THE ALMOST GORENSTEIN REES ALGEBRAS OVER TWO-DIMENSIONAL REGULAR LOCAL RINGS

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ABSTRACT. Let $(R, \mathfrak{m})$ be a two-dimensional regular local ring with infinite residue class field. Then the Rees algebra $\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n$ of I is an almost Gorenstein graded ring in the sense of [6] for every $\mathfrak{m}$-primary integrally closed ideal I in R.

1. Introduction

The purpose of this paper is to study the problem of when the Rees algebras of ideals and modules over two-dimensional regular local rings $(R, \mathfrak{m})$ are almost Gorenstein graded rings. Almost Gorenstein rings in our sense are newcomers and different from those rings studied in [12]. They form a new class of Cohen-Macaulay rings, which are not necessarily Gorenstein, but still good, possibly next to the Gorenstein rings. The notion of these local rings dates back to the paper [2] of V. Barucci and R. Fröberg in 1997, where they dealt with one-dimensional analytically unramified local rings and developed a beautiful theory. Because their notion is not flexible enough to analyze analytically ramified rings, in 2013 S. Goto, N. Matsuoka, and T. T. Phuong [4] extended the notion to arbitrary (but still of dimension one) Cohen-Macaulay local rings. The reader may consult [4] for examples of analytically ramified almost Gorenstein local rings. S. Goto, R. Takahashi, and N. Taniguchi [6] finally gave the definition of almost Gorenstein local/graded rings in our sense. Here let us recall it, which we shall utilize throughout this paper.

Definition 1.1 ([6, Definition 3.3])**.** Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring which possesses the canonical module K_R. Then we say that R is an almost Gorenstein *local* ring, if there exists an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$, where $\mu_R(C)$ denotes the number of elements in a minimal system of generators of C and $\mathrm{e}^0_{\mathfrak{m}}(C)$ is the multiplicity of C with respect to $\mathfrak{m}$.

Definition 1.2 ([6, Definition 8.1])**.** Let $R = \bigoplus_{n \geq 0} R_n$ be a Cohen-Macaulay graded ring such that R_0 is a local ring. Suppose that R possesses the graded canonical module K_R.

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Let $\mathfrak{M}$ be the unique graded maximal ideal of R and $a = \mathrm{a}(R)$ the a-invariant of R. Then we say that R is an almost Gorenstein *graded* ring, if there exists an exact sequence

$$0 \to R \to \mathrm{K}_R(-a) \to C \to 0$$

of graded R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$, where $\mu_R(C)$ denotes the number of elements in a minimal system of generators of C and $\mathrm{e}^0_{\mathfrak{M}}(C)$ is the multiplicity of C with respect to $\mathfrak{M}$. Here $\mathrm{K}_R(-a)$ stands for the graded R-module whose underlying R-module is the same as that of K_R and whose grading is given by $[\mathrm{K}_R(-a)]_n = [\mathrm{K}_R]_{n-a}$ for all $n \in \mathbb{Z}$.

Definition 1.1 (resp. Definition 1.2) means that if R is an almost Gorenstein local (resp. graded) ring, then even though R is not a Gorenstein ring, R can be embedded into the canonical module K_R (resp. $\mathrm{K}_R(-a)$), so that the difference K_R/R (resp. $\mathrm{K}_R(-a)/R$) is an Ulrich R-module ([3]) and behaves well. The reader may consult [6] about the basic theory of almost Gorenstein local/graded rings and the relation between the graded theory and the local theory, as well.

It is shown in [6] that every two-dimensional rational singularity is an almost Gorenstein local ring and all the known examples of Cohen-Macaulay local rings of finite Cohen-Macaulay representation type are almost Gorenstein local rings. The almost Gorenstein local rings which are not Gorenstein are G-regular ([6, Corollary 4.5]) in the sense of [19], that is every totally reflexive module is free, so that the Gorenstein dimension of a finitely generated module is equal to its projective dimension, while over Gorenstein local rings the totally reflexive modules are exactly the maximal Cohen-Macaulay modules. The local rings $R_{\mathfrak{p}}$ ($\mathfrak{p} \in \operatorname{Spec} R$) of almost Gorenstein local rings R are not necessarily almost Gorenstein (see [6, Remark 9.12] for a counterexample). These are particular discrepancies between Gorenstein local rings and almost Gorenstein local rings.

In this paper we are interested in the almost Gorenstein property of Rees algebras and our main result is sated as follows.

Theorem 1.3. *Let $(R, \mathfrak{m})$ be a two-dimensional regular local ring with infinite residue class field and I an $\mathfrak{m}$-primary integrally closed ideal in R. Then the Rees algebra $\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n$ of I is an almost Gorenstein graded ring.*

As a direct consequence we have the following.

Corollary 1.4. *Let $(R, \mathfrak{m})$ be a two-dimensional regular local ring with infinite residue class field. Then $\mathcal{R}(\mathfrak{m}^\ell)$ is an almost Gorenstein graded ring for every integer $\ell > 0$.*

The proof of Theorem 1.3 depends on a result of J. Verma [21] which guarantees the existence of joint reductions with joint reduction number zero. Therefore our method of proof works also for two-dimensional rational singularities, which we shall discuss in the forthcoming paper [7].

Possessing in [5] one of its roots, the theory of Rees algebras has been satisfactorily developed and nowadays one knows many Cohen-Macaulay Rees algebras (see, e.g. [11, 16, 18]). Among them Gorenstein Rees algebras are rather rare ([13]). Nevertheless, although they are not Gorenstein, some of Cohen-Macaulay Rees algebras are still good and could be *almost Gorenstein graded* rings, which we would like to report in this paper and also in the forthcoming papers [7, 8]. Except [6, Theorems 8.2, 8.3] our Theorem 1.3 is the first attempt to answer the question of when the Rees algebras are almost Gorenstein graded rings.

