# A CRITERION FOR REFLEXIVITY OF MODULES

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Abstract. Let $M$ be a finitely generated module over a ring $\Lambda$. With certain mild assumptions on $\Lambda$, it is proven that $M$ is a reflexive $\Lambda$-module, once $M \cong M^{**}$ as a $\Lambda$-module.

Let $\Lambda$ be a ring. For each left $\Lambda$-module $X$, let $X^* = \mathrm{Hom}_\Lambda(X, \Lambda)$ denote the $\Lambda$-dual of $X$. This note aims at reporting the following.

**Proposition 1.** *Let $\Lambda$ be a ring and let $M$ be a finitely generated left $\Lambda$-module. Assume that one of the following conditions is satisfied.*

(1) *$\Lambda$ is a left Noetherian ring.*
(2) *$\Lambda$ is a semi-local ring, that is $\Lambda/\mathrm{J}(\Lambda)$ is semi-simple, where $\mathrm{J}(\Lambda)$ denotes the Jacobson radical of $\Lambda$.*
(3) *$\Lambda$ is a module-finite algebra over a commutative ring $R$.*

*Then, $M$ is a reflexive $\Lambda$-module, that is the canonical map $M \overset{h_M}{\to} M^{**}$ is an isomorphism if and only if there is at least one isomorphism $M \cong M^{**}$ of $\Lambda$-modules.*

To show the above assertion, we need the following. This is well-known, and the proof is standard.

**Lemma 2.** *Let $N$ be a right $\Lambda$-module and set $M = N^*$. Then the composite of the homomorphisms*

$$M \overset{h_M}{\to} M^{**} \overset{(h_N)^*}{\to} M$$

*equals the identity $1_M$.*

*Proof of Proposition* 1. We have only to show the *if* part. Thanks to Lemma 2, we have a split exact sequence

$$0 \to M \overset{h_M}{\to} M^{**} \to X \to 0$$

of left $\Lambda$-modules, so that $M \cong M \oplus X$, since $M \cong M^{**}$. Therefore, we get a surjective homomorphism $\varepsilon : M \to M$ with $\mathrm{Ker}\, \varepsilon = X$. Hence, if Condition (1) is satisfied, then $X = (0)$, so that $M$ is a reflexive $\Lambda$-module. If Condition (2) is satisfied, then, setting $J = \mathrm{J}(\Lambda)$, we get

$$\Lambda/J \otimes_\Lambda M \cong (\Lambda/J \otimes_\Lambda M) \oplus (\Lambda/J \otimes_\Lambda X)$$

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whence $\Lambda/J \otimes_\Lambda X = (0)$ by Krull-Schmidt's theorem, so that $X = (0)$. Suppose that Condition (3) is satisfied. Then, $M \cong M \oplus X$ as an $R$-module, where $M$ is finitely generated also as an $R$-module. Consequently, for every $\mathfrak{p} \in \operatorname{Spec} R$, we get $M_\mathfrak{p} \cong M_\mathfrak{p} \oplus X_\mathfrak{p}$ as an $R_\mathfrak{p}$-module, whence by the case where Condition (2) is satisfied, $X_\mathfrak{p} = (0)$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Thus, $X = (0)$, as claimed. □

**Corollary 3.** *Let $R$ be a commutative ring and $M$ a finitely generated $R$-module. If $M \cong M^{**}$ as an $R$-module, then $M$ is a reflexive $R$-module.*

The following is a direct consequence of [2, Theorem 3.4], where $\mathrm{H}^i_\mathfrak{m}(*)$ denotes the $i$-th local cohomology functor of $R$ with respect to $\mathfrak{m}$.

**Corollary 4** (cf. [2])**.** *Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring of dimension $d \geq 1$ such that $\mathrm{H}^i_\mathfrak{m}(R)$ is a finitely generated $R$-module for all $i \neq d$. Assume that $R$ possesses the canonical module $\mathrm{K}_R$. Then the following conditions are equivalent.*

(1) $\operatorname{depth} R > 0$ *and $R_\mathfrak{p}$ is a Gorenstein ring for all $\mathfrak{p} \in \operatorname{Spec} R$ such that $\mathrm{ht}_R\mathfrak{p} \leq 1$.*
(2) $\mathrm{K}_R \cong \mathrm{K}_R^{**}$ *as an $R$-module.*

*Remark* 5. Let $\Lambda$ be a ring and let $a, b \in \Lambda$ such that $ab = 1$ but $ba \neq 1$. We then have the homomorphism

$$\widehat{b} : {}_\Lambda\Lambda \to {}_\Lambda\Lambda, \quad x \mapsto xb$$

is surjective but not an isomorphism. Therefore, setting $X = \operatorname{Ker} \widehat{b}$, we get

$${}_\Lambda\Lambda \cong {}_\Lambda\Lambda \oplus X.$$

This example shows that $X$ does not necessarily vanish, even if $M \cong M \oplus X$ and $M$ is a finitely generated module. This example seems also to suggest that Proposition 1 doesn't hold true without any specific conditions on $\Lambda$.

Let us note one example in order to show how Corollary 3 works at an actual spot. See [1, p.137, the final step of the proof of (4.35) Proposition] also, where one can find a good opportunity of making use of it, from which the motivation for the present research has come.

**Example 6.** Let $k[s, t]$ be the polynomial ring over a field $k$ and set $R = k[s^3, s^2t, st^2, t^3]$. Then $R$ is a normal ring and the graded canonical module $\mathrm{K}_R$ of $R$ is given by $\mathrm{K}_R = (s^2t, s^3)$. We set $I = (s^2t, s^3)$. Then, since $I$ is a reflexive $R$-module, but not 3-torsionfree in the sense of Auslander-Bridger [1, (2.15) Definition] (because $R$ is not a Gorenstein ring), we must have $\operatorname{Ext}^1_R(R : I, R) \neq (0)$ by [1, (2.17) Theorem]. In what follows, let us check that $\operatorname{Ext}^1_R(R : I, R) \neq (0)$ directly.

