

CHARACTERIZATION OF GENERALIZED GORENSTEIN RINGS

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ABSTRACT. The notion of generalized Gorenstein local ring (GGL ring for short) is one of the generalizations of Gorenstein rings. In this article, there is given a characterization of GGL rings in terms of their canonical ideals and related invariants.

1. INTRODUCTION

The notion of a generalized Gorenstein local ring (GGL ring for short) is one of the generalizations of Gorenstein rings. Similarly, for almost Gorenstein local rings (AGL rings for short), the notion is given in terms of a certain specific embedding of the rings into their canonical modules (see [5] by the first and third authors). However, the research on AGL rings developed in [7] by the first author, N. Matsuoka, and T. T. Phuong for arbitrary one-dimensional Cohen–Macaulay local rings is based on an investigation of the relationship between two invariants: the first Hilbert coefficient of the canonical ideals and the Cohen–Macaulay type of the rings. Therefore, it seems natural to ask for a possible characterization of AGL rings of higher dimension and that of GGL rings in terms of their canonical ideals and some related invariants. For AGL rings, this characterization has been accomplished by the first and fourth authors and R. Takahashi. They have already given a satisfactory result [17, Theorem 5.1]. The present purpose is to perform the task for GGL rings of higher dimension.

Originally, the series of studies in [3, 4, 5, 6, 7, 8, 9, 10, 11, 16, 17, 18, 19, 20] aimed to find a new class of Cohen–Macaulay local rings, which contains the class of Gorenstein rings. AGL rings are one of the candidates for such a class. Historically, the notion of an AGL ring in our sense has its roots in [1] by V. Barucci and R. Fröberg in 1997, where they dealt with one-dimensional analytically unramified local rings. They also explored numerical semigroup rings, starting a very beautiful theory. In [7], the authors relaxed the notion to arbitrary Cohen–Macaulay local rings of dimension one based on a different point of view. Repairing a gap in the proof of [1, Proposition 25], they provided new possible areas of study for one-dimensional Cohen–Macaulay local rings. Among various results in [7], the most striking achievement seems that their arguments have prepared for a possible definition [17, Definition 3.3] of AGL rings of higher dimension. We now have two more notions: the 2-almost Gorenstein local ring ([3]) and GGL ring ([5]), both of

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which are reasonable candidate generalizations of Gorenstein rings and almost Gorenstein rings as well.

As stated above, the present purpose is to find a characterization of GGL rings in terms of the canonical ideals and related invariants. To state our motivation and the results more precisely, let us review the definition of GGL rings. Throughout this article, let (R, \mathfrak{m}) be a Cohen–Macaulay local ring with $d = \dim R > 0$, possessing the canonical module K_R . For simplicity, let us always assume that the residue class field R/\mathfrak{m} of R is infinite. Let \mathfrak{a} be an \mathfrak{m} -primary ideal of R . With this notation, the definition of a GGL ring is stated as follows.

Definition 1.1 ([5]). We say that R is a generalized Gorenstein local ring (GGL for short) if one of the following conditions is satisfied:

- (1) R is a Gorenstein ring.
- (2) R is not a Gorenstein ring, but there exists an exact sequence

$$0 \rightarrow R \xrightarrow{\varphi} K_R \rightarrow C \rightarrow 0$$

of R -modules such that

- (i) C is an Ulrich R -module with respect to \mathfrak{a} , and
- (ii) the induced homomorphism $R/\mathfrak{a} \otimes_R \varphi : R/\mathfrak{a} \rightarrow K_R/\mathfrak{a}K_R$ is injective.

When Case (2) occurs, we especially say that R is a GGL ring with respect to \mathfrak{a} .

Let us further explain Definition 1.1. Let M be a finitely generated R -module of dimension $s \geq 0$. Then, we say that M is an Ulrich R -module with respect to \mathfrak{a} if the following three conditions are satisfied:

- (i) M is a Cohen–Macaulay R -module,
- (ii) $e_{\mathfrak{a}}^0(M) = \ell_R(M/\mathfrak{a}M)$, and
- (iii) $M/\mathfrak{a}M$ is a free R/\mathfrak{a} -module,

where $\ell_R(*)$ is the length, and

$$e_{\mathfrak{a}}^0(M) = \lim_{n \rightarrow \infty} s! \cdot \frac{\ell_R(M/\mathfrak{a}^{n+1}M)}{n^s}$$

denotes the multiplicity of M with respect to \mathfrak{a} ([14]). The notion of an Ulrich R -module with respect to an \mathfrak{m} -primary ideal is a generalization of a maximally generated maximal Cohen–Macaulay R -module (i.e., a maximal Ulrich R -module with respect to \mathfrak{m} ; see [2]). One can consult [5, 14, 15, 17] for the basic properties of Ulrich modules in our sense. Here, let us note one thing. In the setting of Definition 1.1, suppose that there is an exact sequence

$$0 \rightarrow R \rightarrow K_R \rightarrow C \rightarrow 0$$

of R -modules such that $C \neq (0)$. Then, C is a Cohen–Macaulay R -module of dimension $d - 1$ ([17, Lemma 3.1 (2)]), and C is an Ulrich R -module with respect to \mathfrak{a} if and only if $C/\mathfrak{a}C$ is a free R/\mathfrak{a} -module and

$$\mathfrak{a}C = (f_2, f_3, \dots, f_d)C$$

for some elements $f_2, f_3, \dots, f_d \in \mathfrak{a}$ ([5, Proof of Proposition 2.4, Claim]). Therefore, if $\mathfrak{a} = \mathfrak{m}$, Definition 1.1 is exactly the same as that of AGL rings given by [17, Definition

3.3]. In [5], the authors investigate GGL rings, and one can find a report of the basic results for GGL rings, which greatly generalizes several results in [17], clarifying what AGL rings are.

