REMARKS ON ALMOST GORENSTEIN RINGS

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ABSTRACT. This paper investigates the relation between the almost Gorenstein properties for graded rings and for local rings. Once R is an almost Gorenstein graded ring, the localization R_M of R at the graded maximal ideal M is almost Gorenstein as a local ring. The converse does not hold true in general ([7, Theorems 2.7, 2.8], [8, Example 8.8]). However, it does for one-dimensional graded domains with mild conditions, which we clarify in the present paper. We explore the defining ideals of almost Gorenstein numerical semigroup rings as well.

1. INTRODUCTION

An almost Gorenstein ring, which we focus on in the present paper, is one of the attempts to generalize Gorenstein rings. The motivation for this generalization comes from the strong desire to stratify Cohen-Macaulay rings, finding new and interesting classes which naturally include that of Gorenstein rings. The theory of almost Gorenstein rings was introduced by Barucci and Fröberg [1] in the case where the local rings are analytically unramified and of dimension one, e.g., numerical semigroup rings over a field. In 2013, their work inspired Goto, the second author of this paper, and Phuong [6] to extend the notion of almost Gorenstein rings for arbitrary one-dimensional Cohen-Macaulay local rings. More precisely, a one-dimensional Cohen-Macaulay local ring R is called *almost Gorenstein* if R admits a canonical ideal I of R such that $\mathrm{e}_1(I) \le \mathrm{r}(R)$, where $\mathrm{e}_1(I)$ denotes the first Hilbert coefficients of R with respect to I and $\mathrm{r}(R)$ is the Cohen-Macaulay type of R ([6, Definition 3.1]). Two years later, Goto, Takahashi, and the first author of this paper [8] defined the almost Gorenstein graded/local rings of arbitrary dimension. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring. Then R is said to be an *almost Gorenstein local ring* if R admits a canonical module K_R and there exists an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$ ([8, Definition 3.3]). Here, $\mu_R(-)$ (resp. $\mathrm{e}^0_{\mathfrak{m}}(-)$) denotes the number of elements in a minimal system of generators (resp. the multiplicity with respect to $\mathfrak{m}$). When $\dim R = 1$, if R is an almost Gorenstein local ring in the sense of [8], then R is almost Gorenstein in the sense of [6], and vice versa provided the field $R/\mathfrak{m}$ is infinite ([8, Proposition 3.4]). However, the converse does not hold in general ([8, Remark 3.5], see also [6, Remark 2.10]). Similarly as in local rings, a Cohen-Macaulay graded ring $R = \oplus_{n \ge 0} R_n$ with $k = R_0$ a local ring is called an *almost Gorenstein graded ring* if R admits a graded canonical module K_R

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and there exists an exact sequence

$$0 \to R \to \mathrm{K}_R(-a) \to C \to 0$$

of graded R-modules with $\mu_R(C) = \mathrm{e}_M^0(C)$ ([8, Definition 8.1]). Here, M is the graded maximal ideal of R, $a = \mathrm{a}(R)$ is the a-invariant of R, and $\mathrm{K}_R(-a)$ denotes the graded R-module whose underlying R-module is the same as that of K_R and whose grading is given by $[\mathrm{K}_R(-a)]_n = [\mathrm{K}_R]_{n-a}$ for all $n \in \mathbb{Z}$.

Every Gorenstein local/graded ring is almost Gorenstein. The definitions assert that once R is an almost Gorenstein local (resp. graded) ring, either R is a Gorenstein ring, or even though R is not a Gorenstein ring, the local (resp. graded) ring R is embedded into the module K_R (resp. the graded module $\mathrm{K}_R(-a)$) and the difference C behaves well. Moreover, if R is an almost Gorenstein graded ring, then the localization R_M of R at M is an almost Gorenstein local ring, which readily follows from the definition. However, the converse does not hold true in general ([7, Theorems 2.7, 2.8], [8, Example 8.8]), even though it does for determinantal rings of generic, as well as symmetric, matrices over a field ([2, Theorem 1.1], [12, Theorem 1.1]).

In this paper we investigate the question of when the converse holds in one-dimensional rings. Throughout this paper, unless otherwise specified, let $R = \bigoplus_{n \geq 0} R_n$ be a one-dimensional Noetherian $\mathbb{Z}$-graded integral domain admitting a graded canonical module K_R. We assume $k = R_0$ is a field, and $R_n \neq (0)$ and $R_{n+1} \neq (0)$ for some $n \geq 0$. Let M denote the graded maximal ideal of R.

