A CRITERION FOR REFLEXIVITY OF MODULES

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Abstract. Let M be a finitely generated module over a ring Λ. With certain mild assumptions on Λ, it is proven that M is a reflexive Λ-module, once $M \cong M^{**}$ as Λ-modules.

Let Λ be a ring. For each left Λ-module X, let $X^* = \mathrm{Hom}_\Lambda(X, \Lambda)$ denote the Λ-dual of X. Following the terminology of Bass ([3, page 476]), we say that X is *reflexive* if h_X is bijective, *torsionless* if h_X is injective, where $h_X : X \to X^{**}$ is the evaluation map, i.e., $\left[h_X(x)\right](f) = f(x)$ for each $f \in X^*$ and $x \in X$.

This paper investigates a naive question of when does an isomorphism $X \cong X^{**}$ of Λ-modules imply the reflexivity of X, and aims at reporting the following.

Theorem 1. *Let Λ be a ring and let M be a finitely generated left Λ-module. Assume at least one of the following conditions is satisfied.*

(1) *Λ is a left Noetherian ring.*
(2) *Λ is a semi-local ring, that is $\Lambda/\mathrm{J}(\Lambda)$ is semi-simple, where $\mathrm{J}(\Lambda)$ denotes the Jacobson radical of Λ.*
(3) *Λ is a module-finite algebra over a commutative ring R with unity.*

Then M is a reflexive Λ-module if and only if there is at least one isomorphism $M \cong M^{**}$ of Λ-modules.

The motivation of this research comes from various sources. Since X is reflexive (resp. torsionless) if and only if $\mathrm{Ext}^i_\Lambda(\mathrm{D}(X), \Lambda) = (0)$ for all $i = 1, 2$ (resp. $i = 1$), Auslander and Bridger introduced an *n-torsionfree* module X to be $\mathrm{Ext}^i_\Lambda(\mathrm{D}(X), \Lambda) = (0)$ for all $1 \le i \le n$, where $\mathrm{D}(X)$ denotes the Auslander transpose of X ([1, Definition (2.15)]). Particularly, when Λ is an integral domain, X is 1-torsionfree if and only if it is torsionless; equivalently, X is *torsionfree*, i.e., there is no nonzero torsion elements (see e.g., [10, Theorem (A.1)]). We also observe that X is 2-torsionfree if and only if it is reflexive. It turned out to be a prominent property in Foxby's study of n-Gorenstein rings ([7]). With mild assumptions, we point out in Corollary 4 the fact that $\mathrm{K}_R \cong (\mathrm{K}_R)^{**}$ implies R being 2-Gorenstein, where K_R denotes the canonical module of a commutative Noetherian local ring R. Besides, Serre observed that finite reflexive modules over two-dimensional regular local rings are free, and this leads us to obtain an application in the arithmetical property of Iwasawa algebras ([8]). Meanwhile, the notion of reflexivity of modules, in

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general, appears not only commutative algebra but in diverse branches of mathematics, e.g., representation theory and non-commutative resolutions (see [1, 2, 6, 9]), and it plays an essential role in many situations.

One should note that, when the ring Λ is commutative and Noetherian, the equivalence in Theorem 1 is known classically (see e.g., [4, (1.1.9) Proposition (b)]). Yet as we show in Remark 5, the assertion may not hold in general without the appropriate assumption on the ring Λ. This is what makes Theorem 1 attractive and still worth studying.

To show the aforementioned assertion, the following lemma is necessary. This is a well-known fact, and its proof is standard.

Lemma 2. *Let N be a right Λ-module and set $M = N^*$. Then the composite of the homomorphisms*

$$M \overset{h_M}{\to} M^{**} \overset{(h_N)^*}{\to} M$$

equals the identity 1_M.