We now briefly explain how this paper is organized. The proof of Theorem 1.3 shall be given in Section 2. For the Rees algebras of modules over two-dimensional regular local rings we have a similar result, which we give in Section 2 (Corollary 2.7). In Section 3 we explore the case where the ideals are linearly presented over power series rings. The result (Theorem 3.1) seems to suggest that almost Gorenstein Rees algebras are still rather rare, when the dimension of base rings is greater than two, which we shall discuss also in the forthcoming paper [8]. In Section 4 we explore the Rees algebra of the socle ideal $I = Q : \mathfrak{m}$, where Q is a parameter ideal in a two-dimensional regular local ring $(R, \mathfrak{m})$, and show that the Rees algebra $\mathcal{R}(I)$ is an almost Gorenstein graded ring if and only if the order of Q is at most two (Theorem 4.1).

We should note here that for every almost Gorenstein graded ring R with graded maximal ideal $\mathfrak{M}$ the local ring $R_{\mathfrak{M}}$ of R at $\mathfrak{M}$ is by definition an almost Gorenstein local ring, because $[\mathrm{K}_R]_{\mathfrak{M}} \cong \mathrm{K}_{R_{\mathfrak{M}}}$. The converse is not true in general. The typical examples are the Rees algebras $\mathcal{R}(Q)$ of parameter ideals Q in a regular local ring $(R, \mathfrak{m})$ with $\dim R \geq 3$. For this algebra $\mathcal{R} = \mathcal{R}(Q)$ the local ring $\mathcal{R}_{\mathfrak{M}}$ is always an almost Gorenstein *local* ring ([8, Theorem 2.7]) but $\mathcal{R}$ is an almost Gorenstein *graded* ring if and only if $Q = \mathfrak{m}$ ([6, Theorem 8.3]). On the other hand the converse is also true in certain special cases like Theorems 3.1 and 4.1 of the present paper. These facts seem to suggest the property of being an almost Gorenstein graded ring is a rather rigid condition for Rees algebras.

In what follows, unless otherwise specified, let $(R, \mathfrak{m})$ denote a Cohen-Macaulay local ring. For each finitely generated R-module M let $\mu_R(M)$ (resp. $\mathrm{e}^0_{\mathfrak{m}}(M)$) denote the number of elements in a minimal system of generators for M (resp. the multiplicity of M with respect to $\mathfrak{m}$). Let K_R stand for the canonical module of R.

2. Proof of Theorem 1.3

The purpose of this section is to prove Theorem 1.3. Let $(R, \mathfrak{m})$ be a Gorenstein local ring with $\dim R = 2$ and let $I \subsetneq R$ be an $\mathfrak{m}$-primary ideal of R. Assume that I contains a parameter ideal $Q = (a, b)$ of R such that $I^2 = QI$. We set $J = Q : I$. Let

$$\mathcal{R} = R[It] \subseteq R[t] \quad \text{and} \quad T = R[Qt] \subseteq R[t],$$

where t stands for an indeterminate over R. Remember that the Rees algebra $\mathcal{R}$ of I is a Cohen-Macaulay ring ([5]) with $\mathrm{a}(\mathcal{R}) = -1$ and $\mathcal{R} = T + T{\cdot}It$, while the Rees algebra T of Q is a Gorenstein ring of dimension 3 and $\mathrm{a}(T) = -1$ (remember that $T \cong R[x, y]/(bx - ay)$). Hence $\mathrm{K}_T(1) \cong T$ as a graded T-module, where K_T denotes the graded canonical module of T.

Let us begin with the following, which is a special case of [20, Theorem 2.7 (a)]. We note a brief proof.

Proposition 2.1. *$\mathrm{K}_{\mathcal{R}}(1) \cong J\mathcal{R}$ as a graded $\mathcal{R}$-module.*

Proof. Since $\mathcal{R}$ is a module-finite extension of T, we get

$$\mathrm{K}_R(1) \cong \mathrm{Hom}_T(\mathcal{R}, \mathrm{K}_T)(1) \cong \mathrm{Hom}_T(\mathcal{R}, T) \cong T :_F \mathcal{R}$$

as graded $\mathcal{R}$-modules, where $F = \mathrm{Q}(T) = \mathrm{Q}(\mathcal{R})$ is the total ring of fractions. Therefore $T :_F \mathcal{R} = T :_T It$, since $\mathcal{R} = T + T{\cdot}It$. Because $Q^n \cap [Q^{n+1} : I] = Q^n[Q : I]$ for all $n \geq 0$, we have $T :_T It = JT$. Hence $T :_F \mathcal{R} = JT$, so that $JT = J\mathcal{R}$. Thus $\mathrm{K}_{\mathcal{R}}(1) \cong J\mathcal{R}$ as a graded $\mathcal{R}$-module. □

Corollary 2.2. *The ideal $J = Q : I$ in R is integrally closed, if $\mathcal{R}$ is a normal ring.*

Proof. Since $J\mathcal{R} \cong \mathrm{K}_{\mathcal{R}}(1)$, the ideal $J\mathcal{R}$ of $\mathcal{R}$ is unmixed and of height one. Therefore, if $\mathcal{R}$ is a normal ring, $J\mathcal{R}$ must be integrally closed in $\mathcal{R}$, whence J is integrally closed in R because $\overline{J} \subseteq J\mathcal{R}$, where $\overline{J}$ denotes the integral closure of J. □

Let us give the following criterion for $\mathcal{R}$ to be a special kind of almost Gorenstein graded rings. Notice that Condition (2) in Theorem 2.3 requires the existence of joint reductions of $\mathfrak{m}$, I, and J with reduction number zero (cf. [21]).

Theorem 2.3. *With the same notation as above, set $\mathfrak{M} = \mathfrak{m}\mathcal{R} + \mathcal{R}_+$, the graded maximal ideal of $\mathcal{R}$. Then the following conditions are equivalent.*

(1) *There exists an exact sequence*

$$0 \to \mathcal{R} \to \mathrm{K}_{\mathcal{R}}(1) \to C \to 0$$

of graded $\mathcal{R}$-modules such that $\mathfrak{M}C = (\xi, \eta)C$ for some homogeneous elements ξ, η of $\mathfrak{M}$.