With this notation this paper aims at proving the following result.

Theorem 1.1. *There exists a graded canonical ideal J of R containing a parameter ideal as a reduction, and the following conditions are equivalent.*

(1) *R is an almost Gorenstein graded ring.*
(2) *R_M is an almost Gorenstein local ring in the sense of* [6, Definition 3.1].
(3) *R_M is an almost Gorenstein local ring in the sense of* [8, Definition 3.3].

Theorem 1.1 guarantees the existence of a (graded) canonical ideal admitting a parameter ideal as a reduction; hence the proof of [8, Proposition 3.4] shows that the conditions (2) and (3) are equivalent even though the field R/M is finite. As we mentioned, by the definition of almost Gorenstein local/graded rings we only need to verify the implication (2) $\Rightarrow$ (1). What makes (2) $\Rightarrow$ (1) interesting and difficult is that the implication is not true in general.

Let us now explain how this paper is organized. We prove Theorem 1.1 in Section 2 after preparing some auxiliaries. We also explore the explicit generators of defining ideals of almost Gorenstein numerical semigroup rings. Section 3 is devoted to providing examples illustrating Theorem 1.1.

2. Proof of Theorem 1.1

Let S be the set of non-zero homogeneous elements in R. The localization $S^{-1}R = K[t, t^{-1}]$ of R with respect to S is a *simple* graded ring, i.e., every non-zero homogeneous element is invertible, where t is a homogeneous element of degree 1 (remember that $R_n \neq (0)$ and $R_{n+1} \neq (0)$) which is transcendental over k, and $K = [S^{-1}R]_0$ is a field. Let $\overline{R}$ be the integral closure of R in its quotient field $\mathrm{Q}(R)$.

We begin with the following, which has already appeared in [3, Lemma 2.1]. Because it plays an important role in our argument, we include a brief proof for the sake of completeness.

Lemma 2.1. *The equality* $\overline{R} = K[t]$ *holds in* $\mathrm{Q}(R)$.

Proof. As R is an integral domain, we obtain that $\overline{R}$ is a graded ring and $\overline{R} \subseteq S^{-1}R = K[t,t^{-1}]$ ([13, page 157]). Since the field k is Nagata, so is the finitely generated k-algebra R. Hence $\overline{R}$ is a module-finite extension of R. One can verify that $R_n = (0)$ for all $n < 0$ and $R_0 = k$; hence $[\overline{R}]_n = (0)$ for all $n < 0$, $L = [\overline{R}]_0$ is a field, and $k \subseteq L \subseteq K$. We set $N = \bigoplus_{n>0}[\overline{R}]_n$. Since the local ring $\overline{R}_N$ of $\overline{R}$ at the maximal ideal N is a DVR, the ideal N is principal. Choose a homogeneous element $f \in \overline{R}$ of degree $q > 0$ with $N = f\overline{R}$. Then

$$\overline{R} = L[N] = L[f] \subseteq S^{-1}R = K[t,t^{-1}].$$

Besides, because $\overline{R}[f^{-1}] = L[f,f^{-1}]$ is a simple graded ring and $R \subseteq \overline{R}[f^{-1}]$, we have $S^{-1}R \subseteq \overline{R}[f^{-1}] = L[f,f^{-1}]$. Thus $K[t,t^{-1}] = L[f,f^{-1}]$, so that $K = L$ and $q = 1$. Consequently, $\overline{R} = L[f] = K[f] = K[t]$, as desired. $\square$

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Since $S^{-1}\mathrm{K}_R \cong S^{-1}R$ as a graded $S^{-1}R$-modules, we have an injective homomorphism $0 \to \mathrm{K}_R \xrightarrow{\varphi} S^{-1}R$ of graded R-modules. Choose $s \in S$ such that $s \cdot \varphi(\mathrm{K}_R) \subsetneq R$. Set $J = s \cdot \varphi(\mathrm{K}_R)$ and $q = -\deg s$. Then $\mathrm{K}_R(q) \cong J$ as a graded R-module. By setting $a = \mathrm{a}(R)$ and $\ell = -(q+a)$, we then have $J_\ell \neq (0)$ and $J_n = (0)$ for all $n < \ell$. We now choose a non-zero homogeneous element $f \in J_\ell$. Note that $\ell > 0$ and $f \in \left[\overline{R}\right]_\ell$. Therefore

$$J\overline{R} = t^\ell\overline{R} = f\overline{R}.$$

This shows the equality $J^{r+1} = fJ^r$ holds where $r = \mu_R(\overline{R}) - 1$. Here, we recall that $\mu_R(-)$ denotes the number of elements in a minimal system of generators. Hence, J is a graded canonical ideal of R which contains a parameter ideal (f) of R as a minimal reduction.