The present purpose is to give a characterization of GGL rings. Let $r(R)$ stand for the Cohen–Macaulay type of R . We then have the following, which is the main result of this article.

Theorem 1.2. *Let (R, \mathfrak{m}) be a Cohen–Macaulay local ring with $d = \dim R > 0$ and an infinite residue class field, possessing the canonical module K_R . Let $I \subsetneq R$ be an ideal of R such that $I \cong K_R$ as an R -module. Let \mathfrak{a} be an \mathfrak{m} -primary ideal of R . Assume that R is not a Gorenstein ring. Then, the following conditions are equivalent.*

- (1) R is a GGL ring with respect to \mathfrak{a} .
- (2) There exists a parameter ideal $Q = (f_1, f_2, \dots, f_d)$ of R such that $f_1 \in I$ and, setting $J = I + Q$, the following three conditions are satisfied.
 - (i) $\mathfrak{a} = Q :_R J$.
 - (ii) $\mathfrak{a}J = \mathfrak{a}Q$.
 - (iii) $e_J^1(R) = \ell_R(R/\mathfrak{a}) \cdot r(R)$.

When this is the case, R/\mathfrak{a} is a Gorenstein ring, and the following assertions hold true.

- (a) $J^3 = QJ^2$, but $J^2 \neq QJ$.
- (b) $\mathcal{S}_Q(J) \cong (\mathcal{T}/\mathfrak{a}\mathcal{T})(-1)$, as a graded \mathcal{T} -module, where $\mathcal{S}_Q(J)$ (resp. $\mathcal{T} = \mathcal{R}(Q)$) denotes the Sally module of J with respect to Q (see [21]) (resp. the Rees algebra of Q).
- (c) f_2, f_3, \dots, f_d form a super-regular sequence of R with respect to J , and $\text{depth gr}_J(R) = d - 1$, where $\text{gr}_J(R) = \bigoplus_{n \geq 0} J^n/J^{n+1}$ denotes the associated graded ring of J .
- (d) The Hilbert function of R with respect to J is given by

$$\ell_R(R/J^{n+1}) = e_J^0(R) \cdot \binom{n+d}{d} - [e_J^0(R) - \ell_R(R/J) + \ell_R(R/\mathfrak{a})] \cdot \binom{n+d-1}{d-1} + \ell_R(R/\mathfrak{a}) \cdot \binom{n+d-2}{d-2}$$

for $n \geq 1$. Hence, $e_J^2(R) = \ell_R(R/\mathfrak{a})$ if $d \geq 2$, and $e_J^i(R) = 0$ for all $3 \leq i \leq d$ if $d \geq 3$.

The study of GGL rings is still in progress, and Theorem 1.2 now completely generalizes the corresponding assertion [17, Theorem 5.1] of AGL rings to arbitrary GGL rings of higher dimension, certifying that the notion of a GGL ring is a reasonable generalization of AGL rings and that a GGL ring is one of the candidates of broader notion that generalizes Gorenstein rings.

We now briefly explain how this paper is organized. The proof of Theorem 1.2 shall be given in Sections 3 and 4. In Section 2, we summarize some of the known results given by [5], which we use to prove Theorem 1.2. We will explore an example in order to illustrate Theorem 1.2 in Section 5.

2. PRELIMINARIES

In this section we summarize some of the results in [5, Section 4] about GGL rings, which we use to prove Theorem 1.2.

First, let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension one, admitting the canonical module K_R . Let $I \subsetneq R$ be an ideal of R such that $I \cong K_R$ as an R -module. We assume that I contains a parameter ideal (a) of R as a reduction. We set

$$K = \frac{I}{a} = \left\{ \frac{x}{a} \mid x \in I \right\}$$

in the total ring $\mathbb{Q}(R)$ of fractions of R . Hence K is a fractional ideal of R such that $R \subseteq K \subseteq \overline{R}$ and $K \cong K_R$, where \overline{R} denotes the integral closure of R in $\mathbb{Q}(R)$. We set $S = R[K]$ in $\mathbb{Q}(R)$. Hence S is a module-finite birational extension of R . Note that the ring $S = R[K]$ is independent of the choice of canonical ideals I and reductions (a) of I ([3, Theorem 2.5]). We set $\mathfrak{c} = R : S$. We then have the following, which shows the \mathfrak{m} -primary ideal \mathfrak{a} which appears in Definition 1.1 of a GGL ring R is uniquely determined, when $\dim R = 1$. We note a brief proof.