(2) *There exist elements $f \in \mathfrak{m}$, $g \in I$, and $h \in J$ such that*

$$IJ = gJ + Ih \quad \text{and} \quad \mathfrak{m}J = fJ + \mathfrak{m}h.$$

When this is the case, $\mathcal{R}$ is an almost Gorenstein graded ring.

Proof. (2) $\Rightarrow$ (1) Notice that $\mathfrak{M}{\cdot}J\mathcal{R} \subseteq (f, gt){\cdot}J\mathcal{R} + \mathcal{R}h$, since $IJ = gJ + Ih$ and $\mathfrak{m}J = fJ + \mathfrak{m}h$. Consider the exact sequence

$$\mathcal{R} \xrightarrow{\varphi} J\mathcal{R} \to C \to 0$$

of graded $\mathcal{R}$-modules where $\varphi(1) = h$. We then have $\mathfrak{M}C = (f, gt)C$, so that $\dim_{\mathcal{R}_{\mathfrak{M}}} C_{\mathfrak{M}} \leq 2$. Hence by [6, Lemma 3.1] the homomorphism φ is injective and $\mathcal{R}$ is an almost Gorenstein graded ring.

(1) $\Rightarrow$ (2) Suppose that $\mathcal{R}$ is a Gorenstein ring. Then $\mu_R(J) = 1$, since $\mathrm{K}_{\mathcal{R}}(1) \cong J\mathcal{R}$. Hence $J = R$ as $\mathfrak{m} \subseteq \sqrt{J}$, so that choosing $h = 1$ and $f = g = 0$, we get $IJ = gJ + Ih$ and $\mathfrak{m}J = fJ + \mathfrak{m}h$.

Suppose that $\mathcal{R}$ is not a Gorenstein ring and consider the exact sequence

$$0 \to \mathcal{R} \xrightarrow{\varphi} J\mathcal{R} \to C \to 0$$

of graded $\mathcal{R}$-modules with $C \neq (0)$ and $\mathfrak{M}C = (\xi, \eta)C$ for some homogeneous elements ξ, η of $\mathfrak{M}$. Hence $\mathcal{R}_{\mathfrak{M}}$ is an almost Gorenstein local ring in the sense of [6, Definition 3.3]. We set $h = \varphi(1) \in J$, $m = \deg \xi$, and $n = \deg \eta$; hence $C = J\mathcal{R}/\mathcal{R}h$. Remember that $h \notin \mathfrak{m}J$, since $\mathcal{R}_{\mathfrak{M}}$ is not a regular local ring (see [6, Corollary 3.10]). If $\min\{m, n\} > 0$, then $\mathfrak{M}C \subseteq \mathcal{R}_+C$, whence $\mathfrak{m}C_0 = (0)$ (notice that $[\mathcal{R}_+C]_0 = (0)$, as $C = \mathcal{R}C_0$). Therefore

$\mathfrak{m}J \subseteq (h)$, so that we have $J = (h) = R$. Thus $\mathcal{R}h = J\mathcal{R}$ and $\mathcal{R}$ is a Gorenstein ring, which is impossible. Assume $m = 0$. If $n = 0$, then $\mathfrak{M}C = \mathfrak{m}C$ since $\xi, \eta \in \mathfrak{m}$, so that

$$C_1 \subseteq \mathcal{R}_+ C_0 \subseteq \mathfrak{m}C$$

and therefore $C_1 = (0)$ by Nakayama's lemma. Hence $IJ = Ih$ as $[J\mathcal{R}]_1 = \varphi(\mathcal{R}_1)$, which shows (h) is a reduction of J, so that $(h) = R = J$. Therefore $\mathcal{R}$ is a Gorenstein ring, which is impossible. If $n \geq 2$, then because

$$\mathfrak{M}{\cdot}J\mathcal{R} \subseteq \xi{\cdot}J\mathcal{R} + \eta{\cdot}J\mathcal{R} + \mathcal{R}h,$$

we get $IJ \subseteq \xi IJ + Ih$, whence $IJ = Ih$. This is impossible as we have shown above. Hence $n = 1$. Let us write $\eta = gt$ with $g \in I$ and take $f = \xi$. We then have

$$\mathfrak{M}{\cdot}J\mathcal{R} \subseteq (f, gt){\cdot}J\mathcal{R} + \mathcal{R}h,$$

whence $\mathfrak{m}J \subseteq fJ + Rh$. Because $h \notin \mathfrak{m}J$, we get $\mathfrak{m}J \subseteq fJ + \mathfrak{m}h$, so that $\mathfrak{m}J = fJ + \mathfrak{m}h$, while $IJ = gJ + Ih$, because $IJ \subseteq fIJ + gJ + Ih$. This completes the proof of Theorem 2.3. $\square$

Let us explore two examples to show how Theorem 2.3 works.

Example 2.4. Let $S = k[[x, y, z]]$ be the formal power series ring over an infinite field k. Let $\mathfrak{n} = (x, y, z)$ and choose $f \in \mathfrak{n}^2 \setminus \mathfrak{n}^3$. We set $R = S/(f)$ and $\mathfrak{m} = \mathfrak{n}/(f)$. Then for every integer $\ell > 0$ the Rees algebra $\mathcal{R}(\mathfrak{m}^\ell)$ of $\mathfrak{m}^\ell$ is an almost Gorenstein graded ring and $\mathrm{r}(\mathcal{R}) = 2\ell + 1$, where $\mathrm{r}(\mathcal{R})$ denotes the Cohen-Macaulay type of $\mathcal{R}$.