As for the equivalent conditions, we only need to show the implication $(2) \Rightarrow (1)$.

$(2) \Rightarrow (1)$ We consider the exact sequence

$$0 \to R \xrightarrow{\psi} J(\ell) \to C \to 0$$

of graded R-modules with $\psi(1) = f$. Since $\mathfrak{m}I = \mathfrak{m}f$, we get $MJ = Mf$ and hence $MC = (0)$, i.e. C is an Ulrich R-module. Therefore R is an almost Gorenstein graded ring because $J(\ell) \cong \mathrm{K}_R(-a)$. This completes the proof. $\square$

Let $\mathbb{N}$ be the set of non-negative integers. A *numerical semigroup* is a non-empty subset H of $\mathbb{N}$ which is closed under addition, contains the zero element, and whose complement in $\mathbb{N}$ is finite. Every numerical semigroup H admits a finite minimal system of generators, i.e., there exist positive integers $a_1, a_2, \dots, a_\ell \in H$ $(\ell \geq 1)$ with $\gcd(a_1, a_2, \dots, a_\ell) = 1$ such that $H = \langle a_1, a_2, \dots, a_\ell \rangle = \left\{\sum_{i=1}^{\ell} c_i a_i \;\middle|\; c_i \in \mathbb{N} \text{ for all } 1 \leq i \leq \ell\right\}$. For a field k, the ring $k[H] = k[t^{a_1}, t^{a_2}, \dots, t^{a_\ell}]$ (or $k[[H]] = k[[t^{a_1}, t^{a_2}, \dots, t^{a_\ell}]]$) is called the *numerical semigroup ring* of H over k, where t denotes an indeterminate over k. Note that the ring $k[H]$ satisfies the assumption of Theorem 1.1. Let M be the graded maximal ideal of R. Since every non-zero ideal in numerical semigroup rings admits its minimal reduction, the two definitions for almost Gorenstein local rings [6, Definition 3.1] and [8, Definition 3.3] are equivalent. In addition, the local ring $k[H]_M$ is almost Gorenstein if and only if $k[[H]]$ is an almost Gorenstein local ring; equivalently, the semigroup H is almost symmetric ([1, Proposition 29], [10, Theorem 2.4]).

Hence we have the following, which recovers a result of Goto, Kien, Matsuoka, and Truong.

Corollary 2.2 ([5, Proposition 2.3]). *A numerical semigroup ring $k[[H]]$ is an almost Gorenstein local ring if and only if $k[H]$ is an almost Gorenstein graded ring, or equivalently, H is almost symmetric.*

In the rest of this section, let $R = k[H]$ be the numerical semigroup ring over k and

$$\mathrm{c}(H) = \min\{n \in \mathbb{Z} \mid m \in H \text{ for all } m \in \mathbb{Z} \text{ with } m \geq n\}$$

the conductor of H. We set $\mathrm{f}(H) = \mathrm{c}(H) - 1$. Then $\mathrm{f}(H) = \max\ (\mathbb{Z} \setminus H)$, which is called the Frobenius number of H. Let

$$\mathrm{PF}(H) = \{n \in \mathbb{Z} \setminus H \mid n + a_i \in H \text{ for all } 1 \leq i \leq \ell\}$$

be the set of pseudo-Frobenius numbers of H. The graded canonical module K_R has the form

$$\mathrm{K}_R = \sum_{c \in \mathrm{PF}(H)} Rt^{-c}$$

whence $\mathrm{f}(H) = \mathrm{a}(R)$ and $\#\mathrm{PF}(H) = \mathrm{r}(R)$ ([9, Example (2.1.9), Definition (3.1.4)]). Here, $\mathrm{r}(R)$ denotes the Cohen-Macaulay type of R. We write $\mathrm{PF}(H) = \{c_1 < c_2 < \cdots < c_r\}$; hence $r = \mathrm{r}(R)$ and $c_r = \mathrm{a}(R)$. For each $1 \leq i \leq r$, we set $b_i = -c_{r+1-i}$. Thus $\mathrm{K}_R = \sum_{i=1}^{r} Rt^{b_i}$.

