $\mathrm{a}(R)$ of the ring $R = k[H]$ ([6, Example (2.1.9)]). Besides, the semigroup $H$ is *symmetric*, i.e., the equality $\#\{n \in H \mid n < \mathrm{c}(H)\} = \#(\mathbb{N} \setminus H)$ holds, if and only if its semigroup ring $R = k[H]$ is Gorenstein, where $\#(-)$ denotes the cardinality of a set and $\mathrm{c}(H) = \mathrm{f}(H) + 1$ is the conductor of $H$. See [8, Proposition 2.21] or [10, Theorem] for the proof of this fact.

With this notation, the main result of this paper is stated as follows.

**Theorem 1.2.** *Suppose that $R = k[H]$ is a Gorenstein ring. Then the equality*

$$\#\{I \mid I \text{ is a graded ideal of } R \text{ such that } R/I \text{ is Gorenstein and } \mu_R(I) \geq 2\} = \mathrm{c}(H)$$

*holds.*

Let us now explain how this paper is organized. To show Theorem 1.2, we need several auxiliaries which we will prepare in Section 2. We actually provide them in a more general setting, not only for numerical semigroup rings. We shall prove Theorem 1.2 in Section 3 starting with the case where $\mathrm{a}(R/I) < \mathrm{a}(R)$. In Section 4 we finally provide a complete list of non-principal graded ideals $I$ in $R$ whose quotient rings $R/I$ are Gorenstein. As an application of Theorem 1.2, we consider such ideals in the associated graded ring with respect to a certain filtration of ideals. Examples are explored as well.

## 2. Preliminaries

Let $R = \bigoplus_{n \geq 0} R_n$ be a one-dimensional Noetherian graded integral domain. Throughout this section, we assume $k = R_0$ is a field, and $R_n \neq (0)$ and $R_{n+1} \neq (0)$ for some $n \geq 0$. Let $W$ be the set of non-zero homogeneous elements in $R$. Note that the localization $W^{-1}R = K[t, t^{-1}]$ of $R$ with respect to $W$ is a *simple* graded ring, i.e., every non-zero homogeneous element is invertible, where $t$ is a homogeneous element of degree 1 which is transcendental over $k$, and $K = [W^{-1}R]_0$ is a field. There is an exact sequence

$$0 \to R \to K[t, t^{-1}] \to \mathrm{H}^1_{\mathfrak{m}}(R) \to 0$$

of graded $R$-modules, where $\mathfrak{m}$ denotes the graded maximal ideal of $R$ and $\mathrm{H}^1_{\mathfrak{m}}(R)$ is the 1st graded local cohomology module of $R$ with respect to $\mathfrak{m}$. As $R_0 = k$ and $[\mathrm{H}^1_{\mathfrak{m}}(R)]_0$ is a finite-dimensional $k$-vector space (remember that $\mathrm{H}^1_{\mathfrak{m}}(R)$ is an Artinian $R$-module), the field extension

$K/k$ is finite. Hence $k = K$, if $k$ is an algebraically closed field. Let $\overline{R}$ be the integral closure of $R$ in its quotient field $\mathrm{Q}(R)$.

We begin with the following which was pointed out by S. Goto.

**Lemma 2.1.** *The equality* $\overline{R} = K[t]$ *holds in* $\mathrm{Q}(R)$.

*Proof.* Note that $\overline{R}$ is a graded ring and $\overline{R} \subseteq W^{-1}R = K[t,t^{-1}]$; see e.g., [14, page 157]. As the field $k$ is Nagata, so is the finitely generated $k$-algebra $R$. Thus $\overline{R}$ is a finite $R$-module. As $R_n = (0)$ for all $n < 0$ and $R_0 = k$, we see that $[\overline{R}]_n = (0)$ for all $n < 0$, $L = [\overline{R}]_0$ is a field, and $k \subseteq L \subseteq K$. Set $N = \bigoplus_{n>0}[\overline{R}]_n$. Since the local ring $\overline{R}_N$ of $\overline{R}$ at the maximal ideal $N$ is a DVR, the ideal $N$ is principal. We choose a homogeneous element $f \in \overline{R}$ of degree $q > 0$ such that $N = f\overline{R}$. Hence $\overline{R} = L[N] = L[f] \subseteq W^{-1}R = K[t,t^{-1}]$. Besides, because $\overline{R}[f^{-1}] = L[f, f^{-1}]$ is a simple graded ring and $R \subseteq \overline{R}[f^{-1}]$, we have $W^{-1}R \subseteq \overline{R}[f^{-1}] = L[f, f^{-1}]$. Therefore

$$K[t,t^{-1}] = L[f, f^{-1}]$$

so that $K = L$ and $q = 1$. This shows $\overline{R} = L[f] = K[f] = K[t]$, as claimed. $\square$

For $R$-submodules $X$ and $Y$ of $\mathrm{Q}(R)$, let $X : Y = \{a \in \mathrm{Q}(R) \mid aY \subseteq X\}$. If we consider ideals $I, J$ of $R$, we set $I :_R J = \{a \in R \mid aJ \subseteq I\}$. Hence $I :_R J = (I : J) \cap R$.