Proof of Theorem 1. We need only prove the "forward implication". Thanks to Lemma 2, we have a split exact sequence

$$0 \to M \overset{h_M}{\to} M^{**} \to X \to 0$$

of left Λ-modules, so that $M \cong M \oplus X$, since $M \cong M^{**}$. Therefore, we get a surjective homomorphism $\varepsilon : M \to M$ with $\operatorname{Ker} \varepsilon = X$. Hence, if Condition (1) is satisfied, then $X = (0)$, so that M is a reflexive Λ-module. In the case of Condition (2), where $J = \mathrm{J}(\Lambda)$, we have:

$$\Lambda/J \otimes_\Lambda M \cong (\Lambda/J \otimes_\Lambda M) \oplus (\Lambda/J \otimes_\Lambda X).$$

By Krull-Schmidt's theorem, this implies $X = (0)$. Suppose Condition (3) is met. Then, $M \cong M \oplus X$ as R-modules, where M is also finitely generated as an R-module. Consequently, for every $\mathfrak{p} \in \operatorname{Spec} R$, we have $M_\mathfrak{p} \cong M_\mathfrak{p} \oplus X_\mathfrak{p}$ as $R_\mathfrak{p}$-modules. Applying the case where Condition (2) is satisfied, we get that $X_\mathfrak{p} = (0)$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Thus, this shows that $X = (0)$, as asserted. □

Corollary 3. *Let R be a commutative ring and let M be a finitely generated R-module. If $M \cong M^{**}$ as R-modules, then M is a reflexive R-module.*

The following is a direct consequence of [5, Theorem 3.6], where $\mathrm{H}^i_\mathfrak{m}(-)$ denotes the ith local cohomology functor of R with respect to $\mathfrak{m}$.

Corollary 4 (cf. [5])**.** *Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring of dimension $d \geq 1$ such that $\mathrm{H}^i_\mathfrak{m}(R)$ is a finitely generated R-module for all $i \neq d$. Assume that R possesses the canonical module K_R. Then the following conditions are equivalent.*

(1) $\operatorname{depth} R > 0$ *and $R_\mathfrak{p}$ is a Gorenstein ring for all $\mathfrak{p} \in \operatorname{Spec} R$ such that* $\mathrm{ht}_R \mathfrak{p} \leq 1$.
(2) $\mathrm{K}_R \cong (\mathrm{K}_R)^{**}$ *as R-modules.*

Remark 5. Let Λ be a ring with elements $a, b \in \Lambda$ such that $ab = 1$ but $ba \neq 1$. We then have the homomorphism

$$\widehat{b} : {}_\Lambda\Lambda \to {}_\Lambda\Lambda, \quad x \mapsto xb$$

is surjective but not an isomorphism. Therefore, setting $X = \operatorname{Ker} \widehat{b}$, we get

$$ {}_\Lambda\Lambda \cong {}_\Lambda\Lambda \oplus X. $$

This remark shows that X does not necessarily vanish, even if $M \cong M \oplus X$ and M is a finitely generated module. Additionally, this example suggests that Theorem 1 may not hold without specific conditions on Λ.

To demonstrate the practical application of Corollary 3, let us consider an example that highlights its utility. Refer to [1, p.137, the final step of the proof of Proposition (4.35)] for a relevant instance where Corollary 3 is effectively employed. This particular example also serves as the inspiration for our current research.

Example 6. Let $k[s,t]$ be the polynomial ring over a field k and set $R = k[s^3, s^2t, st^2, t^3]$. Then R is a normal ring and the graded canonical module K_R of R is given by $\mathrm{K}_R = (s^2t, s^3)$. We set $I = (s^2t, s^3)$. Then, since I is a reflexive R-module, but not 3-torsionfree (because R is not a Gorenstein ring), we must have $\operatorname{Ext}^1_R(R : I, R) \neq (0)$ by [1, Theorem (2.17)]. In what follows, let us check that $\operatorname{Ext}^1_R(R : I, R) \neq (0)$ directly.

First, consider the exact sequence

$$ (\sharp) \qquad 0 \to R \to R : I \to \operatorname{Ext}^1_R(R/I, R) \to 0 $$

induced from the sequence $0 \to I \to R \to R/I \to 0$. Taking the R-dual of $(\sharp)$ again, we get the exact sequence

$$ 0 \to R : (R : I) \to R \to \operatorname{Ext}^1_R(\operatorname{Ext}^1_R(R/I, R), R) \to \operatorname{Ext}^1_R(R : I, R) \to 0, $$

that is,

$$ 0 \to R/I \xrightarrow{\sigma} \operatorname{Ext}^1_R(\operatorname{Ext}^1_R(R/I, R), R) \to \operatorname{Ext}^1_R(R : I, R) \to 0. $$