Let $S = k[x_1, x_2, ..., x_\ell]$ be the weighted polynomial ring over the field k with $x_i \in S_{a_i}$ for all $1 \leq i \leq \ell$. Consider the homomorphism $\varphi : S \to R$ of graded k-algebras defined by $\varphi(x_i) = t^{a_i}$ for all $1 \leq i \leq \ell$.

Suppose R is almost Gorenstein, but not a Gorenstein ring. Then $r \geq 2$. Since $R \subseteq t^{-b_1}\mathrm{K}_R = \sum_{i=1}^{r} Rt^{b_i - b_1} \subseteq \overline{R}$, we have $M\mathrm{K}_R \subseteq Rt^{b_1}$, where $M = (t^h \mid 0 < h \in H)$ is the graded maximal ideal of R. Hence, $c_r - c_i = c_{r-i}$ for all $1 \leq i \leq r-1$ (see e.g., [5, Proposition 2.3], [6, Theorem 3.11], [8, Definition 8.1], and [10, Theorem 2.4]). We consider the graded S-linear map

$$F = \begin{array}{c} S(-b_1) \\ \oplus \\ S(-b_2) \\ \oplus \\ \vdots \\ \oplus \\ S(-b_r) \end{array} \xrightarrow{\varepsilon} \mathrm{K}_R \longrightarrow 0$$

defined by $\varepsilon(\boldsymbol{e}_i) = t^{b_i}$ for each $2 \leq i \leq \ell$ and $\varepsilon(\boldsymbol{e}_1) = -t^{b_1}$, where $\{\boldsymbol{e}_i\}_{1 \leq i \leq r}$ denotes the standard basis of F. Set $L = \mathrm{Ker}\,\varepsilon$. For each $2 \leq i \leq r$ and $1 \leq j \leq \ell$, we have $x_j t^{b_i} \in Rt^{b_1}$. Choose a homogeneous element $y_{ij} \in S$ of degree $a_j + b_i - b_1$ such that $x_j t^{b_i} - y_{ij} t^{b_1} = 0$; hence $x_j \boldsymbol{e}_i + y_{ij}\boldsymbol{e}_1 \in L$. Since $\{x_j \boldsymbol{e}_i + y_{ij}\boldsymbol{e}_1\}_{2 \leq i \leq r,\ 1 \leq j \leq \ell}$ forms a part of minimal basis of L, we have $q \geq (r-1)\ell$, where $q = \mu_S(L)$. Let $m = q - (r-1)\ell$. If $m > 0$, then we can choose homogeneous elements $z_1, z_2, \ldots, z_m \in S$ such that $\{x_j \boldsymbol{e}_i + y_{ij}\boldsymbol{e}_1, z_1\boldsymbol{e}_1, z_2\boldsymbol{e}_1, \ldots, z_m\boldsymbol{e}_1\}_{2 \leq i \leq r,\ 1 \leq j \leq \ell}$ is a minimal basis of L. Hence

$$G \xrightarrow{\mathbb{M}} F \xrightarrow{\varepsilon} \mathrm{K}_R \longrightarrow 0$$

gives a minimal presentation of K_R, where G denotes a graded free S-module of rank q and the $r \times q$ matrix $\mathbb{M}$ has the following form

$$\mathbb{M} = \left[\begin{array}{cccc|cccc|c|cccc|cccc} y_{21} & y_{22} & \cdots & y_{2\ell} & y_{31} & y_{32} & \cdots & y_{3\ell} & \cdots & y_{r1} & y_{r2} & \cdots & y_{r\ell} & z_1 & z_2 & \cdots & z_m \\ x_1 & x_2 & \cdots & x_\ell & & 0 & & & 0 & & 0 & & & & 0 & & \\ & 0 & & & x_1 & x_2 & \cdots & x_\ell & 0 & & 0 & & & & 0 & & \\ & \vdots & & & & \vdots & & & \ddots & & \vdots & & & & \vdots & & \\ & 0 & & & & 0 & & & 0 & x_1 & x_2 & \cdots & x_\ell & & 0 & & \end{array}\right].$$