*Remark* 2.2. Let $I$ be a non-zero graded ideal of $R$. It is straightforward to check that $R : I$ is a graded $R$-submodule of $K[t,t^{-1}]$ which contains $R$. In addition, the natural isomorphism $R : I \xrightarrow{\cong} \mathrm{Hom}_R(I, R)$, $\alpha \mapsto (x \mapsto \alpha x)$ is graded. Thus, provided $k = K$, every homogeneous component of $\mathrm{Hom}_R(I, R)$ has dimension, as a $k$-vector space, at most 1.

For a Cohen-Macaulay graded ring $A = \bigoplus_{n \geq 0} A_n$ such that $A_0$ is a local ring, we set $\mathrm{a}(A) = \max\{n \in \mathbb{Z} \mid [\mathrm{H}^d_{\mathfrak{M}}(A)]_n \neq (0)\}$ which is called the a-*invariant* of $A$ ([6, Definition (3.1.4)]). Here, $\mathfrak{M}$ denotes the unique graded maximal ideal of $A$, $d = \dim A$, and $\{[\mathrm{H}^d_{\mathfrak{M}}(A)]_n\}_{n \in \mathbb{Z}}$ is the homogeneous components of the $d$-th graded local cohomology module $\mathrm{H}^d_{\mathfrak{M}}(A)$ of $A$ with respect to $\mathfrak{M}$. When $A$ admits the graded canonical module $\mathrm{K}_A$, one has $\mathrm{a}(A) = -\min\{n \in \mathbb{Z} \mid [\mathrm{K}_A]_n \neq (0)\}$.

Let $M$ be a graded $R$-module and $\ell$ an integer. Let $M(\ell)$ denote the graded $R$-module whose underlying $R$-module is the same as that of the $R$-module $M$ and the grading is given by $[M(\ell)]_n = M_{\ell+n}$ for all $n \in \mathbb{Z}$. When $M$ is finitely generated, we denote by $\mu_R(M)$ the minimal number of generators of $M$.

With this notation, we furthermore assume $R$ admits a graded canonical module $\mathrm{K}_R$. Let $(-)^\vee = \mathrm{Hom}_R(-, \mathrm{K}_R)$ denote the canonical dual functor. We then have the following.

**Lemma 2.3.** *Suppose that $R$ is a Gorenstein ring. Let $I$ be a graded ideal of $R$ such that $R/I$ is Gorenstein and $\mu_R(I) \geq 2$. Then the following assertions hold true.*

(1) $[I^\vee]_{-\mathrm{a}(R)} \neq (0)$ *and* $[I^\vee]_{-\mathrm{a}(R/I)} \neq (0)$.
(2) $\min\{n \in \mathbb{Z} \mid [I^\vee]_n \neq (0)\} = \min\{-\mathrm{a}(R), -\mathrm{a}(R/I)\}$.
(3) $\mu_R(I^\vee) = 2$.
(4) *If* $k = K$, *then* $\mathrm{a}(R) \neq \mathrm{a}(R/I)$.
(5) $I^\vee = Rf + Rg$ *for some* $f \in [I^\vee]_{-\mathrm{a}(R)}$ *and* $g \in [I^\vee]_{-\mathrm{a}(R/I)}$.

*Proof.* We set $a = \mathrm{a}(R)$, $b = \mathrm{a}(R/I)$, and $n = \min\{n \in \mathbb{Z} \mid [I^\vee]_n \neq (0)\}$. By taking the functor $(-)^\vee$ to the exact sequence $0 \to I \to R \to R/I \to 0$, we get the sequence

$$(*) \qquad\qquad 0 \to R(a) \to I^\vee \to (R/I)(b) \to 0$$

of graded $R$-modules, because $\mathrm{K}_R \cong R(a)$ and $\mathrm{Ext}^1_R(R/I, \mathrm{K}_R) \cong \mathrm{K}_{(R/I)} \cong (R/I)(b)$. This shows $[I^\vee]_{-a} \neq (0)$, $[I^\vee]_{-b} \neq (0)$, and $n = \min\{-a, -b\}$. Besides, the exact sequence $(*)$ implies $\mu_R(I^\vee) \leq 2$. As $I^{\vee\vee} \cong I$ and $\mu_R(I) \geq 2$, we get $\mu_R(I^\vee) = 2$. This proves the assertions (1), (2), and (3).

If $k = K$, then all the homogeneous components of $I^\vee \cong \mathrm{Hom}_R(I, R)(a)$, as a $k$-vector space, have dimension at most 1. Thus $a \neq b$, and the assertion (4) holds.