Proposition 2.1 ([5]). *Suppose R is not a Gorenstein ring but R is a GGL ring with respect to an \mathfrak{m} -primary ideal \mathfrak{a} of R . Then $\mathfrak{a} = \mathfrak{c}$.*

Proof. We choose an exact sequence

$$0 \rightarrow R \xrightarrow{\varphi} I \rightarrow C \rightarrow 0$$

of R -modules such that C is an Ulrich R -module with respect to \mathfrak{a} and the induced homomorphism $R/\mathfrak{a} \otimes_R \varphi : R/\mathfrak{a} \rightarrow I/\mathfrak{a}I$ is injective. We set $f = \varphi(1)$ and identify $C = I/(f)$. Then $\mathfrak{a}I \subseteq (f)$ since $\mathfrak{a} \cdot (I/(f)) = (0)$, while $(f) \cap \mathfrak{a}I = \mathfrak{a}f$ since the homomorphism $R/\mathfrak{a} \otimes_R \varphi$ is injective. Consequently, $\mathfrak{a}I = \mathfrak{a}f$, whence (f) is a reduction of I . We consider $L = \frac{I}{f}$ and set $S = R[L]$. Then $\mathfrak{a}L = \mathfrak{a}$ since $\mathfrak{a}I = \mathfrak{a}f$, so that $\mathfrak{a}S = \mathfrak{a}$ since $S = L^n$ for all $n \gg 0$. Therefore, $\mathfrak{a} \subseteq \mathfrak{c} = R : S$, so that $\mathfrak{a} = \mathfrak{c}$, because $\mathfrak{c} \subseteq R : L = (f) :_R I = \mathfrak{a}$. \square

In general we have the following.

Fact 2.2 ([5]). Let $\mathfrak{a} = R : K$. Then the following conditions are equivalent.

- (1) $K^2 = K^3$.
- (2) $\mathfrak{a} = \mathfrak{c}$.
- (3) $\mathfrak{a}K = \mathfrak{a}$.

The key in the theory of one-dimensional GGL rings is the following, which we shall freely use in the present article. See [5, Section 4] for the proof.

Theorem 2.3 ([5]). *Suppose that R is not a Gorenstein ring. Then the following conditions are equivalent.*

- (1) R is a GGL ring (necessarily with respect to \mathfrak{c}).
- (2) K/R is a free R/\mathfrak{c} -module.
- (3) $K/\mathfrak{c} = K/\mathfrak{c}K$ is a free R/\mathfrak{c} -module.
- (4) S/R is a free R/\mathfrak{c} -module.
- (5) $S/\mathfrak{c} = S/\mathfrak{c}S$ is a free R/\mathfrak{c} -module.
- (6) $e_1(I) = \ell_R(R/\mathfrak{c}) \cdot \mathfrak{r}(R)$.

When this is the case, the following assertions hold true.

- (i) $K^2 = K^3$.

- (ii) R/\mathfrak{c} is a Gorenstein ring.
- (iii) $S/K \cong R/\mathfrak{c}$.

Let us note here the non-zero-divisor characterization given by [5] for GGL rings of higher dimension. To state it, let (R, \mathfrak{m}) be a Cohen-Macaulay local ring with $d = \dim R > 0$ and infinite residue class field, possessing the canonical module K_R . We then have the following.

Theorem 2.4 ([5]). *Suppose that R is not a Gorenstein ring. Then the following assertions hold true.*

- (1) *Let R be a GGL ring with respect to \mathfrak{a} and $d \geq 2$. Consider the exact sequence*

$$0 \rightarrow R \xrightarrow{\varphi} K_R \rightarrow C \rightarrow 0$$

of R -modules such that C is an Ulrich R -module with respect to \mathfrak{a} and the induced homomorphism $R/\mathfrak{a} \otimes_R \varphi : R/\mathfrak{a} \rightarrow K_R/\mathfrak{a}K_R$ is injective. Choose a superficial element $f \in \mathfrak{a}$ for C with respect to \mathfrak{a} and assume that f is R -regular. Then $R/(f)$ is a GGL ring with respect to $\mathfrak{a}/(f)$, possessing

$$0 \rightarrow R/(f) \xrightarrow{R/(f) \otimes_R \varphi} K_R/fK_R \rightarrow C/fC \rightarrow 0$$

to be a defining exact sequence.

- (2) *Let $f \in \mathfrak{m}$ be R -regular and assume that $R/(f)$ is a GGL ring with respect to $[\mathfrak{a} + (f)]/(f)$. Then R is a GGL ring with respect to $\mathfrak{a} + (f)$ and $f \notin \mathfrak{m}[\mathfrak{a} + (f)]$.*

3. PROOF OF THE MAIN PART OF THEOREM 1.2

Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring with $d = \dim R > 0$ and infinite residue class field, possessing the canonical module K_R . Let $I \subsetneq R$ be an ideal of R such that $I \cong K_R$ as an R -module. We choose a parameter ideal $Q = (f_1, f_2, \dots, f_d)$ of R so that $f_1 \in I$. We set $\mathfrak{q} = (f_2, f_3, \dots, f_d)$ and $J = I + \mathfrak{q}$. Let \mathfrak{a} be an \mathfrak{m} -primary ideal of R . In Sections 3 and 4 we throughout assume that R is not a Gorenstein ring. The purpose is to prove the equivalence between Conditions (1) and (2) in Theorem 1.2.

Let us begin with the following.

Proposition 3.1. $\mathfrak{q} \cap I = \mathfrak{q}I$ and $J \neq Q$.