Proof. Since $\mathrm{e}^0_{\mathfrak{m}}(R) = 2$, we have $\mathfrak{m}^2 = (a, b)\mathfrak{m}$ for some elements $a, b \in \mathfrak{m}$. Let $\ell > 0$ be an integer and set $I = \mathfrak{m}^\ell$ and $Q = (a^\ell, b^\ell)$. We then have $I^2 = QI$ and $Q : I = I$, so that $\mathcal{R} = \mathcal{R}(I)$ is a Cohen-Macaulay ring and $\mathrm{K}_{\mathcal{R}}(1) \cong I\mathcal{R}$ by Proposition 2.1, whence $\mathrm{r}(\mathcal{R}) = \mu_R(I) = 2\ell + 1$. Because $\mathfrak{m}^{\ell+1} = a\mathfrak{m}^\ell + b^\ell\mathfrak{m}$ and $Q : I = I = \mathfrak{m}^\ell$, by Theorem 2.3 $\mathcal{R}$ is an almost Gorenstein graded ring. $\square$

Example 2.5. Let $(R, \mathfrak{m})$ be a two-dimensional regular local ring with $\mathfrak{m} = (x, y)$. Let $1 \leq m \leq n$ be integers and set $I = (x^m) + \mathfrak{m}^n$. Then $\mathcal{R}(I)$ is an almost Gorenstein graded ring.

Proof. We may assume $m > 1$. We set $Q = (x^m, y^n)$ and $J = Q : I$. Then $Q \subseteq I$ and $I^2 = QI$. Since $I = (x^m) + (x^i y^{n-i} \mid 0 \leq i \leq m-1)$, we get

$$\begin{aligned} J = Q : (x^i y^{n-i} \mid 0 \leq i \leq m-1) &= \bigcap_{i=1}^{m-1} \left[(x^m, y^n) : x^i y^{n-i}\right] \\ &= \bigcap_{i=1}^{m-1} (x^{m-i}, y^i) \\ &= \mathfrak{m}^{m-1}. \end{aligned}$$

Take $f = x \in \mathfrak{m}$, $g = x^m \in I$, and $h = y^{m-1} \in J = \mathfrak{m}^{m-1}$. We then have $\mathfrak{m}J = fJ + \mathfrak{m}h$ and $IJ = Ih + gJ$, so that by Theorem 2.3 $\mathcal{R}(I)$ is an almost Gorenstein graded ring. $\square$

To prove Theorem 1.3 we need a result of J. Verma [21] about joint reductions of integrally closed ideals. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let I and J be ideals of R and let $a \in I$ and $b \in J$. Then we say that a, b are a *joint reduction* of I, J if $aJ + Ib$ is a reduction of IJ. Joint reductions always exist (see, e.g., [17]) if the residue class field of R is infinite. We furthermore have the following.

Theorem 2.6 ([21, Theorem 2.1])**.** *Let $(R, \mathfrak{m})$ be a two-dimensional regular local ring. Let I and J be $\mathfrak{m}$-primary ideals of R. Assume that a, b are a joint reduction of I, J. Then $IJ = aJ + Ib$, if I and J are integrally closed.*

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let $(R, \mathfrak{m})$ be a two-dimensional regular local ring with infinite residue class field and let I be an $\mathfrak{m}$-primary integrally closed ideal in R. We choose a parameter ideal Q of R so that $Q \subseteq I$ and $I^2 = QI$ (this choice is possible; see [23, Appendix 5] or [10]). Therefore the Rees algebra $\mathcal{R} = \mathcal{R}(I)$ is a Cohen-Macaulay ring ([5]). Because $\mathcal{R}$ is a normal ring ([23]), by Corollary 2.2 $J = Q : I$ is an integrally closed ideal in R. Consequently, choosing three elements $f \in \mathfrak{m}$, $g \in I$, and $h \in J$ so that f, h are a joint reduction of $\mathfrak{m}, J$ and g, h are a joint reduction of I, J, we readily get by Theorem 2.6 the equalities

$$\mathfrak{m}J = fJ + \mathfrak{m}g \quad \text{and} \quad IJ = gJ + Ih$$

stated in Condition (2) of Theorem 2.3. Thus $\mathcal{R} = \mathcal{R}(I)$ is an almost Gorenstein graded ring. $\square$

We now explore the almost Gorenstein property of the Rees algebras of modules. To state the result we need additional notation. For the rest of this section let $(R, \mathfrak{m})$ be a two-dimensional regular local ring with infinite residue class field. Let $M \neq (0)$ be a finitely generated torsion-free R-module and assume that M is non-free. Let $(-)^* = \mathrm{Hom}_R(-, R)$. Then $F = M^{**}$ is a finitely generated free R-module and we get a canonical exact sequence

$$0 \to M \xrightarrow{\varphi} F \to C \to 0$$

of R-modules with $C \neq (0)$ and $\ell_R(C) < \infty$. Let $\mathrm{Sym}(M)$ and $\mathrm{Sym}(F)$ denote the symmetric algebras of M and F respectively and let $\mathrm{Sym}(\varphi) : \mathrm{Sym}(M) \to \mathrm{Sym}(F)$ be the homomorphism induced from $\varphi : M \to F$. Then the Rees algebra $\mathcal{R}(M)$ of M is defined by

$$\mathcal{R}(M) = \mathrm{Im}\left[\mathrm{Sym}(M) \xrightarrow{\mathrm{Sym}(\varphi)} \mathrm{Sym}(F)\right]$$

([18]). Hence $\mathcal{R}(M) = \mathrm{Sym}(M)/T$ where $T = t(\mathrm{Sym}(M))$ denotes the R-torsion part of $\mathrm{Sym}(M)$, so that $M = [\mathcal{R}(M)]_1$ is an R-submodule of $\mathcal{R}(M)$. Let $x \in F$. Then we say that x is integral over M, if it satisfies an integral equation

$$x^n + c_1 x^{n-1} + \cdots + c_n = 0$$

in the symmetric algebra $\mathrm{Sym}(F)$ with $n > 0$ and $c_i \in M^i$ for each $1 \leq i \leq n$. Let $\overline{M}$ be the set of elements of F which are integral over M. Then $\overline{M}$ forms an R-submodule of F, which is called the integral closure of M. We say that M is *integrally closed*, if $\overline{M} = M$.