Recall that $\mathfrak{m}$ is the graded maximal ideal of $R$. By applying the functor $R/\mathfrak{m} \otimes_R -$ to the sequence $(*)$, we have the exact sequence of the form:

$$(R/\mathfrak{m})(a) \xrightarrow{\xi} I^\vee/\mathfrak{m}I^\vee \xrightarrow{\eta} (R/\mathfrak{m})(b) \to 0.$$

As $\mu_R(I^\vee) = 2$, the map $\xi$ is injective. We choose $f \in [I^\vee]_{-a}$ and $g \in [I^\vee]_{-b}$ such that $\overline{f} = \xi(1)$ and $\eta(\overline{g}) = 1$, where $\overline{*}$ denotes the image in $I^\vee/\mathfrak{m}I^\vee$. Then the images of $f, g$ form a $k$-basis of $I^\vee/\mathfrak{m}I^\vee$. Hence $I^\vee = Rf + Rg$ by Nakayama's lemma. $\square$

*Remark* 2.4. If $R$ is a numerical semigroup ring over a field $k$, then $k = K$. Whereas, if $k = K$, e.g., $k$ is an algebraically closed field, then the ring $R$ is isomorphic to a semigroup ring of a numerical semigroup ([6, Proposition (2.2.11)]).

## 3. Proof of Theorem 1.2

We first fix the notation on which all the results in this section are based.

**Setup 3.1.** Let $\mathbb{N}$ be the set of non-negative integers and $a_1, a_2, \ldots, a_\ell \in \mathbb{Z}$ $(\ell \geq 1)$ be positive integers such that $\gcd(a_1, a_2, \ldots, a_\ell) = 1$. We set

$$H = \langle a_1, a_2, \ldots, a_\ell \rangle = \left\{ \sum_{i=1}^{\ell} c_i a_i \;\middle|\; c_i \in \mathbb{N} \text{ for all } 1 \leq i \leq \ell \right\}$$

and call it the *numerical semigroup* generated by $\{a_i\}_{1 \leq i \leq \ell}$. The reader may consult the book [12] for the fundamental results on numerical semigroups. Let $S = k[t]$ denote the polynomial ring over a field $k$, and define

$$k[H] = k[t^{a_1}, t^{a_2}, \ldots, t^{a_\ell}] \subseteq S$$

which we call the *semigroup ring* of $H$ over $k$. The ring $R = k[H]$ forms a Noetherian integral domain with $\dim R = 1$ and is a $\mathbb{Z}$-graded subring of $S$ whose grading $\{R_n\}_{n \in \mathbb{Z}}$ is given by

$$R_n = \begin{cases} kt^n & \text{if } n \in H, \\ (0) & \text{otherwise.} \end{cases}$$

In addition, $S$ is a birational module-finite extension of $R$, so that $\overline{R} = S$, where $\overline{R}$ denotes the integral closure of $R$ in its quotient field $\mathrm{Q}(R)$. Let

$$\mathrm{c}(H) = \min\{n \in \mathbb{Z} \mid m \in H \text{ for all } m \in \mathbb{Z} \text{ such that } m \geq n\},$$

and set $\mathrm{f}(H) = \max\ (\mathbb{Z} \setminus H)$ which is called the *Frobenius number* of $H$. By [6], we get

$$R : S = t^{\mathrm{c}(H)}S \text{ and } \mathrm{f}(H) = \mathrm{c}(H) - 1 = \mathrm{a}(R).$$

Note that, for each non-zero ideal $I$ in $R$, we have $\mathrm{a}(R/I) \in H$, whence $\mathrm{a}(R/I) \neq \mathrm{a}(R)$; see also Lemma 2.3 (4). We set $a = \mathrm{a}(R)$ and $c = \mathrm{c}(H)$.

The following plays a key in our argument.

**Proposition 3.2.** *Suppose that $R = k[H]$ is a Gorenstein ring. Then the equality*

$$\#\{I \mid I \textit{ is a graded ideal of } R \textit{ such that } \mathrm{a}(R/I) < a \textit{ and } \mu_R(I) \geq 2\} = \frac{c}{2}$$

*holds.*

*Proof.* Let $\mathscr{Y}_R$ be the set of graded ideals $I$ of $R$ such that $R/I$ is Gorenstein, $\mathrm{a}(R/I) < a$, and $\mu_R(I) \geq 2$. To show the required equality, we may assume $R \neq \overline{R}$. Thus $\mathscr{Y}_R \neq \emptyset$. For each $I \in \mathscr{Y}_R$, since $\mathrm{a}(R/I) < a$, we then have $R_m \subseteq I$ for all $m \geq c = a+1$. By setting $J = R : I$, we see that $J$ is a graded ideal of $R$ and

$$R \subseteq J \subseteq R : \mathfrak{c} = R : (R : \overline{R}) = \overline{R}$$

where the second inclusion follows from $\mathfrak{c} \subseteq I$ and the last equality holds by [9, Bemerkung 2.5] (remember that $R$ is a Gorenstein ring). This shows $J = (1, t^m)$ for some $m \in \mathbb{N} \setminus H$. So we can consider the map

$$\Phi : \mathscr{Y}_R \to \mathbb{N} \setminus H$$

defined by $\Phi(I) = m$ for each $I \in \mathscr{Y}_R$, where $R : I = (1, t^m)$.