Proof. We get $\mathfrak{q} \cap I = \mathfrak{q}I$, since \mathfrak{q} is a parameter ideal of the Gorenstein ring R/I . If $J = Q$, then $I = Q \cap I = (f_1) + (\mathfrak{q} \cap I)$, whence $I = (f_1)$ by the first equality. Therefore, R is a Gorenstein ring, which is impossible. \square

Theorem 3.2. *The following conditions are equivalent.*

- (1) $\mathfrak{a} = Q :_R J$, $\mathfrak{a}J = \mathfrak{a}Q$, and $e_J^1(R) = \ell_R(R/\mathfrak{a}) \cdot \mathfrak{r}(R)$.
- (2) $\mathfrak{a}J = \mathfrak{a}Q$ and the R/\mathfrak{a} -module J/Q is free.

When this is the case, R/\mathfrak{q} is a GGL ring with respect to $\mathfrak{a}/\mathfrak{q}$, whence so is the ring R with respect to \mathfrak{a} .

Proof. We may assume $\mathfrak{a}J = \mathfrak{a}Q$. Hence, Q is a reduction of J . Because $J/Q \neq (0)$ by Proposition 3.1, we get $\mathfrak{a} = Q :_R J$, once J/Q is R/\mathfrak{a} -free. Consequently, we may also

assume that $\mathfrak{a} = Q :_R J$. First, suppose that $d = 1$. Hence $J = I$. We set $K = \frac{I}{f_1}$ in the total ring of fractions of R . Then, since $\mathfrak{a}K = \mathfrak{a}$, we have $\mathfrak{a} = R : R[K]$ by Fact 2.2. Consequently, by Theorem 2.3, R is a GGL ring (necessarily with respect to \mathfrak{c} ; see Proposition 2.1) if and only if $e_I^1(R) = \ell_R(R/\mathfrak{a}) \cdot r(R)$. By Theorem 2.3, the former condition is also equivalent to saying that $I/Q (\cong K/R)$ is a free R/\mathfrak{a} -module, whence the equivalence of Conditions (1) and (2) follows.

Let us consider the case where $d \geq 2$. Assume that Condition (2) is satisfied. Let us check that $\bar{R} = R/\mathfrak{q}$ is a GGL ring. Set

$$\bar{R} = R/\mathfrak{q}, \quad \bar{Q} = Q/\mathfrak{q}, \quad \bar{J} = J/\mathfrak{q}, \quad \text{and} \quad \bar{\mathfrak{a}} = \mathfrak{a}/\mathfrak{q}.$$

We then have $\bar{J} = (I + \mathfrak{q})/\mathfrak{q} \cong I/\mathfrak{q}I = K_{\bar{R}}$, since $I \cong K_R$ and f_2, f_3, \dots, f_d is a regular sequence for the R -module I . Consequently, because $\bar{\mathfrak{a}} \cdot \bar{J} = \bar{\mathfrak{a}} \cdot \bar{Q}$ and \bar{J}/\bar{Q} is $\bar{R}/\bar{\mathfrak{a}}$ -free, from the case of $d = 1$ it follows that \bar{R} is a GGL ring (Fact 2.2 and Theorem 2.3), whence so is R with respect to \mathfrak{a} (Theorem 2.4 (2)).

We now assume that the implication (2) \Rightarrow (1) holds true for $d - 1$. Since Q is a reduction of J and the field R/\mathfrak{m} is infinite, there exist elements $h_1, h_2, \dots, h_d \in Q$ such that (i) $h_1 \in I$, (ii) $Q = (h_1, h_2, \dots, h_d)$, and (iii) h_2 is superficial for R with respect to J . This time, we consider the ring $\bar{R} = R/(h_2)$ and let $\bar{*}$ denote the reduction mod (h_2) . Then $\bar{I} = [I + (h_2)]/(h_2) \cong I/h_2I = K_{\bar{R}}$ and $\bar{h}_1 \in \bar{I}$. Condition (2) is clearly satisfied for the ring \bar{R} as for the ideals $\bar{\mathfrak{a}}, \bar{Q}$, and \bar{J} . Therefore, by the hypothesis of induction on d we get

$$e_{\bar{J}}^1(\bar{R}) = \ell_{\bar{R}}(\bar{R}/\bar{\mathfrak{a}}) \cdot r(\bar{R}).$$

Consequently, $e_J^1(R) = \ell_R(R/\mathfrak{a}) \cdot r(R)$, because $e_J^1(R) = e_{\bar{J}}^1(\bar{R})$ (remember that h_2 is superficial for R with respect to J).

The reverse implication (1) \Rightarrow (2) also follows by induction on d , chasing the above argument in the opposite direction. \square

Let us suppose, with the same notation as in Theorem 3.2, that the equivalent conditions of Theorem 3.2 are satisfied. Then we get the following.

Proposition 3.3. $J \subseteq \mathfrak{a}$. Hence $J^2 \subseteq Q$.

Proof. Suppose $d = 1$. We set $K = \frac{I}{f_1}$, $S = R[K]$, and $\mathfrak{c} = R : S$. Then since R is a GGL ring with respect to \mathfrak{a} , we get $\mathfrak{a} = \mathfrak{c}$ by Proposition 2.1 and $\mathfrak{c} = R : K$ (see Fact 2.2 and Theorem 2.3). Therefore, $I \subseteq \mathfrak{c}$, because \mathfrak{c} is an ideal of S and $f_1 \in \mathfrak{c}$ (note that $I = f_1K \subseteq R$). If $d > 1$, then passing to R/\mathfrak{q} , we have $J/\mathfrak{q} \subseteq \mathfrak{a}/\mathfrak{q}$, whence $J \subseteq \mathfrak{a}$. Therefore $J^2 \subseteq Q$, because $\mathfrak{a} = Q :_R J$. \square

We are now ready to prove the equivalence of Conditions (1) and (2) in Theorem 1.2.