First, consider the exact sequence

$$0 \to R \to R : I \to \operatorname{Ext}^1_R(R/I, R) \to 0$$

induced from the sequence $0 \to I \to R \to R/I \to 0$. Taking the $R$-dual of it again, we get the exact sequence

$$0 \to R : (R : I) \to R \to \operatorname{Ext}^1_R(\operatorname{Ext}^1_R(R/I, R), R) \to \operatorname{Ext}^1_R(R : I, R) \to 0,$$

that is

$$0 \to R/I \xrightarrow{\sigma} \operatorname{Ext}^1_R(\operatorname{Ext}^1_R(R/I, R), R) \to \operatorname{Ext}^1_R(R : I, R) \to 0.$$

Therefore, the homomorphism

$$\sigma : R/I \to \mathrm{Ext}^1_R(\mathrm{Ext}^1_R(R/I,R),R)$$

should not be an isomorphism. Because

$$\mathrm{Hom}_{R/(f)}(\mathrm{Hom}_{R/(f)}(R/I,R/(f)),R/(f)) \cong \mathrm{Ext}^1_R(\mathrm{Ext}^1_R(R/I,R),R)$$

for every $0 \neq f \in I$, thanks to Corollary 3, the assertion that $\sigma$ is not an isomorphism is equivalent to saying that $R/I$ is not a reflexive $R/(f)$-module for some $0 \neq f \in I$. In the following, we shall confirm that $R/I$ is not a reflexive $R/(s^3)$-module. Before starting work, we would like to note here and emphasize that if we do not make use of Corollary 3, we must certify the above homomorphism $\sigma$ to be induced from the canonical map

$$R/I \overset{h_{R/I}}{\to} \mathrm{Hom}_{R/(s^3)}(\mathrm{Hom}_{R/(s^3)}(R/I,R/(s^3)),R/(s^3)),$$

which provably makes a tedious calculation necessary.

We set $T = R/(s^3)$ and $J = (\overline{s^2t},\overline{st^2})$ in $T$, where $\overline{*}$ denotes the image in $T$. Notice that $\mathrm{Hom}_T(R/I,T) \cong (0) :_T I = J$ and $\mathrm{Hom}_T(T/J,T) \cong (0) :_T J = (\overline{s^2t})$. Therefore, from the exact sequence

$$0 \to J \to T \to T/J \to 0,$$

we get the exact sequence

$$0 \to (\overline{s^2t}) \to T \to \mathrm{Hom}_T(J,T) \to \mathrm{Ext}^1_T(T/J,T) \to 0,$$

that is the exact sequence

$$\text{(E)} \qquad 0 \to R/I \to \mathrm{Hom}_T(J,T) \to \mathrm{Ext}^1_T(T/J,T) \to 0,$$

which guarantees it suffices to show $\mathrm{Ext}^1_T(T/J,T) \neq (0)$, since $\mathrm{Hom}_T(J,T) = \mathrm{Hom}_T(\mathrm{Hom}_T(R/I,T),T)$. We now identify

$$R = k[X,Y,Z,W]/\mathbf{I}_2\left(\begin{smallmatrix} X & Y & Z \\ Y & Z & W \end{smallmatrix}\right),$$

where $k[X,Y,Z,W]$ denotes the polynomial ring over $k$, $\mathbf{I}_2(\mathbb{M})$ stands for the ideal of $k[X,Y,Z,W]$ generated by the $2 \times 2$ minors of a matrix $\mathbb{M}$, and $X,Y,Z,W$ correspond to $s^3, s^2t, st^2, t^3$, respectively. We denote by $x,y,z,w$ the images of $X,Y,Z,W$ in $T$. Then, $T/J$ has a $T$-free resolution

$$\ldots \to T^{\oplus 6} \xrightarrow{\left(\begin{smallmatrix} y & z & 0 & 0 & 0 & 0 \\ -x & 0 & w & y & z & 0 \\ 0 & -x & -z & 0 & 0 & y \end{smallmatrix}\right)} T^3 \xrightarrow{(x\ y\ z)} T \to T/J \to 0.$$

Taking the $T$-dual of the resolution, we have $\left(\begin{smallmatrix} x \\ 0 \\ 0 \end{smallmatrix}\right) \in \mathrm{Ker}\ [T^{\oplus 3} \xrightarrow{\left(\begin{smallmatrix} y & -x & 0 \\ z & 0 & -x \\ 0 & w & -z \\ 0 & y & 0 \\ 0 & z & 0 \\ 0 & 0 & y \end{smallmatrix}\right)} T^{\oplus 6}]$, but $\left(\begin{smallmatrix} y \\ 0 \\ 0 \end{smallmatrix}\right) \neq \alpha \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)$ for any $\alpha \in T$. Thus, $\mathrm{Ext}^1_T(T/J,T) \neq (0)$, so that the exact sequence (E) shows $R/I$ is not a reflexive $T$-module. Hence, by Corollary 3 the homomorphism

$$\sigma : R/I \to \mathrm{Ext}^1_R(\mathrm{Ext}^1_R(R/I,R),R)$$

is not an isomorphism. Thus, $\mathrm{Ext}^1_R(\mathrm{K}^*_R,R) \neq (0)$, and $\mathrm{K}_R$ is not 3-torsionfree.

## References

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