Conversely, for each $m \in \mathbb{N} \setminus H$, we set $J = (1, t^m)$. Then $R \subsetneq J \subseteq \overline{R} = k[t]$. By setting $I = R : J$, we have

$$\mathfrak{c} = R : \overline{R} \subseteq R : J = I \subsetneq R$$

which yield that $\mathrm{a}(R/I) < a$, $\mu_R(I) \geq 2$, and the ring $R/I$ is Gorenstein. Indeed, since $t^c\overline{R} = \mathfrak{c} \subseteq I$ and $\mathrm{a}(R/I) \in H$, we have $\mathrm{a}(R/I) < a$. The $\mathrm{K}_R$-dual $(-)^\vee$ of the exact sequence $0 \to I \to R \to R/I \to 0$ induces the sequence

$$0 \to R(a) \xrightarrow{\varphi} I^\vee \to \mathrm{Ext}^1_R(R/I, \mathrm{K}_R) \to 0$$

of graded $R$-modules, because $\mathrm{K}_R \cong R(a)$. Let $f = \varphi(1)$. Then $f \in [I^\vee]_{-a}$ forms a part of a minimal basis of $I^\vee$. As $I^\vee \cong J$ and $\mu_R(J) = 2$, we get $\mu_R(I) \geq 2$; while the $R$-module $\mathrm{Ext}^1_R(R/I, \mathrm{K}_R)$ is cyclic, so that $R/I$ is a Gorenstein ring, because $\mathrm{Ext}^1_R(R/I, \mathrm{K}_R) \cong \mathrm{K}_{(R/I)}$ is the canonical module of $R/I$. Here, the proof of above especially shows that if $\mathrm{a}(R/I) < a$ then $R/I$ is Gorenstein. Hence, we define the map

$$\Psi : \mathbb{N} \setminus H \to \mathscr{Y}_R, \ \ m \mapsto R : (1, t^m)$$

and it is straightforward to check the composite maps $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are identity. In particular, the map $\Phi$ is bijective. Therefore

$$\#\{I \mid I \text{ is a graded ideal of } R, \mathrm{a}(R/I) < a, \text{ and } \mu_R(I) \geq 2\} = \#\mathscr{Y}_R = \#(\mathbb{N} \setminus H) = \frac{c}{2}$$

where the last equality follows from the fact that $H$ is symmetric, i.e., $R$ is Gorenstein. This completes the proof. $\square$

We are ready to prove Theorem 1.2.

*Proof of Theorem 1.2.* Let $\mathscr{X}_R$ be the set of graded ideals $I$ of $R$ such that $R/I$ is Gorenstein and $\mu_R(I) \geq 2$. Similarly as in the proof of Proposition 3.2, we may assume $R \neq \overline{R}$. So $\mathscr{X}_R \neq \emptyset$. By Proposition 3.2, it suffices to show that the number of ideals $I \in \mathscr{X}_R$ with $\mathrm{a}(R/I) > a$ is a half of the conductor $c$ of $H$.

For each $I \in \mathscr{X}_R$, we have $\mu_R(I^\vee) = 2$, so we can write

$$I^\vee = Rf + Rg \quad \text{for some } f \in [I^\vee]_{-a} \text{ and } g \in [I^\vee]_{-\mathrm{a}(R/I)}.$$

Set $b = \mathrm{a}(R/I)$. Since $(0) :_R g = (0)$, we get the exact sequence

$$0 \to R(b) \xrightarrow{\xi} I^\vee \to C \to 0$$

of graded $R$-modules, where $\xi(1) = g$ and $C = \mathrm{Coker}\,\xi$. We consider the graded ideal $J = (0) :_R C$ of $R$. As $C$ has a finite length and $C \cong I^\vee/Rg \neq (0)$, we get $(0) \neq J \subsetneq R$. Besides, we have the isomorphism

$$C \cong R\overline{f} \cong (R/J)(a)$$

as a graded $R$-module, where $\overline{*}$ denotes the image in $I^\vee/Rg$. Hence we obtain the sequence

$$0 \to R(b) \to I^\vee \to (R/J)(a) \to 0$$

of graded $R$-modules. By applying the functor $(-)^\vee$ to the above sequence, we have