Proof of the main part in Theorem 1.2. See Theorem 3.2 for the implication (2) \Rightarrow (1). To see the implication (1) \Rightarrow (2), we consider the exact sequence

$$(\#) \quad 0 \rightarrow R \xrightarrow{\varphi} I \rightarrow C \rightarrow 0$$

of R -modules such that C is an Ulrich R -module with respect to \mathfrak{a} and the induced homomorphism $R/\mathfrak{a} \otimes_R \varphi : R/\mathfrak{a} \rightarrow I/\mathfrak{a}I$ is injective. Let $f_1 = \varphi(1) \in I$. Then f_1 is a non-zerodivisor of R . Choose elements $f_2, f_3, \dots, f_d \in \mathfrak{a}$ so that in the ring $R' = R/(f_1)$

these elements generate a reduction of $\mathfrak{a} \cdot R'$. Then f_1, f_2, \dots, f_d is a system of parameters of R with $f_1 \in I$. Set $\mathfrak{q} = (f_2, f_3, \dots, f_d)$, $Q = (f_1) + \mathfrak{q}$, and $J = I + \mathfrak{q}$. Then, \mathfrak{q} is a parameter ideal of R/I , and because

$$\ell_{R'}(C/\mathfrak{q}C) = e_{\mathfrak{a} \cdot R'}^0(C) = e_{\mathfrak{a}}^0(C) = \ell_R(C/\mathfrak{a}C),$$

we get $\mathfrak{a}C = \mathfrak{q}C$. Therefore, since $\mathfrak{a}^n C = \mathfrak{q}^n C$ for all $n \in \mathbb{Z}$, f_2, f_3, \dots, f_d forms a super-regular sequence for C with respect to \mathfrak{a} , whence it is a superficial sequence for C with respect to \mathfrak{a} . We set $\bar{R} = R/\mathfrak{q}$, $\bar{I} = I/\mathfrak{q}I$, and $\bar{C} = C/\mathfrak{q}C$. Then, from exact sequence (#) we get the exact sequence

$$(\bar{\#}) \quad 0 \rightarrow \bar{R} \xrightarrow{\bar{\varphi}} \bar{I} \rightarrow \bar{C} \rightarrow 0$$

of \bar{R} -modules. By Theorem 2.4 (1) the ring \bar{R} is a GGL ring with respect to $\mathfrak{a}/\mathfrak{q}$, possessing sequence ($\bar{\#}$) to be a defining exact sequence. Consequently, because

$$J/\mathfrak{q} = (I + \mathfrak{q})/\mathfrak{q} \cong \bar{I} = K_{\bar{R}},$$

by Theorem 2.3 it follows that $\bar{\mathfrak{a}} \cdot \bar{J} = \bar{\mathfrak{a}} \cdot \bar{Q}$ and \bar{J}/\bar{Q} is $\bar{R}/\bar{\mathfrak{a}}$ -free, where

$$\bar{\mathfrak{a}} = \mathfrak{a}/\mathfrak{q}, \quad \bar{J} = J/\mathfrak{q}, \quad \text{and} \quad \bar{Q} = Q/\mathfrak{q}.$$

Hence J/Q is R/\mathfrak{a} -free and $\mathfrak{a}J \subseteq \mathfrak{a}Q + \mathfrak{q}$. Therefore

$$\mathfrak{a}J = (\mathfrak{a}Q + \mathfrak{q}) \cap \mathfrak{a}J = \mathfrak{a}Q + [\mathfrak{q} \cap \mathfrak{a}J] = \mathfrak{a}Q + [\mathfrak{q} \cap \mathfrak{a}I] \subseteq \mathfrak{a}Q + [\mathfrak{q} \cap I] = \mathfrak{a}Q + \mathfrak{q}I,$$

where the third equality follows from the fact that $\mathfrak{a}J = \mathfrak{a}I + \mathfrak{a}\mathfrak{q}$. Hence $\mathfrak{a}J = \mathfrak{a}Q$, because $I \subseteq \mathfrak{a}$ by Proposition 3.3. Thus Theorem 3.2 certifies that Condition (2) in Theorem 1.2 is satisfied for the ideals \mathfrak{a} , Q , and J . This completes the proof of the equivalence of Conditions (1) and (2) in Theorem 1.2. \square

4. PROOF OF THE LAST ASSERTIONS OF THEOREM 1.2

Let us show the last assertions of Theorem 1.2. In what follows, assume that our ideals Q and J satisfy the equivalent conditions in Theorem 3.2. Hence R (resp. R/\mathfrak{q}) is a GGL ring with respect to \mathfrak{a} (resp. $\mathfrak{a}/\mathfrak{q}$), and R/\mathfrak{a} is a Gorenstein ring by Theorem 2.3 (ii). To prove the last assertions of Theorem 1.2, we need some preliminaries. Let us maintain the same notation as in the proof of Theorem 3.2.

We begin with the following.