Therefore, the homomorphism

$$ \sigma : R/I \to \operatorname{Ext}^1_R(\operatorname{Ext}^1_R(R/I, R), R) $$

cannot be an isomorphism. Since

$$ \operatorname{Hom}_{R/(f)}(\operatorname{Hom}_{R/(f)}(R/I, R/(f)), R/(f)) \cong \operatorname{Ext}^1_R(\operatorname{Ext}^1_R(R/I, R), R) $$

for every nonzero $f \in I$, as per Corollary 3, the assertion that σ is not an isomorphism is equivalent to stating that R/I is not a reflexive $R/(f)$-module for some nonzero $f \in I$. Subsequently, we will verify that R/I is not a reflexive $R/(s^3)$-module. Before proceeding, it is important to highlight that without applying Corollary 3, one would need to validate that the homomorphism σ originates from the canonical map

$$ h_{R/I} : R/I \to \operatorname{Hom}_{R/(s^3)}(\operatorname{Hom}_{R/(s^3)}(R/I, R/(s^3)), R/(s^3)), $$

which would inevitably require tedious calculations.

We set $T = R/(s^3)$ and $J = (\overline{s^2t}, \overline{st^2})$ in T, where $\overline{}$ denotes the image in T. Notice that $\operatorname{Hom}_T(R/I, T) \cong (0) :_T I = J$ and $\operatorname{Hom}_T(T/J, T) \cong (0) :_T J = (\overline{s^2t})$. Therefore, from the exact sequence

$$ 0 \to J \to T \to T/J \to 0, $$

we get the exact sequence

$$ 0 \to (\overline{s^2t}) \to T \to \operatorname{Hom}_T(J, T) \to \operatorname{Ext}^1_T(T/J, T) \to 0, $$

that is, the exact sequence

$$(\mathcal{E}) \qquad 0 \to R/I \to \mathrm{Hom}_T(J,T) \to \mathrm{Ext}^1_T(T/J,T) \to 0,$$

which indicates that it suffices to show $\mathrm{Ext}^1_T(T/J,T) \neq (0)$, since $\mathrm{Hom}_T(J,T) = \mathrm{Hom}_T(\mathrm{Hom}_T(R/I,T),T)$. We now express

$$R = k[X,Y,Z,W]/\mathbf{I}_2\left(\begin{smallmatrix} X & Y & Z \\ Y & Z & W\end{smallmatrix}\right),$$

where $k[X,Y,Z,W]$ denotes the polynomial ring over k, $\mathbf{I}_2(\mathbb{M})$ stands for the ideal of $k[X,Y,Z,W]$ generated by the 2×2 minors of a matrix $\mathbb{M}$, and X,Y,Z,W correspond to s^3, s^2t, st^2, t^3, respectively. We denote by x,y,z,w the images of X,Y,Z,W in T. Then, T/J has a T-free resolution

$$\ldots \to T^{\oplus 6} \xrightarrow{\begin{pmatrix} y & z & 0 & 0 & 0 & 0 \\ -x & 0 & w & y & z & 0 \\ 0 & -x & -z & 0 & 0 & y\end{pmatrix}} T^3 \xrightarrow{(x\ y\ z)} T \to T/J \to 0.$$

Taking the T-dual of the resolution, we have $\begin{pmatrix} x \\ 0 \\ 0\end{pmatrix} \in \mathrm{Ker}\ [T^{\oplus 3} \xrightarrow{\begin{pmatrix} y & -x & 0 \\ z & 0 & -x \\ 0 & w & -z \\ 0 & y & 0 \\ 0 & z & 0 \\ 0 & 0 & y\end{pmatrix}} T^{\oplus 6}]$, but $\begin{pmatrix} y \\ 0 \\ 0\end{pmatrix} \neq \alpha \begin{pmatrix} x \\ y \\ z\end{pmatrix}$ for any $\alpha \in T$. Thus, $\mathrm{Ext}^1_T(T/J,T) \neq (0)$, so that the exact sequence $(\mathcal{E})$ shows R/I is not a reflexive T-module. Hence, by Corollary 3, the homomorphism

$$\sigma : R/I \to \mathrm{Ext}^1_R(\mathrm{Ext}^1_R(R/I,R),R)$$

is not an isomorphism. Thus, $\mathrm{Ext}^1_R({\mathrm{K}_R}^*, R) \neq (0)$, and K_R is not 3-torsionfree.

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