$$0 \to I \to R(a-b) \to \mathrm{Ext}^1_R(R/J, \mathrm{K}_R)(-a) \to 0$$

because $I^{\vee\vee} \cong I$ and $R(b)^\vee = \mathrm{Hom}_R(R(b), \mathrm{K}_R) \cong \mathrm{Hom}_R(R(b), R(a)) \cong R(a-b)$. In particular, $\mathrm{K}_{(R/J)} \cong \mathrm{Ext}^1_R(R/J, \mathrm{K}_R)$ is cyclic; hence $R/J$ is Gorenstein. By letting $\alpha = \mathrm{a}(R/J)$, we have $\mathrm{K}_{(R/J)} \cong (R/J)(\alpha)$. Therefore, by changing the shift by $b-a$, we get the exact sequence

$$0 \to I(b-a) \xrightarrow{\psi} R \to (R/J)(\alpha - a + b - a) \to 0$$

of graded $R$-modules. The degree 0 part of the following isomorphism

$$R/\operatorname{Im}\psi \cong (R/J)(\alpha - 2a + b)$$

induces $\alpha - 2a + b = 0$; while $I(b-a) \cong \operatorname{Im}\psi \cong J$. Hence, $\mathrm{a}(R/J) = \alpha = 2a - b$ and $I \cong J(a-b)$ as a graded $R$-module. In particular, $\mu_R(J) \geq 2$. Thus $J \in \mathscr{X}_R$.

Let $W$ be the set of non-zero homogeneous elements in $R$. Consider the simple graded ring $W^{-1}R = k[t, t^{-1}]$, where $t$ is a homogeneous element of degree 1 which is transcendental over $k$. Hence we have the commutative diagram below:

$$\begin{array}{ccc}
k[t,t^{-1}](a-b) = W^{-1}(J(a-b)) & \xrightarrow{\cong} & W^{-1}I = k[t,t^{-1}] \\
\uparrow & & \uparrow \\
J(a-b) & \xrightarrow{\cong} & I = t^{b-a}J
\end{array}$$

Note that the induced isomorphism $k[t,t^{-1}](a-b) \xrightarrow{\cong} k[t,t^{-1}]$ is given by the homothety of homogeneous element of degree $b-a$. Therefore $I = t^{b-a}J$.

To sum up this argument, for each $I \in \mathscr{X}_R$, there exists a graded ideal $J \in \mathscr{X}_R$ satisfying

$$\mathrm{a}(R/J) = 2a - \mathrm{a}(R/I) \quad \text{and} \quad I = t^{\mathrm{a}(R/I)-a}J.$$

This shows, if $\mathrm{a}(R/I) > a$ (resp. $\mathrm{a}(R/I) < a$), then $\mathrm{a}(R/J) < a$ (resp. $\mathrm{a}(R/J) > a$). So, there is a one-to-one correspondence between the set of ideals $I \in \mathscr{X}_R$ with $\mathrm{a}(R/I) > a$, and the set of ideals $J \in \mathscr{X}_R$ with $\mathrm{a}(R/J) < a$. Finally we conclude that

$$\#\mathscr{X}_R = \#\{I \in \mathscr{X}_R \mid \mathrm{a}(R/I) > a\} + \#\{I \in \mathscr{X}_R \mid \mathrm{a}(R/I) < a\} = \frac{c}{2} + \frac{c}{2} = c$$

as desired. $\square$

## 4. Corollaries and examples

We summarize some consequences of Theorem 1.2. In this section we maintain the notation as in Setup 3.1. Let $\mathscr{X}_R$ be the set of graded ideals $I$ of $R$ such that $R/I$ is Gorenstein and $\mu_R(I) \geq 2$. Recall that $a = \mathrm{a}(R)$ and $c = \mathrm{c}(H)$.

The direct consequence of the proof of Theorem 1.2 is stated as follows, which is useful to compute concrete examples.

**Corollary 4.1.** *Suppose that $R = k[H]$ is a Gorenstein ring. For each $I \in \mathscr{X}_R$, we set $J = t^{a-\mathrm{a}(R/I)}I$. Then the following assertions hold true.*

(1) *$J \in \mathscr{X}_R$ and $\mathrm{a}(R/J) = 2a - \mathrm{a}(R/I)$. Hence, if $\mathrm{a}(R/I) < a$ (resp. $\mathrm{a}(R/I) > a$), then $\mathrm{a}(R/J) > a$ (resp. $\mathrm{a}(R/J) < a$).*
(2) *$\mathrm{a}(R/I) \in H$, $a \neq \mathrm{a}(R/I)$, and $a - \mathrm{a}(R/I) \in \mathbb{Z} \setminus H$.*
(3) *If $\mathrm{a}(R/I) < a$, then $a - \mathrm{a}(R/I) \in \mathbb{N} \setminus H$.*
(4) *If $\mathrm{a}(R/I) > a$, then $\mathrm{a}(R/I) - a \in \mathbb{N} \setminus H$.*