Note that $b_i - b_1 = c_{i-1}$ and $a_j + (b_i - b_1) \in H$ (remember that $c_r - c_i = c_{r-i}$). We write $a_j + (b_i - b_1) = d_1a_1 + d_2a_2 + \cdots + d_\ell a_\ell$ for some $d_i \in \mathbb{N}$. Then, because $b_i - b_1 \notin H$, we have $d_j = 0$. As y_{ij} has degree $a_j + b_i - b_1$, we may choose

$$y_{ij} = \prod_{1 \le k \le \ell,\ k \ne j} x_k^{d_k} \quad \text{for all} \quad 2 \le i \le r,\ 1 \le j \le \ell.$$

With this notation we reach the following, where, for each $t \ge 1$, $I_t(\mathbb{X})$ denotes the ideal of S generated by the $t \times t$ minors of a matrix $\mathbb{X}$.

Theorem 2.3. *Suppose that $R = k[H]$ is almost Gorenstein, but not a Gorenstein ring. Then, for each $2 \le i \le r$, the difference $\deg y_{ij} - \deg x_j\ (= c_{i-1})$ is constant for every $1 \le j \le \ell$, and the defining ideal of R has the following form*

$$\operatorname{Ker}\varphi = \sum_{i=2}^{r} I_2\begin{pmatrix} y_{i1} & y_{i2} & \cdots & y_{i\ell} \\ x_1 & x_2 & \cdots & x_\ell \end{pmatrix} + (z_1, z_2, \cdots, z_m).$$

Remark 2.4. For a higher dimensional *semi-Gorenstein* ring A, i.e., a special class of almost Gorenstein ring, the form of the defining ideals can be determined by the minimal free resolution of A ([8, Theorem 7.8]). Our contribution in Theorem 2.3 is that we succeeded in writing y_{ij} concretely in case of numerical semigroup rings.

Remark 2.5. When the almost symmetric semigroup H is minimally generated by four elements, Eto provided an explicit minimal system of generators of defining ideals of the semigroup rings $k[H]$ by using the notion of RF-matrices ([4, Section 5]).

Example 2.6. The semigroup ring $R = k[H]$ for a numerical semigroup H described below is an almost Gorenstein graded ring, and its defining ideal $\operatorname{Ker}\varphi$ is given by the following form.

(1) Let $H = \langle 7, 8, 13, 17, 19 \rangle$. Then $\mathrm{PF}(H) = \{6, 9, 12, 18\}$ and

$$\begin{aligned} \operatorname{Ker}\varphi &= I_2\begin{pmatrix} x_3 & x_1^2 & x_5 & x_1x_2^2 & x_2x_4 \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix} + I_2\begin{pmatrix} x_2^2 & x_4 & x_1^2x_2 & x_3^2 & x_1x_2x_3 \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix} \\ &+ I_2\begin{pmatrix} x_5 & x_1x_3 & x_2x_4 & x_1^3x_2 & x_1^2x_4 \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix}. \end{aligned}$$

(2) Let $H = \langle 11, 13, 14, 16, 31 \rangle$. Then $\mathrm{PF}(H) = \{15, 17, 19, 34\}$ and

$$\begin{aligned} \operatorname{Ker}\varphi &= I_2\begin{pmatrix} x_2^2 & x_3^2 & x_2x_4 & x_5 & x_1^3x_2 \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix} + I_2\begin{pmatrix} x_3^2 & x_3x_4 & x_5 & x_1^3 & x_4^3 \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix} \\ &+ I_2\begin{pmatrix} x_3x_4 & x_4^2 & x_1^3 & x_1^2x_2 & x_1x_2^3 \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix} + (x_1x_4 - x_2x_3). \end{aligned}$$

(3) Let $H = \langle 13,15,16,18,19\rangle$. Then $\mathrm{PF}(H) = \{17,20,23,40\}$ and

$$\begin{aligned}\operatorname{Ker}\varphi &= \mathrm{I}_2\begin{pmatrix} x_2^2 & x_3^2 & x_2x_4 & x_3x_5 & x_4^2 \\ x_1 & x_2 & x_3 & x_4 & x_5\end{pmatrix} + \mathrm{I}_2\begin{pmatrix} x_2x_4 & x_3x_5 & x_4^2 & x_5^2 & x_1^3 \\ x_1 & x_2 & x_3 & x_4 & x_5\end{pmatrix} \\ &+ \mathrm{I}_2\begin{pmatrix} x_4^2 & x_5^2 & x_1^3 & x_1^2x_2 & x_1^2x_3 \\ x_1 & x_2 & x_3 & x_4 & x_5\end{pmatrix} + (x_1x_4 - x_2x_3, x_1x_5 - x_3^2, x_2x_5 - x_3x_4).\end{aligned}$$

The next provides an explicit minimal system of generators for defining ideals of $R = k[H]$ when R has *minimal multiplicity*, i.e., the embedding dimension is equal to the multiplicity.