Lemma 4.1. $\mathfrak{q} \cap J^2 = \mathfrak{q}J$.

Proof. Remember that $J^2 = \mathfrak{q}J + I^2$, since $J = I + \mathfrak{q}$. We then have $\mathfrak{q} \cap J^2 = \mathfrak{q}J + (\mathfrak{q} \cap I^2)$, so that $\mathfrak{q} \cap J^2 = \mathfrak{q}J$, because $\mathfrak{q} \cap I^2 \subseteq \mathfrak{q} \cap I = \mathfrak{q}I$. \square

Proposition 4.2. *We set $L = Q :_R \mathfrak{a}$. Then the following assertions hold true.*

- (1) $L = J :_R \mathfrak{a}$ and $L^2 \subseteq Q$.
- (2) $L/J \cong R/\mathfrak{a}$ as an R -module.
- (3) $J^2/QJ \cong R/\mathfrak{a}$ as an R -module.
- (4) $\mathfrak{a}L = \mathfrak{a}Q$.
- (5) $L^2 = QL$.
- (6) $J^3 = QJ^2$ but $J^2 \neq QJ$.

Proof. (1), (2), (3) First, consider the case where $d = 1$. Let us maintain the notation of the proof of Proposition 3.3. Then $Q :_R \mathfrak{c} = I :_R \mathfrak{c} = f_1 S$. In fact, we have $\mathfrak{c} = K : S = R : K$ (see Fact 2.2 and Theorem 2.3) and hence $f_1 \in \mathfrak{c}$. Let $x \in R$. Then $x \cdot \mathfrak{c} \subseteq I$ if and only if $\frac{x}{f_1} \cdot \mathfrak{c} \subseteq K$. The latter condition is equivalent to saying that $\frac{x}{f_1} \in S$, since $K : \mathfrak{c} = K : (K : S) = S$. Thus $I :_R \mathfrak{c} = f_1 S$. Because $f_1 S \cdot \mathfrak{c} = f_1 \mathfrak{c} \subseteq Q = (f_1)$, we get $I :_R \mathfrak{c} = f_1 S \subseteq Q :_R \mathfrak{c}$. Hence $Q :_R \mathfrak{c} = I :_R \mathfrak{c} = f_1 S$. Consequently

$$(Q :_R \mathfrak{c})^2 = f_1(f_1 S) \subseteq Q = (f_1),$$

and $[Q :_R \mathfrak{c}]/I = f_1 S/f_1 K \cong S/K \cong R/\mathfrak{c}$ by Theorem 2.3, which proves Assertions (1) and (2), because $\mathfrak{a} = \mathfrak{c}$. Assertion (3) is now clear, since

$$I^2/f_1 I \cong K^2/K \cong R/\mathfrak{c}$$

by Theorem 2.3. Now consider the case where $d \geq 2$. To show Assertions (1) and (2), passing to the ring R/\mathfrak{q} , we can safely assume that $d = 1$, and we have already done with the case. Consider Assertion (3). We set $\bar{R} = R/\mathfrak{q}$ and denote by $\bar{*}$ the reduction mod \mathfrak{q} . Let $\varphi : J^2/QJ \rightarrow \bar{J}^2/f_1 \bar{J}$ be the canonical epimorphism. We then have $\text{Ker } \varphi = [J^2 \cap (\mathfrak{q} + f_1 J)]/QJ$. Hence, because $J^2 \cap \mathfrak{q} = \mathfrak{q}J$ by Lemma 4.1, we have

$$J^2 \cap (\mathfrak{q} + f_1 J) = f_1 J + (J^2 \cap \mathfrak{q}) = f_1 J + \mathfrak{q}J = QJ,$$

whence the required isomorphism $J^2/QJ \cong R/\mathfrak{a}$ follows.

(4) Suppose $d = 1$. Then $\mathfrak{c}L = \mathfrak{c} \cdot f_1 S = f_1 \cdot \mathfrak{c}$, whence the assertion follows. Suppose that $d \geq 2$ and that Assertion (4) holds true for $d-1$. Note that $Q = (f_1, f_1 + f_2) + (f_3, \dots, f_d)$. Then, because $R/(f_1 + f_2)$ and $R/(f_2)$ are GGL rings with respect to $\mathfrak{a}/(f_1 + f_2)$ and $\mathfrak{a}/(f_2)$ respectively, thanks to the hypothesis of induction on d , we get

$$\mathfrak{a}L \subseteq [\mathfrak{a}Q + (f_1 + f_2)] \cap [\mathfrak{a}Q + (f_2)] = \mathfrak{a}Q + \{(f_1 + f_2) \cap [\mathfrak{a}Q + (f_2)]\}.$$

Since $\mathfrak{a}Q + (f_2) \subseteq \mathfrak{a} \cdot (f_1 + f_2) + (f_2, f_3, \dots, f_d)$, we furthermore have that

$$\mathfrak{a}L \subseteq \mathfrak{a}Q + \{(f_1 + f_2) \cap [\mathfrak{a} \cdot (f_1 + f_2) + (f_2, f_3, \dots, f_d)]\} = \mathfrak{a}Q + (f_1 + f_2) \cdot (f_2, f_3, \dots, f_d) = \mathfrak{a}Q.$$

Hence $\mathfrak{a}L = \mathfrak{a}Q$.