*Proof.* We already proved the assertion (1) in the proof of Theorem 1.2. Recall that $\mathrm{a}(R/I) \in H$ and $a \notin H$. So $a \neq \mathrm{a}(R/I)$. As $H$ is symmetric and $\mathrm{a}(R/I) \in H$, we see that $a - \mathrm{a}(R/I) \in \mathbb{Z} \setminus H$. In particular, if $\mathrm{a}(R/I) < a$, then $a - \mathrm{a}(R/I) \in \mathbb{N} \setminus H$. On the other hand, we assume $\mathrm{a}(R/I) > a$. Since $J \in \mathscr{X}_R$ and $\mathrm{a}(R/J) < a$, we conclude that $\mathrm{a}(R/I) - a = a - \mathrm{a}(R/J) \in \mathbb{N} \setminus H$, as claimed. $\square$

The next provides a complete list of graded ideals in $\mathscr{X}_R$.

**Corollary 4.2.** *Suppose that $R = k[H]$ is a Gorenstein ring. Then the equality*

$$\mathscr{X}_R = \{R :_R t^m, t^m(R :_R t^m) \mid m \in \mathbb{N} \setminus H\}$$

*holds. Moreover, for each $m \in \mathbb{N} \setminus H$, one has*

$$\mathrm{a}(R/R :_R t^m) = a - m \ \ \textit{and} \ \ \mathrm{a}(R/t^m(R :_R t^m)) = a + m.$$

*Proof.* Note that $R : (1, t^m) = R :_R t^m$ for all $m \in \mathbb{N} \setminus H$. By Proposition 3.2, there is a one-to-one correspondence below:

$$\mathbb{N} \setminus H \longleftrightarrow \{I \in \mathscr{X}_R \mid \mathrm{a}(R/I) < a\}, \ m \longmapsto R : (1, t^m).$$

This shows the equality $\{I \in \mathscr{X}_R \mid \mathrm{a}(R/I) < a\} = \{R :_R t^m \mid m \in \mathbb{N} \setminus H\}$. Besides, the proof of Theorem 1.2 guarantees that the map

$$\{I \in \mathscr{X}_R \mid \mathrm{a}(R/I) < a\} \longleftrightarrow \{I \in \mathscr{X}_R \mid \mathrm{a}(R/I) > a\}, \ I \longmapsto t^{a-\mathrm{a}(R/I)}I$$

is bijective. Hence $\{I \in \mathscr{X}_R \mid \mathrm{a}(R/I) > a\} = \{t^{a-\mathrm{a}(R/R:_R t^m)}(R :_R t^m) \mid m \in \mathbb{N} \setminus H\}$ holds. Since $H$ is symmetric and $c = a+1$, it is straightforward to check that $\mathrm{a}(R/R :_R t^m) = a - m$ for all $m \in \mathbb{N} \setminus H$. Therefore the equality

$$\mathscr{X}_R = \{R :_R t^m, t^m(R :_R t^m) \mid m \in \mathbb{N} \setminus H\}$$

holds. Furthermore, by Corollary 4.1 (1), we have the equalities

$$\mathrm{a}(R/t^m(R :_R t^m)) = 2a - \mathrm{a}(R/R :_R t^m) = 2a - (a-m) = a + m$$

which complete the proof. $\square$

The ideals of the forms $R :_R t^m$ and $t^m(R :_R t^m)$ are easy to compute, especially in numerical semigroup rings, and provide numerous examples illustrating Theorem 1.2.

**Example 4.3.** Let $k[t]$ be the polynomial ring over a field $k$ and $R = k[H]$ the semigroup ring of a numerical semigroup $H$. Then the following assertions hold.

(1) Let $H = \langle 2, 2\ell+1 \rangle$ $(\ell \geq 1)$. Then $\mathrm{c}(H) = 2\ell$ and the equality

$$\mathscr{X}_R = \{(t^2, t^{2\ell+1}), (t^4, t^{2\ell+1}), \ldots, (t^{2\ell}, t^{2\ell+1}), (t^{2\ell+1}, t^{4\ell}), (t^{2\ell+1}, t^{4\ell-2}), \ldots, (t^{2\ell+1}, t^{2\ell+2})\}$$

holds.

(2) Let $H = \langle 3, 4 \rangle$. Then $\mathrm{c}(H) = 6$ and the equality

$$\mathscr{X}_R = \{(t^3, t^4), (t^4, t^6), (t^3, t^8), (t^8, t^9), (t^6, t^8), (t^4, t^9)\}$$

holds.