With this notation we have the following.

Corollary 2.7. *Let $\mathfrak{M} = \mathfrak{m}\mathcal{R}(M) + \mathcal{R}(M)_+$ be the unique graded maximal ideal of $\mathcal{R}(M)$ and suppose that M is integrally closed. Then $\mathcal{R}(M)_{\mathfrak{M}}$ is an almost Gorenstein local ring in the sense of* [6, Definition 3.3].

Proof. Let $U = R[x_1, x_2, \dots, x_n]$ be the polynomial ring with sufficiently large $n > 0$ and set $S = U_{\mathfrak{m}U}$. We denote by $\mathfrak{n}$ the maximal ideal of S. Then thanks to [18, Theorem 3.5] and [9, Theorem 3.6], we can find some elements $f_1, f_2, \dots, f_{r-1} \in S \otimes_R M$ ($r = \mathrm{rank}_R F$) and an $\mathfrak{n}$-primary integrally closed ideal I in S, so that $f_1, f_2, \dots, f_{r-1}$ form a regular sequence in $\mathcal{R}(S \otimes_R M)$ and

$$\mathcal{R}(S \otimes_R M)/(f_1, f_2, \dots, f_{r-1}) \cong \mathcal{R}(I)$$

as a graded S-algebra. Therefore, because $\mathcal{R}(I)$ is an almost Gorenstein graded ring by Theorem 1.3, $S \otimes_R \mathcal{R}(M) = \mathcal{R}(S \otimes_R M)$ is an almost Gorenstein graded ring (cf. [6, Theorem 3.7 (1)]). Consequently $\mathcal{R}(M)_{\mathfrak{M}}$ is an almost Gorenstein local ring by [6, Theorem 3.9]. $\square$

3. Almost Gorenstein property in Rees algebras of ideals with linear presentation matrices

Let $R = k[[x_1, x_2, \dots, x_d]]$ ($d \geq 2$) be the formal power series ring over an infinite field k. Let I be a perfect ideal of R with $\mathrm{grade}_R I = 2$, possessing a linear presentation matrix φ

$$0 \to R^{\oplus(n-1)} \xrightarrow{\varphi} R^{\oplus n} \to I \to 0,$$

that is each entry of the matrix φ is contained in $\sum_{i=1}^{d} kx_i$. We set $n = \mu_R(I)$ and $\mathfrak{m} = (x_1, x_2, \dots, x_d)$; hence $I = \mathfrak{m}^{n-1}$ if $d = 2$. In what follows we assume that $n > d$ and that our ideal I satisfies the condition (G_d) of [1], that is $\mu_{R_\mathfrak{p}}(IR_\mathfrak{p}) \leq \dim R_\mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{V}(I) \setminus \{\mathfrak{m}\}$. Then thanks to [16, Theorem 1.3] and [11, Proposition 2.3], the Rees algebra $\mathcal{R} = \mathcal{R}(I)$ of I is a Cohen-Macaulay ring with $\mathrm{a}(\mathcal{R}) = -1$ and

$$\mathrm{K}_{\mathcal{R}}(1) \cong \mathfrak{m}^{n-d}\mathcal{R}$$

as a graded $\mathcal{R}$-module.

We are interested in the question of when $\mathcal{R}$ is an almost Gorenstein graded ring. Our answer is the following, which suggests that almost Gorenstein Rees algebras might be rare in dimension greater than two.

Theorem 3.1. *With the same notation as above, set $\mathfrak{M} = \mathfrak{m}\mathcal{R} + \mathcal{R}_+$, the graded maximal ideal of $\mathcal{R}$. Then the following conditions are equivalent.*

(1) *$\mathcal{R}$ is an almost Gorenstein graded ring*
(2) *$\mathcal{R}_{\mathfrak{M}}$ is an almost Gorenstein local ring*
(3) *$d = 2$.*

Proof. (1) $\Rightarrow$ (2) This follows from the definition, since $[\mathrm{K}_{\mathcal{R}}]_{\mathfrak{M}} \cong \mathrm{K}_{\mathcal{R}_{\mathfrak{M}}}$.

(3) $\Rightarrow$ (1) We have $I = \mathfrak{m}^{n-1}$ since $d = 2$ and so $\mathcal{R}$ is an almost Gorenstein graded ring (Corollary 1.4).

(2) $\Rightarrow$ (3) Let $\Delta_i = (-1)^{i+1} \det \varphi_i$ for each $1 \le i \le n$, where φ_i stands for the $(n-1) \times (n-1)$ matrix which is obtained from φ by deleting the i-th row. Hence $I = (\Delta_1, \Delta_2, \dots, \Delta_n)$ and the ideal I has a presentation

$$(P) \qquad 0 \to R^{\oplus(n-1)} \xrightarrow{\varphi} R^{\oplus n} \xrightarrow{[\Delta_1\ \Delta_2\ \cdots\ \Delta_n]} I \to 0.$$

Notice that $\mathcal{R}$ is not a Gorenstein ring, since $\mathrm{r}(\mathcal{R}) = \mu_R(\mathfrak{m}^{n-d}) = \binom{n-1}{d-1} > 1$. We set $A = \mathcal{R}_{\mathfrak{M}}$ and $\mathfrak{n} = \mathfrak{M}A$; hence $\mathrm{K}_A = [\mathrm{K}_{\mathcal{R}}]_{\mathfrak{M}}$. We take an exact sequence

$$0 \to A \xrightarrow{\phi} \mathrm{K}_A \to C \to 0$$

of A-modules such that $C \neq (0)$ and C is an Ulrich A-module. Let $f = \phi(1)$. Then $f \notin \mathfrak{n}\mathrm{K}_A$ by [6, Corollary 3.10] and we get the exact sequence

$$(E) \qquad 0 \to \mathfrak{n}f \to \mathfrak{n}\mathrm{K}_A \to \mathfrak{n}C \to 0.$$