Corollary 2.7 (cf. [8, Corollary 7.10])**.** *Suppose that $R = k[H]$ is almost Gorenstein, but not a Gorenstein ring. If R has minimal multiplicity, the defining ideal of R has the following form*

$$\operatorname{Ker}\varphi = \sum_{i=2}^{r} \mathrm{I}_2\begin{pmatrix} y_{i1} & y_{i2} & \cdots & y_{i\ell} \\ x_1 & x_2 & \cdots & x_\ell\end{pmatrix}.$$

Proof. We maintain the notation as in this section. Since R has minimal multiplicity, by [11, Theorem 1] $q = (\ell-2)\binom{\ell}{\ell-1} = (\ell-2)\ell = (r-1)\ell$, so that $m = 0$. $\square$

3. Examples of Theorem 1.1

We close this paper by providing some examples. In this section, the almost Gorenstein property for local rings refers to the definition in the sense of [8, Definition 3.3]. The first example indicates that Theorem 1.1 does not hold unless R is an integral domain.

Example 3.1 ([8, Example 8.8])**.** Let $U = k[s,t]$ be the polynomial ring over a field k and set $R = k[s, s^3t, s^3t^2, s^3t^3]$. We regard U as a $\mathbb{Z}$-graded ring under the grading $k = U_0$ and $s,t \in U_1$. Let M be the graded maximal ideal of R. Then the following assertions hold true.

(1) R, R/sR are not almost Gorenstein graded rings.
(2) R_M, R_M/sR_M are almost Gorenstein local rings.

Proof. Let $S = k[X,Y,Z,W]$ be the polynomial ring over k. We consider S as a $\mathbb{Z}$-graded ring with $k = S_0$, $X \in S_1$, $Y \in S_4$, $Z \in S_5$, and $W \in S_6$. Let $\varphi : S \to R$ be the graded k-algebra map such that $\varphi(X) = s$, $\varphi(Y) = s^3t$, $\varphi(Z) = s^3t^2$, and $\varphi(W) = s^3t^3$. By [8, Example 8.8], R is not almost Gorenstein graded ring with $\mathrm{a}(R) = -2$, but the local ring R_M is almost Gorenstein.

The exact sequence $0 \to R(-1) \xrightarrow{s} R \to R/sR \to 0$ of graded R-modules induces the isomorphism $\mathrm{K}_{(R/sR)} \cong (\mathrm{K}_R/s\mathrm{K}_R)(1)$. Note that $\mathrm{a}(R/sR) = -1$. If R/sR is an almost Gorenstein graded ring, we can choose an exact sequence

$$0 \to R/sR \xrightarrow{\Psi} (\mathrm{K}_R/s\mathrm{K}_R)(2) \to D \to 0$$

of graded R-modules such that $MD = (0)$. Write $\Psi(1) = \overline{\xi}$ with $\xi \in [\mathrm{K}_R]_2$. We consider

$$R \xrightarrow{\Phi} \mathrm{K}_R(2) \to C \to 0$$

where $\Phi(1) = \xi$. Then $C/sC \cong D$. As $MD = (0)$, we get $\dim_R C \le 1$, Hence the map Φ is injective ([8, Lemma 3.1]), and s is a non-zerodivisor on C because $R/sR \otimes_R \Phi = \Psi$. Thus $\mu_R(C) = \mathrm{e}_M^0(C)$. This makes a contradiction. As X is superficial for $S/(Y,Z,W)$ with respect to the maximal ideal of S, by [8, Theorem 3.7 (2)], we conclude that R_M/sR_M is almost Gorenstein as a local ring. $\square$