(3) Let $H = \langle 3, 5 \rangle$. Then $\mathrm{c}(H) = 8$ and the equality

$$\mathscr{X}_R = \{(t^3, t^5), (t^5, t^6), (t^3, t^{10}), (t^5, t^9), (t^{10}, t^{12}), (t^9, t^{10}), (t^5, t^{12}), (t^6, t^{10})\}$$

holds.

(4) Let $H = \langle n, n+1, \ldots, 2n-2 \rangle$ $(n \geq 4)$. Then $\mathrm{c}(H) = 2n$ and the equality

$$\begin{aligned} \mathscr{X}_R &= \{(t^n, t^{n+1}, \ldots, t^{2n-2}), (t^{n+1}, t^{n+2}, \ldots, t^{2n-2}, t^{2n})\} \\ &\bigcup \{(t^n, t^{n+1}, \ldots, t^{n+i-1}, t^{n+i+1}, \ldots, t^{2n-2}) \mid 1 \leq i \leq n-2\} \\ &\bigcup \{(t^{3n-1}, t^{3n}, \ldots, t^{4n-3}), (t^{2n}, t^{2n+1}, \ldots, t^{3n-3}, t^{3n-1})\} \\ &\bigcup \{(t^{2n-i-1}, t^{2n-i}, \ldots, t^{2n-2}, t^{2n}, \ldots, t^{3n-i-3}) \mid 1 \leq i \leq n-2\} \end{aligned}$$

holds.

Let $H_1 = \langle a_1, a_2, \ldots, a_\ell \rangle$ and $H_2 = \langle b_1, b_2, \ldots, b_m \rangle$ $(\ell, m \geq 1)$ be numerical semigroups. We choose $d_1 \in H_2 \setminus \{b_1, b_2, \ldots, b_m\}$ and $d_2 \in H_1 \setminus \{a_1, a_2, \ldots, a_\ell\}$ such that $\gcd(d_1, d_2) = 1$. We say that

$$H = \langle d_1 H_1, d_2 H_2 \rangle = \langle d_1 a_1, d_1 a_2, \ldots, d_1 a_\ell, d_2 b_1, d_2 b_2, \ldots, d_2 b_m \rangle$$

is a *gluing* of $H_1$ and $H_2$ with respect to $d_1 \in H_2$ and $d_2 \in H_1$.

Note that every three-generated symmetric numerical semigroup $H$ is obtained by gluing of a two-generated numerical semigroup $H_1$ and $\mathbb{N}$ ([7, Section 3], [13, Proposition 3]). Let $a, b \in \mathbb{Z}$ be positive integers with $\gcd(a, b) = 1$. We set $H_1 = \langle a, b \rangle$ and assume that $H_1$ is minimally generated by two-elements. Choose $c \in H_1$ and $d \in \mathbb{N}$ so that $c, d$ satisfy the conditions that $c > 0$, $d > 1$, $c \notin \{a, b\}$, and $\gcd(c, d) = 1$. Hence, $\gcd(da, db, c) = 1$. We consider a gluing

$H = \langle dH_1, c\mathbb{N}\rangle$ of $H_1$ and $\mathbb{N}$ with respect to $d \in \mathbb{N}$ and $c \in H_1$. Let $k$ be a field. We then have the isomorphism

$$k[H] \cong k[X,Y,Z]/(X^b - Y^a, Z^d - X^m Y^n)$$

of $k$-algebras, where $c = am + bn$ with $m, n \in \mathbb{N}$. Hence, $\mathrm{a}(k[H]) = d(ab-a-b)+(d-1)c$.

**Corollary 4.4.** *Let $H$ be a three-generated symmetric numerical semigroup. Under the same notation of above, the equality*

$$\#\mathscr{X}_{k[H]} = d(ab-a-b)+(d-1)c+1$$

*holds.*

**Example 4.5.** Let $k$ be a field and $H = \langle 4,6,7\rangle$. Then $H = \langle 2\langle 2,3\rangle, 7\mathbb{N}\rangle$ and

$$R = k[H] \cong k[X,Y,Z]/(X^3 - Y^2, Z^2 - X^2 Y).$$

In particular, $\mathrm{a}(R) = 9$ and $\#\mathscr{X}_R = \mathrm{c}(H) = 10$. Indeed, we have the equality

$$\begin{aligned} \mathscr{X}_R &= \{(t^4,t^6,t^7),(t^6,t^7,t^8),(t^4,t^7),(t^4,t^6),(t^6,t^7)\} \\ &\bigcup \{(t^{13},t^{15},t^{16}),(t^{11},t^{12},t^{13}),(t^7,t^{10}),(t^6,t^8),(t^7,t^8)\}. \end{aligned}$$

For an $R$-module $M$, we denote by $[M]$ the isomorphism class of $M$.