Because $\mathfrak{n}C = (f_1, f_2, \dots, f_d)C$ for some $f_1, f_2, \dots, f_d \in \mathfrak{n}$ ([6, Proposition 2.2]) and $\mu_A(\mathfrak{n}) = d + n$, we get by the exact sequence (E) that

$$\mu_{\mathcal{R}}(\mathfrak{M}\mathrm{K}_{\mathcal{R}}) = \mu_A(\mathfrak{n}\mathrm{K}_A) \le (d+n) + d\cdot(\mathrm{r}(A) - 1) = d\binom{n-1}{d-1} + n,$$

while

$$\mu_{\mathcal{R}}(\mathfrak{M}\mathrm{K}_{\mathcal{R}}) = \mu_R(\mathfrak{m}^{n-d+1}) + \mu_R(\mathfrak{m}^{n-d}I) = \binom{n}{d-1} + \mu_R(\mathfrak{m}^{n-d}I)$$

since $\mathfrak{M} = (\mathfrak{m}, It)\mathcal{R}$ and $\mathrm{K}_{\mathcal{R}}(1) = \mathfrak{m}^{n-d}\mathcal{R}$. Consequently we have

$$(*) \qquad \mu_R(\mathfrak{m}^{n-d}I) \le d\binom{n-1}{d-1} + n - \binom{n}{d-1}.$$

To estimate the number $\mu_R(\mathfrak{m}^{n-d}I)$ from below, we consider the homomorphism

$$\psi : \mathfrak{m}^{n-d} \otimes_R I \to \mathfrak{m}^{n-d}I$$

defined by $x \otimes y \mapsto xy$ and set $X = \operatorname{Ker}\psi$. Let $x \in X$ and write $x = \sum_{i=1}^{d} x_i \otimes \Delta_i$ with $x_i \in \mathfrak{m}^{n-d}$. Then since $\sum_{i=1}^{d} x_i\Delta_i = 0$ in R and since every entry of the matrix φ is linear, the presentation (P) of I guarantees the existence of elements $y_j \in \mathfrak{m}^{n-d-1}$ $(1 \le j \le n-1)$ such that

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \varphi \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix}.$$

Hence X is a homomorphic image of $[\mathfrak{m}^{n-d-1}]^{\oplus(n-1)}$. Therefore in the exact sequence

$$0 \to X \to \mathfrak{m}^{n-d} \otimes_R I \to \mathfrak{m}^{n-d}I \to 0$$

we get

$$\mu_R(X) \le (n-1)\binom{n-2}{d-1}.$$

Consequently

$$(**) \qquad \mu_R(\mathfrak{m}^{d-n}I) \ge n\binom{n-1}{d-1} - (n-1)\binom{n-2}{d-1},$$

so that combining the estimations (∗) and (∗∗), we get

$$\begin{aligned} 0 &\leq \left[d\binom{n-1}{d-1} + n - \binom{n}{d-1}\right] - \left[n\binom{n-1}{d-1} - (n-1)\binom{n-2}{d-1}\right] \\ &= \left[(d-n)\binom{n-1}{d-1} + (n-1)\binom{n-2}{d-1}\right] + \left[n - \binom{n}{d-1}\right] \\ &= n - \binom{n}{d-1}. \end{aligned}$$

Hence $d = 2$, because $n < \binom{n}{d-1}$ if $n > d \geq 3$. □

Before closing this section, let us note one concrete example.

Example 3.2. Let $R = k[[x, y, z]]$ be the formal power series ring over an infinite field k. We set $I = (x^2y, y^2z, z^2x, xyz)$ and $Q = (x^2y, y^2z, z^2x)$. Then Q is a minimal reduction of I with $\mathrm{red}_Q(I) = 2$. The ideal I has a presentation of the form

$$0 \to R^{\oplus 3} \xrightarrow{\varphi} R^{\oplus 4} \to I \to 0$$

with $\varphi = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \\ y & z & x \end{pmatrix}$ and it is direct to check that I satisfies all the conditions required for Theorem 3.1. Hence Theorem 3.1 shows that $\mathcal{R}(I)$ cannot be an almost Gorenstein graded ring, while Q is not a perfect ideal of R but its Rees algebra $\mathcal{R}(Q)$ is an almost Gorenstein graded ring with $\mathrm{r}(\mathcal{R}) = 2$; see [14].

4. The Rees algebras of socle ideals $I = (a, b) : \mathfrak{m}$

Throughout this section let $(R, \mathfrak{m})$ denote a two-dimensional regular local ring with infinite residue class field. Let $Q = (a, b)$ be a parameter ideal of R. We set $I = Q : \mathfrak{m}$ and $\mathcal{R} = \mathcal{R}(I)$. For each ideal $\mathfrak{a}$ in R we set $\mathrm{o}(\mathfrak{a}) = \sup\{n \in \mathbb{Z} \mid \mathfrak{a} \subseteq \mathfrak{m}^n\}$. We are interested in the question of when $\mathcal{R}$ is an almost Gorenstein graded ring. Our answer is the following. Notice that the implication (1) ⇒ (2) follows from the definition.

Theorem 4.1. *With the same notation as above assume that $Q \neq \mathfrak{m}$. Then the following conditions are equivalent.*

(1) *$\mathcal{R}$ is an almost Gorenstein graded ring.*
(2) *$\mathcal{R}_{\mathfrak{M}}$ is an almost Gorenstein local ring*
(3) $\mathrm{o}(Q) \leq 2$.