(5) Let $x \in L^2$. Then, $x \in Q$, since $L^2 \subseteq Q$ by Assertion (1). We write $x = \sum_{i=1}^d f_i x_i$ with $x_i \in R$. Let $\alpha \in \mathfrak{a}$. Then, because

$$\alpha x = \sum_{i=1}^d f_i (\alpha x_i) \in \mathfrak{a}L^2 \subseteq Q^2$$

by Assertion (4), we get $\alpha x_i \in Q$ for all $1 \leq i \leq d$, whence $x_i \in Q :_R \mathfrak{a} = L$. Thus $L^2 = QL$.

(6) The equality $J^3 = QJ^2$ is a direct consequence of [13, Proposition 2.6], since $\mu_R(L/J) = 1$ by Assertion (2). Suppose that $J^2 = QJ$ and let $\bar{*}$ denote the reduction mod \mathfrak{q} . Then since $\bar{J} \cong K_{\bar{R}}$ and $\bar{J}^2 = \bar{Q} \cdot \bar{J}$, by [7, Theorem 3.7] \bar{R} is a Gorenstein ring, which is impossible. Hence $J^2 \neq QJ$. \square

Proposition 4.3. *The sequence f_2, f_3, \dots, f_d is a super-regular sequence of R with respect to J . Hence $\text{depth gr}_J(R) = d - 1$.*

Proof. To see the first assertion, it suffices to show that $\mathfrak{q} \cap J^{n+1} = \mathfrak{q}J^n$ for all $n \geq 1$. By Lemma 4.1 we may assume that $n \geq 2$ and that our assertion holds true for $n - 1$. Then, since $J^{n+1} = QJ^n = \mathfrak{q}J^n + f_1J^n$ by Proposition 4.2 (5), we have

$$\mathfrak{q} \cap J^{n+1} = \mathfrak{q}J^n + (\mathfrak{q} \cap f_1J^n).$$

Consequently, because $\mathfrak{q} \cap f_1J^n = f_1(\mathfrak{q} \cap J^n)$ (remember that f_1, f_2, \dots, f_d is an R -regular sequence), by the hypothesis of induction on n we have $\mathfrak{q} \cap f_1J^n \subseteq \mathfrak{q}J^n$. Hence $\mathfrak{q} \cap J^{n+1} = \mathfrak{q}J^n$. Consequently, $\text{depth gr}_J(R) \geq d - 1$. Suppose that $\text{depth gr}_J(R) = d$. Then, f_1, f_2, \dots, f_d is a super-regular sequence of R with respect to J , so that $Q \cap J^2 = QJ$. Therefore, $J^2 = QJ$, because $J^2 \subseteq Q$ by Proposition 4.2 (1), which contradicts Proposition 4.2 (6). Hence $\text{depth gr}_J(R) = d - 1$. \square

Let $\mathcal{T} = \mathcal{R}(Q)$ and $\mathcal{R} = \mathcal{R}(J)$ be the Rees algebras of Q and J respectively. We now consider the Sally module $\mathcal{S}_Q(J) = J\mathcal{R}/J\mathcal{T}$ of J with respect to Q (see [21]).

Theorem 4.4. $\mathcal{S}_Q(J) \cong (\mathcal{T}/\mathfrak{a}\mathcal{T})(-1)$ as a graded \mathcal{T} -module.

Proof. We set $\mathcal{S} = \mathcal{S}_Q(J)$ and denote, for each $n \in \mathbb{Z}$, by $[\mathcal{S}]_n$ the homogeneous component of \mathcal{S} of degree n . Then $[\mathcal{S}]_1 = J^2/QJ$ and $\mathcal{S} = \mathcal{T} \cdot [\mathcal{S}]_1$ ([12, Lemma 2.1]). Hence by Proposition 4.2 (3), we get an epimorphism $\varphi : (\mathcal{T}/\mathfrak{a}\mathcal{T})(-1) \rightarrow \mathcal{S}$ of graded \mathcal{T} -modules. Let $X = \text{Ker } \varphi$ and assume that $X \neq (0)$. We choose an element $\mathfrak{p} \in \text{Ass}_{\mathcal{T}} X$. Then, since $\mathfrak{p} \in \text{Ass}_{\mathcal{T}} \mathcal{T}/\mathfrak{a}\mathcal{T}$, and $\mathcal{T}/\mathfrak{a}\mathcal{T} = (R/\mathfrak{a})[X_1, X_2, \dots, X_d]$ is the polynomial ring over R/\mathfrak{a} (remember that \mathcal{T} is isomorphic to the symmetric algebra of Q over R), we have $\mathfrak{p} = \mathfrak{m}\mathcal{T}$. Then $\ell_{\mathcal{T}_{\mathfrak{p}}}((\mathcal{T}/\mathfrak{a}\mathcal{T})_{\mathfrak{p}}) = \ell_R(R/\mathfrak{a})$, while by [12, Proposition 2.2] we have

$$\ell_{\mathcal{T}_{\mathfrak{p}}}(\mathcal{S}_{\mathfrak{p}}) = e_J^1(R) - \ell_R(J/Q).$$