Remark 3.2. We maintain the same notation as in Example 3.1. Let $T = k[Y,Z,W]$ be the polynomial ring over k. Note that $R/sR \cong T/(YW - Z^2, YZ, Y^2) = T/\mathrm{I}_2\left(\begin{smallmatrix} 0 & Y & Z \\ Y & Z & W \end{smallmatrix}\right) = V$. If we consider T as a $\mathbb{Z}$-graded ring under the grading $k = T_0$, $Y \in T_4$, $Z \in T_5$, and $W \in T_6$, as we have shown in Example 3.1 the ring V is not almost Gorenstein as a graded ring. Whereas, if we consider T as a $\mathbb{Z}$-graded ring with $k = T_0$ and $Y,Z,W \in T_1$, the T-module V has a graded minimal free resolution of the form

$$0 \longrightarrow \begin{matrix} T(-3) \\ \oplus \\ T(-3) \end{matrix} \xrightarrow{\left[\begin{smallmatrix} 0 & Y \\ Y & Z \\ Z & W \end{smallmatrix}\right]} \begin{matrix} T(-2) \\ \oplus \\ T(-2) \\ \oplus \\ T(-2) \end{matrix} \xrightarrow{[\Delta_1 \ -\Delta_2 \ \Delta_3]} T \xrightarrow{\varepsilon} V \longrightarrow 0$$

where $\Delta_1 = YW - Z^2$, $\Delta_2 = -YZ$, and $\Delta_3 = -Y^2$. Taking K_T-dual, we get the resolution

$$0 \longrightarrow T(-3) \xrightarrow{\left[\begin{smallmatrix} \Delta_1 \\ -\Delta_2 \\ \Delta_3 \end{smallmatrix}\right]} \begin{matrix} T(-1) \\ \oplus \\ T(-1) \\ \oplus \\ T(-1) \end{matrix} \xrightarrow{\left[\begin{smallmatrix} 0 & Y & Z \\ Y & Z & W \end{smallmatrix}\right]} \begin{matrix} T \\ \oplus \\ T \end{matrix} \xrightarrow{\varepsilon} \mathrm{K}_V \longrightarrow 0$$

of K_V as a graded T-module. We then consider the homomorphism

$$V \xrightarrow{\Phi} \mathrm{K}_V \to C \to 0$$

of graded T-modules defined by $\Phi(1) = \xi$, where $\xi = \varepsilon\left(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right)$. The isomorphisms $C \cong \mathrm{K}_V/V\xi \cong T/(Y,Z,W)$ guarantee that Φ is injective and $NC = (0)$ where $N = (Y,Z,W)T$. Thus V is an almost Gorenstein graded ring. Hence the almost Gorenstein property for graded rings depends on the choice of its gradings.

Example 3.3. Let $S = k[X,Y,Z]$ be the polynomial ring over a field k. We consider S as a $\mathbb{Z}$-graded ring under the grading $k = S_0$, $X \in S_3$, $Y \in S_1$, and $Z \in S_2$. Set $R = S/(Z^3 - X^2, XY, YZ)$. Then R is not an almost Gorenstein graded ring, but the local ring R_M is almost Gorenstein, where M denotes the graded maximal ideal of R.

Proof. Let $I = (Z^3 - X^2, XY, YZ)$. Note that $I = (X,Z) \cap (Z^3 - X^2, Y) = \mathrm{I}_2\left(\begin{smallmatrix} Z^2 & X & Y \\ X & Z & 0 \end{smallmatrix}\right)$. Thus R is a Cohen-Macaulay reduced ring with $\dim R = 1$. Note that

$$0 \longrightarrow \begin{matrix} S(-7) \\ \oplus \\ S(-6) \end{matrix} \xrightarrow{\left[\begin{smallmatrix} Z^2 & X \\ X & Z \\ Y & 0 \end{smallmatrix}\right]} \begin{matrix} S(-3) \\ \oplus \\ S(-4) \\ \oplus \\ S(-6) \end{matrix} \xrightarrow{[\Delta_1 \ -\Delta_2 \ \Delta_3]} S \xrightarrow{\varepsilon} R \longrightarrow 0$$

gives a graded minimal free resolution of R, where $\Delta_1 = -YZ$, $\Delta_2 = XY$, and $\Delta_3 = Z^3 - X^2$. Hence we get the resolution of K_R below

$$0 \longrightarrow S(-6) \xrightarrow{\left[\begin{smallmatrix} \Delta_1 \\ -\Delta_2 \\ \Delta_3 \end{smallmatrix}\right]} \begin{matrix} S(-3) \\ \oplus \\ S(-2) \\ \oplus \\ S \end{matrix} \xrightarrow{\left[\begin{smallmatrix} Z^2 & X & Y \\ X & Z & 0 \end{smallmatrix}\right]} \begin{matrix} S(-1) \\ \oplus \\ S \end{matrix} \xrightarrow{\varepsilon} \mathrm{K}_R \longrightarrow 0.$$