Proof of the implication (3) ⇒ (1). If $\mathrm{o}(Q) = 1$, then $Q = (x, y^q)$ $(q \geq 2)$ for some regular system x, y of parameters of R. Hence $I = (x, y^{q-1})$ and $\mathcal{R}$ is a Gorenstein ring. Suppose that $\mathrm{o}(Q) = 2$. Then $\mathrm{o}(I) = \mathrm{o}(Q) = 2$ since $I^2 = QI$ ([22]). Because $\mu_R(I) = 3 = \mathrm{o}(I)+1$, there exists an element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $I : x = I : \mathfrak{m}$ (cf. [10], [23, App. 5]). We set $\overline{R} = R/(x)$. Then $Q\overline{R} = I\overline{R}$, since Q is a reduction of I and $\overline{R}$ is a DVR. We may assume $a\overline{R} = I\overline{R}$, whence $I \subseteq (a, x)$. Let us write $b = af + xg$ with $f, g \in R$. Then $Q = (a, xg)$. Therefore $I\mathfrak{m} = Q\mathfrak{m} = (g\mathfrak{m})x + a\mathfrak{m}$ (remember that $I\mathfrak{m} = Q\mathfrak{m}$; see, e.g. [22]). Notice that $g\mathfrak{m} \subseteq I$, because $g \in Q : x \subseteq I : x = I : \mathfrak{m}$. Therefore $I\mathfrak{m} = Ix + a\mathfrak{m}$, whence $\mathcal{R}$ is an almost Gorenstein graded ring by Theorem 2.3. □

To prove the implication (2) $\Rightarrow$ (3) we need Theorem 4.2 below. From now we write $\mathfrak{m} = (x, y)$ and let $Q = (a, b)$ be a parameter ideal of R such that $\mathrm{o}(Q) \geq 2$. Hence $I^2 = QI$ with $\mu_R(I) = 3$. Let $I = (a, b, c)$. Then since $xc, yc \in Q$, we get equations

$$f_1a + f_2b + xc = 0 \quad \text{and} \quad g_1a + g_2b + yc = 0$$

with $f_i, g_i \in \mathfrak{m}$ $(i = 1, 2)$.

Theorem 4.2. *With the notation above, if $(f_1, f_2, g_1, g_2) \subseteq \mathfrak{m}^2$, then $\mathcal{R}_\mathfrak{M}$ is not an almost Gorenstein local ring.*

We divide the proof of Theorem 4.2 into three steps. Let us begin with the following.

Lemma 4.3. *Let $\mathbb{M} = \begin{pmatrix} f_1 & f_2 & x \\ g_1 & g_2 & y \end{pmatrix}$. Then R/I has a minimal free resolution*

$$0 \to R^{\oplus 2} \xrightarrow{{}^t\mathbb{M}} R^{\oplus 3} \xrightarrow{[a\ b\ c]} R \to R/I \to 0.$$

Proof. Let $\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ x \end{pmatrix}$ and $\mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \\ y \end{pmatrix}$. Then $\mathbf{f}, \mathbf{g} \in \mathfrak{m}{\cdot}R^{\oplus 3}$. As $\mathbf{f}, \mathbf{g}$ mod $\mathfrak{m}^2{\cdot}R^{\oplus 3}$ are linearly independent over $R/\mathfrak{m}$, the complex

$$0 \to R^{\oplus 2} \xrightarrow{{}^t\mathbb{M}} R^{\oplus 3} \xrightarrow{[a\ b\ c]} \longrightarrow R \to R/I \to 0$$

is exact and gives rise to a minimal free resolution of R/I. $\square$

Let $\mathcal{S} = R[X, Y, Z]$ be the polynomial ring and let $\varphi : \mathcal{S} \to \mathcal{R} = R[It]$ (t an indeterminate) be the R-algebra map defined by $\varphi(X) = at$, $\varphi(Y) = bt$, and $\varphi(Z) = ct$. Let $K = \operatorname{Ker} \varphi$. Since $c^2 \in QI$, we have a relation of the form

$$c^2 = a^2 f + b^2 g + abh + bci + caj$$

with $f, g, h, i, j \in R$. We set

$$\begin{aligned} F &= Z^2 - \left(fX^2 + gY^2 + hXY + iYZ + jZX\right), \\ G &= f_1X + f_2Y + xZ, \\ H &= g_1X + g_2Y + yZ. \end{aligned}$$

Notice that $F \in \mathcal{S}_2$ and $G, H \in \mathcal{S}_1$.

Proposition 4.4. *$\mathcal{R}$ has a minimal graded free resolution of the form*

$$0 \to \mathcal{S}(-2) \oplus \mathcal{S}(-2) \xrightarrow{{}^t\mathbb{N}} \mathcal{S}(-2) \oplus \mathcal{S}(-1) \oplus \mathcal{S}(-1) \xrightarrow{[F\ G\ H]} \mathcal{S} \to \mathcal{R} \to 0,$$

so that the graded canonical module $\mathrm{K}_\mathcal{R}$ of $\mathcal{R}$ has a presentation

$$\mathcal{S}(-1) \oplus \mathcal{S}(-2) \oplus \mathcal{S}(-2) \xrightarrow{\mathbb{N}} \mathcal{S}(-1) \oplus \mathcal{S}(-1) \to \mathrm{K}_\mathcal{R} \to 0.$$