**Corollary 4.6.** *Suppose that $R = k[H]$ is a Gorenstein ring. Then the equalities*

$$\{[I] \mid I \in \mathscr{X}_R\} = \{[R :_R t^m] \mid m \in \mathbb{N}\setminus H\} \text{ and } \#\{[I] \mid I \in \mathscr{X}_R\} = \frac{c}{2}$$

*hold.*

*Proof.* Note that $R :_R t^m \cong t^m(R :_R t^m)$ as an $R$-module for each $m \in \mathbb{N}\setminus H$. This shows the equality $\{[I] \mid I \in \mathscr{X}_R\} = \{[R :_R t^m] \mid m \in \mathbb{N}\setminus H\}$ and its cardinality is at most a half of $c$. We now assume $R :_R t^m = t^\ell(R :_R t^{m'})$ for some $m, m' \in \mathbb{N}\setminus H$ and $\ell \in \mathbb{Z}$. Then

$$R : (1,t^m) = t^\ell(R : (1,t^{m'})) = R : (t^{-\ell}(1,t^{m'})) = R : (t^{-\ell}, t^{-\ell+m'}).$$

As $R$ is Gorenstein, we have $(1,t^m) = (t^{-\ell}, t^{-\ell+m'})$ in $k[t,t^{-1}]$. Since $-\ell < -\ell+m'$, we have $\ell = 0$ and $m = -\ell+m' = m'$. Hence the cardinality of $\{[I] \mid I \in \mathscr{X}_R\}$ is the half of $c$. $\square$

*Remark* 4.7. There exists a one-dimensional local Gorenstein numerical semigroup ring $A$ with infinite residue class field (e.g., $\mathbb{Q}[[t^3,t^7]]$, $\mathbb{C}[[t^4,t^5,t^6]]$) admitting infinitely many two-generated Ulrich ideals. Hence $\mathscr{X}_A = \infty$.

When $A$ is a local ring, although the set $\mathscr{X}_A$ is not necessarily finite, there is an associated graded ring $G$ with respect to a filtration of ideals such that $\mathscr{X}_G$ is a finite set.

Let $(A,\mathfrak{m})$ be a Noetherian local ring with $\dim A = 1$ and $V = \overline{A}$ the integral closure of $A$ in its total ring $\mathrm{Q}(A)$ of fractions. Assume that $V$ is a DVR which is a module-finite extension of $A$ and $A/\mathfrak{m} \cong V/\mathfrak{n}$, where $\mathfrak{n} = tV$ $(t \in V)$ denotes the maximal ideal of $V$. Let $\mathrm{o}(-)$ denote the $\mathfrak{n}$-adic valuation (or the order function) of $V$ and set

$$v(A) = \{\mathrm{o}(f) \mid 0 \neq f \in A\}.$$

Then, $H_A = v(A)$ is called the *value semigroup* of $A$, which is indeed a numerical semigroup. Let $\mathfrak{c} = A : V$ denote the conductor of $A$. Then $\mathfrak{c} = t^{\mathrm{c}(H_A)}V$ and $\mathrm{c}(H_A) = \ell_A(V/\mathfrak{c})$. Note that $A$ is Gorenstein if and only if $H_A = v(A)$ is symmetric ([10, Theorem]).

For each $\ell \in \mathbb{Z}$, we set $F_\ell = \mathfrak{n}^\ell \cap A$. Then $\mathscr{F} = \{F_\ell\}_{\ell \in \mathbb{Z}}$ is a filtration of ideals in $A$. We define

$$G = G(\mathscr{F}) = \bigoplus_{\ell \geq 0} F_\ell / F_{\ell+1} = \bigoplus_{\ell \geq 0} (\mathfrak{n}^\ell \cap A)/(\mathfrak{n}^{\ell+1} \cap A)$$

and call it the *associated graded ring of $A$ with respect to $\mathscr{F}$*. Note that, for each $\ell \geq 0$, $G_\ell \neq (0)$ if and only if $\ell \in H_A$. This shows $H_A = \{\ell \geq 0 \mid G_\ell \neq (0)\}$ and the isomorphism below:

$$G = \bigoplus_{\ell \geq 0} F_\ell / F_{\ell+1} \cong (A/\mathfrak{m})[H_A].$$

With this notation we have the following.

**Corollary 4.8.** *Let $(A, \mathfrak{m})$ be a one-dimensional Gorenstein complete local domain with algebraically closed residue class field. Let $G = G(\mathscr{F})$ be the associated graded ring of $A$ with respect to the filtration $\mathscr{F} = \{\mathfrak{n}^\ell \cap A\}_{\ell \in \mathbb{Z}}$, where $\mathfrak{n}$ denotes the maximal ideal of $V = \overline{A}$. Then the equality*

$$\#\{I \mid I \text{ is a graded ideal of } G \text{ such that } G/I \text{ is Gorenstein and } \mu_G(I) \geq 2\} = \mathrm{c}(H_A)$$

*holds, where $H_A = v(A)$ denotes the value semigroup of $A$.*

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