Therefore, $\ell_{\mathcal{T}_{\mathfrak{p}}}(\mathcal{S}_{\mathfrak{p}}) = \ell_R(R/\mathfrak{a})$, because $e_J^1(R) = \ell_R(R/\mathfrak{a}) \cdot r(R)$ by Theorem 3.2 and $\ell_R(J/Q) = \ell_R(R/\mathfrak{a}) \cdot (r(R) - 1)$ by Theorem 2.3. Thus $\ell_{\mathcal{T}_{\mathfrak{p}}}((\mathcal{T}/\mathfrak{a}\mathcal{T})_{\mathfrak{p}}) = \ell_{\mathcal{T}_{\mathfrak{p}}}(\mathcal{S}_{\mathfrak{p}})$, which forces $X_{\mathfrak{p}} = (0)$. This is absurd. Thus $(\mathcal{T}/\mathfrak{a}\mathcal{T})(-1) \cong \mathcal{S}$ as a graded \mathcal{T} -module. \square

Because

$$\ell_R(R/J^{n+1}) = e_J^0(R) \cdot \binom{n+d}{d} - [e_J^0(R) - \ell_R(R/J)] \cdot \binom{n+d-1}{d-1} - \ell_R([\mathcal{S}]_n)$$

for all $n \in \mathbb{Z}$ (see [12, Proposition 2.2]), by Theorem 4.4 we readily get the following.

Corollary 4.5. *The Hilbert function of R with respect to J is given by*

$$\ell_R(R/J^{n+1}) = e_J^0(R) \cdot \binom{n+d}{d} - [e_J^0(R) - \ell_R(R/J) + \ell_R(R/\mathfrak{a})] \cdot \binom{n+d-1}{d-1} + \ell_R(R/\mathfrak{a}) \cdot \binom{n+d-2}{d-2}$$

for $n \geq 1$. Hence, $e_J^2(R) = \ell_R(R/\mathfrak{a})$ if $d \geq 2$, and $e_J^i(R) = 0$ for all $3 \leq i \leq d$ if $d \geq 3$.

5. EXAMPLE

Let $S = k[[X, Y, Z, V]]$ be the formal power series ring over an infinite field k and let $\mathfrak{b} = \mathbb{I}_2 \begin{pmatrix} X^2 & Y+V & Z \\ Y & Z & X^3 \end{pmatrix}$ denote the ideal of S generated by 2×2 minors of the matrix $\begin{pmatrix} X^2 & Y+V & Z \\ Y & Z & X^3 \end{pmatrix}$. We set $R = S/\mathfrak{b}$. We denote by x, y, z, v the images of X, Y, Z, V in R , respectively. Then we have the following.

Example 5.1. The following assertions hold true.

- (1) R is a two-dimensional GGL ring with respect to $\mathfrak{a} = (x^2, y, z, v)$.
- (2) $\mathfrak{r}(R) = 2$ and $I = (x^2, y)$ is a canonical ideal of R .
- (3) Set $Q = (x^2, v)$ and $J = I + Q$. Then Q is a parameter ideal of R with $x^2 \in I$.
- (4) We have $\mathfrak{a} = Q :_R J$, $\mathfrak{a}J = \mathfrak{a}Q$, and $e_J^1(R) = \ell_R(R/\mathfrak{a}) \cdot \mathfrak{r}(R) = 4$.

Proof. Since

$$R/(v) \cong k[[X, Y, Z]]/\mathbb{I}_2\left(\begin{smallmatrix} X^2 & Y & Z \\ Y & Z & X^3 \end{smallmatrix}\right) \cong k[[t^3, t^7, t^8]]$$

where t denotes an indeterminate over k , we have $\dim R/(v) = 1$. Hence $\text{ht}_S \mathfrak{b} \geq 2$, so that R is a Cohen-Macaulay ring with $\dim R = 2$. Because $R/vR = k[[t^3, t^7, t^8]]$ is a GGL ring with respect to (t^6, t^7, t^8) and v is a non-zerodivisor of R , by Theorem 2.4 (2) R is a GGL ring with respect to \mathfrak{a} . To see that $I \cong K_R$, note that (t^6, t^7) is a canonical ideal of $k[[t^3, t^7, t^8]]$. Since $R/I = S/(X^2, Y, Z^2)$, the element v acts on R/I as a non-zerodivisor, so that $(v) \cap I = vI$. Hence $J/(v) = [I + (v)]/(v) \cong I/vI$. Because the ideal $J/(v)$ corresponds to (t^6, t^7) under the identification

$$R/(v) = k[[X, Y, Z]]/\mathbb{I}_2\left(\begin{smallmatrix} X^2 & Y & Z \\ Y & Z & X^3 \end{smallmatrix}\right) = k[[t^3, t^7, t^8]],$$

we see $\mathfrak{r}_R(I/vI) = 1$, where $\mathfrak{r}_R(*)$ stands for the Cohen-Macaulay type. Therefore $\mathfrak{r}_R(I) = 1$, whence $I \cong K_R$ because $(0) :_R I = (0)$. Since

$$R/Q \cong k[X, Y, Z, V]/(X^2, Y^2, Z^2, YZ, V),$$

we get $Q :_R J = Q :_R y = (x^2, y, z, v) = \mathfrak{a} \supseteq J$. It is direct to check that $\mathfrak{a}J = \mathfrak{a}Q$. The equality $e_J^1(R) = \ell_R(R/\mathfrak{a}) \cdot \mathfrak{r}(R) = 4$ follows from the fact that $\ell_R(R/(x^2, y, z, v)) = \mathfrak{r}(R) = 2$. \square

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