Proof. We have $K = \mathcal{S}K_1 + (F)$ (cf., e.g. [15, Theorem 4.1]; use the fact that $I^2 = QI$ and $c^2 \in QI$). Hence $\mathcal{R}$ has a minimal graded free resolution of the form

$$(*) \quad 0 \to \mathcal{S}(-m) \oplus \mathcal{S}(-\ell) \xrightarrow{{}^t\mathbb{N}} \mathcal{S}(-2) \oplus \mathcal{S}(-1) \oplus \mathcal{S}(-1) \xrightarrow{[F\ G\ H]} \mathcal{S} \to \mathcal{R} \to 0$$

with $m, \ell \geq 1$. We take the $\mathcal{S}(-3)$-dual of the resolution $(*)$. Then as $\mathrm{K}_{\mathcal{S}} = \mathcal{S}(-3)$, we get the presentation

$$\mathcal{S}(-1) \oplus \mathcal{S}(-2) \oplus \mathcal{S}(-2) \xrightarrow{\mathbb{N}} \mathcal{S}(m-3) \oplus \mathcal{S}(\ell-3) \to \mathrm{K}_{\mathcal{R}} \to 0$$

of the canonical module $\mathrm{K}_{\mathcal{R}}$ of $\mathcal{R}$. Hence $m, \ell \leq 2$ because $\mathrm{a}(\mathcal{R}) = -1$. Assume that $m = 1$. Then the matrix ${}^t\mathbb{N}$ has the form ${}^t\mathbb{N} = \begin{pmatrix} 0 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{pmatrix}$ with $\alpha_2, \alpha_3 \in R$. We have $\alpha_2 G + \alpha_3 H = 0$, or equivalently $\alpha_2 \begin{pmatrix} f_1 \\ f_2 \\ x \end{pmatrix} + \alpha_3 \begin{pmatrix} g_1 \\ g_2 \\ y \end{pmatrix} = \mathbf{0}$, whence $\alpha_2 = \alpha_3 = 0$ by Lemma 4.3. This is impossible, whence $m = 2$. We similarly have $\ell = 2$ and the assertion follows. $\square$

We are ready to prove Theorem 4.2.

Proof of Theorem 4.2. Let $\mathbb{N}$ be the matrix given by Proposition 4.4 and write $\mathbb{N} = \begin{pmatrix} \alpha & F_1 & F_2 \\ \beta & G_1 & G_2 \end{pmatrix}$. Then Proposition 4.4 shows that $F_i, G_i \in \mathcal{S}_1$ $(i = 1, 2)$ and $\alpha, \beta \in \mathfrak{m}$. We write $F_i = \alpha_{i1}X + \alpha_{i2}Y + \alpha_{i3}Z$ and $G_i = \beta_{i1}X + \beta_{i2}Y + \beta_{i3}Z$ with $\alpha_{ij}, \beta_{ij} \in R$. Let Δ_j denote the determinant of the matrix obtained by deleting the j-th column from $\mathbb{N}$. Then by the theorem of Hilbert-Burch we have $G = -\varepsilon\Delta_2$ and $H = \varepsilon\Delta_3$ for some unit ε of R, so that

$$\begin{pmatrix} f_1 \\ f_2 \\ x \end{pmatrix} = (\varepsilon\beta) \begin{pmatrix} \alpha_{21} \\ \alpha_{22} \\ \alpha_{23} \end{pmatrix} - (\varepsilon\alpha) \begin{pmatrix} \beta_{21} \\ \beta_{22} \\ \beta_{23} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} g_1 \\ g_2 \\ y \end{pmatrix} = (\varepsilon\alpha) \begin{pmatrix} \beta_{11} \\ \beta_{12} \\ \beta_{13} \end{pmatrix} - (\varepsilon\beta) \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{13} \end{pmatrix}.$$

Hence

$$x = (\varepsilon\beta)\alpha_{23} - (\varepsilon\alpha)\beta_{23} \quad \text{and} \quad y = (\varepsilon\alpha)\beta_{13} - (\varepsilon\beta)\alpha_{13},$$

which shows $(x, y) = (\varepsilon\alpha, \varepsilon\beta) = \mathfrak{m}$ because $(x, y) \subseteq (\varepsilon\alpha, \varepsilon\beta) \subseteq \mathfrak{m}$.

Since $f_1 = (\varepsilon\beta)\alpha_{21} - (\varepsilon\alpha)\beta_{21}$ and $\varepsilon\alpha, \varepsilon\beta$ is a regular system of parameters of R, we get $\alpha_{21}, \beta_{21} \in \mathfrak{m}$ if $f_1 \in \mathfrak{m}^2$. Therefore if $(f_1, f_2, g_1, g_2) \subseteq \mathfrak{m}^2$, then $\alpha_{ij}, \beta_{ij} \in \mathfrak{m}$ for all $i, j = 1, 2$, whence

$$\mathbb{N} \equiv \begin{pmatrix} \alpha & \alpha_{13}Z & \alpha_{23}Z \\ \beta & \beta_{13}Z & \beta_{23}Z \end{pmatrix} \bmod \mathfrak{N}^2$$

where $\mathfrak{N} = \mathfrak{m}\mathcal{S} + \mathcal{S}_+$ denotes the graded maximal ideal of $\mathcal{S}$. We set $B = \mathcal{S}_{\mathfrak{N}}$. Then it is clear that after any elementary row and column operations the matrix $\mathbb{N}$ over the regular local ring B of dimension 5 is not equivalent to a matrix of the form

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{pmatrix}$$

with $\alpha_1, \alpha_2, \alpha_3$ a part of a regular system of parameters of B. Hence by [6, Theorem 7.8] $\mathcal{R}_{\mathfrak{M}}$ cannot be an almost Gorenstein local ring. $\square$

We are now in a position to finish the proof of Theorem 4.1.

Proof of the implication (2) $\Rightarrow$ (3) *in Theorem 4.1.* It suffices to show that $\mathcal{R}_{\mathfrak{M}}$ is not an almost Gorenstein local ring if $\mathrm{o}(Q) \geq 3$. We write $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ with $f_{ij} \in \mathfrak{m}^2$ $(i,j=1,2)$ and set $c = \det \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$. Then $Q : c = \mathfrak{m}$ and $Q : \mathfrak{m} = Q + (c)$. We have

$$(-f_{22})a + f_{12}b + cx = 0 \quad \text{and} \quad f_{21}a + (-f_{11})b + cy = 0.$$

Hence by Theorem 4.2 $\mathcal{R}_{\mathfrak{M}}$ is not an almost Gorenstein local ring, which completes the proof of Theorem 4.1. $\square$

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