This shows $\mathrm{a}(R) = 1$ and $[\mathrm{K}_R]_{-1} = k\xi$, where $\xi = \varepsilon\left(\binom{1}{0}\right)$. Hence, for each homomorphism $\varphi : R \to \mathrm{K}_R(-1)$ of graded S-modules with $\varphi \neq 0$, we see that $\operatorname{Im}\varphi = R\xi$. Therefore

$$(\mathrm{K}_R/R\xi)(-1) \cong S/(X,Z)$$

which implies the map φ is not injective; see [8, Lemma 3.1 (2)]. So R is not almost Gorenstein as a graded ring. On the flip side, the elementary row operation

$$\begin{pmatrix} Z^2 & X & Y \\ X & Z & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} Z^2+X & X+Z & Y \\ X & Z & 0 \end{pmatrix}$$

and the equality $(Z^2+X, X+Z, Y) = (X,Y,Z)$ in the local ring S_N where $N = (X,Y,Z)S$ guarantee that R_M is an almost Gorenstein local ring by [8, Theorem 7.8]. $\square$

Example 3.3 shows Theorem 1.1 is no longer true even when R is reduced. As we show next, there is a counterexample of Theorem 1.1 in case of homogenous reduced rings as well.

Example 3.4. Let $S = k[X,Y,Z]$ be the polynomial ring over a field k. We consider S as a $\mathbb{Z}$-graded ring under the grading $k = S_0$ and $X,Y,Z \in S_1$. Set $R = S/I$, where $I = (X,Y) \cap (Y,Z) \cap (Z,X) \cap (X, Y+Z)$. Then R is not an almost Gorenstein graded ring, but the local ring R_M is almost Gorenstein, where M denotes the graded maximal ideal of R.

Proof. Note that I is an radical ideal of R and $I = (XY, XZ, YZ(Y+Z)) = \mathrm{I}_2\begin{pmatrix} Y+Z & 0 & Y \\ 0 & X & YZ \end{pmatrix}$. Then the homogeneous ring R is Cohen-Macaulay, reduced, and of dimension one. Set $\Delta_1 = XY$, $\Delta_2 = YZ(Y+Z)$, and $\Delta_3 = X(Y+Z)$. Since

$$0 \longrightarrow \begin{matrix} S(-3) \\ \oplus \\ S(-4) \end{matrix} \xrightarrow{\begin{bmatrix} Y+Z & 0 \\ 0 & X \\ Y & YZ \end{bmatrix}} \begin{matrix} S(-2) \\ \oplus \\ S(-3) \\ \oplus \\ S(-2) \end{matrix} \xrightarrow{[\Delta_1\ -\Delta_2\ \Delta_3]} S \xrightarrow{\varepsilon} R \longrightarrow 0$$

forms a graded minimal free resolution of R, we get the resolution

$$0 \longrightarrow S(-3) \xrightarrow{\begin{bmatrix} \Delta_1 \\ -\Delta_2 \\ \Delta_3 \end{bmatrix}} \begin{matrix} S(-1) \\ \oplus \\ S \\ \oplus \\ S(-1) \end{matrix} \xrightarrow{\begin{bmatrix} Y+Z & 0 & Y \\ 0 & X & YZ \end{bmatrix}} \begin{matrix} S \\ \oplus \\ S(-1) \end{matrix} \xrightarrow{\varepsilon} \mathrm{K}_R \longrightarrow 0.$$

of K_R as a graded S-module. Thus $\mathrm{a}(R) = 1$ and $[\mathrm{K}_R]_{-1} = k\xi$, where $\xi = \varepsilon\left(\binom{0}{1}\right)$. We have the elementary row operation

$$\begin{pmatrix} Y+Z & 0 & Y \\ 0 & X & YZ \end{pmatrix} \longrightarrow \begin{pmatrix} Y+Z & X & Y+YZ \\ 0 & X & YZ \end{pmatrix}$$

and the equality $(Y+Z, X, Y+YZ) = (X,Y,Z)$ in S_N where $N = (X,Y,Z)S$. Similarly as in the proof of Example 3.3, we conclude that R is not almost Gorenstein as a graded ring; while the local ring R_M is almost Gorenstein. $\square$

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