

EFFICIENT GENERATION OF IDEALS IN CORE SUBALGEBRAS OF THE POLYNOMIAL RING $k[t]$ OVER A FIELD k

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ABSTRACT. This note aims at finding explicit and efficient generation of ideals in subalgebras R of the polynomial ring $S = k[t]$ (k a field) such that $t^{c_0}S \subseteq R$ for some integer $c_0 > 0$. The class of these subalgebras which we call cores of S includes the semigroup rings $k[H]$ of numerical semigroups H , but much larger than the class of numerical semigroup rings. For $R = k[H]$ and $M \in \text{Max } R$, our result eventually shows that $\mu_R(M) \in \{1, 2, \mu(H)\}$ where $\mu_R(M)$ (resp. $\mu(H)$) stands for the minimal number of generators of M (resp. H), which covers in the specific case the classical result of O. Forster–R. G. Swan.

1. INTRODUCTION

This note aims at finding efficient systems of generators for ideals in certain subalgebras R of the polynomial ring $S = k[t]$ with one indeterminate t over a field k . The class of subalgebras which this note concerns naturally includes the semigroup rings $k[H]$ of numerical semigroups H .

Investigation on the numbers of generators of ideals and modules is one of the classical subjects of great interest in commutative algebra. One of the main problems was to look for a bound, in terms of the local data, on the minimal number of generators for a finitely generated module M over a commutative Noetherian ring R . This problem was solved by O. Forster [3] in 1964, and subsequently in 1967, R. G. Swan [6] gave an efficient bound for the number of generators, also generalizing Forster’s argument to the non-commutative case. The reader may consult [1, 2], where D. Eisenbud and E. G. Evans, Jr. extended various stability theorems for projective modules to the context of arbitrary finitely generated modules, developing a beautiful theory of basic elements. Let us note one of the results in the following form. Throughout, let $\mu_R(*)$ stand for the minimal number of generators.

Theorem 1.1 (Forster-Swan Theorem on the number of generators of a module, [1, Corollary 3], [2, Corollary 5]). *Let M be a finitely generated module over a commutative Noetherian ring R of finite dimension and set*

$$b(M) = \sup_{\mathfrak{p} \in \text{Spec } R} [\dim R/\mathfrak{p} + \mu_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})].$$

Then $\mu_R(M) \leq b(M)$.

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On the other hand, in 1978 J. Sally [5] explored the case where the base rings are local, giving several fundamental results about the numbers of generators of ideals in Cohen-Macaulay local rings of small dimension. Combining her results with Theorem 1.1, we nowadays have satisfactorily general methods to estimate the numbers of generators for the ideals in rings of small dimension. For example, if $R = k[H] = \sum_{h \in H} kt^h$ is the semigroup ring of a numerical semigroup H where t denotes an indeterminate over a field k , then $\text{Spec } R$ has at most an isolated singularity, so that for a maximal ideal M of R we get $\mu_R(M) \leq 2$ if $M \neq P_0$ and $\mu_R(P_0) = \mu(H)$, where $P_0 = (t^h \mid 0 < h \in H)$ and $\mu(H)$ stands for the minimal number of generators of H . Nevertheless, even for the maximal ideals M of $k[H]$, it would be a different and interesting problem to look for explicit and efficient systems of generators, which we shall pursue in this note.

We explain how this note is organized. In Section 2 we study k -subalgebras R of the polynomial ring $S = k[t]$ over a field k such that $t^{c_0}S \subseteq R$ for some integer $c_0 > 0$. The class of these algebras which we call cores of S naturally includes (but much larger than) the class of numerical semigroup rings. We consider the ideals I of the form $I = fS \cap R$ where $f \in S = k[t]$ such that $f(0) = 1$, and show that I is at most 2-generated, giving explicit generators of I which depend only on c_0 and f (Theorem 2.4). The result leads to the study of the integral closures \bar{I} of arbitrary ideals I of R , and in Section 3, we will give an estimation of the minimal number of generators of \bar{I} . In Section 4, we shall focus on the case where $R = k[H]$, giving rather incredible systems of generators of k -rational closed points M of $\text{Spec } R$ (Theorem 4.2, Corollary 4.4), which eventually shows that $\mu_R(M) \in \{2, \mu(H)\}$, if $1 \notin H$ and the field k is algebraically closed (Corollary 4.5).

2. MAIN RESULT

Let $S = k[t]$ be the polynomial ring over a field k . Let $0 < a_1, a_2, \dots, a_v \in \mathbb{Z}$ ($v > 0$) be integers such that $\text{GCD}(a_1, a_2, \dots, a_v) = 1$. We denote by

$$H = \langle a_1, a_2, \dots, a_v \rangle = \left\{ \sum_{i=1}^v c_i a_i \mid 0 \leq c_i \in \mathbb{Z} \text{ for } 1 \leq \forall i \leq v \right\}$$

the numerical semigroup generated by a_i 's. The reader may consult [4] for a general reference about numerical semigroups. We set $k[H] = k[t^{a_1}, t^{a_2}, \dots, t^{a_v}] \subseteq S$ and call it the semigroup ring of H . Let

$$c(H) = \min\{n \in \mathbb{Z} \mid m \in H \text{ for all } m \in \mathbb{Z} \text{ such that } m \geq n\}$$

denote the conductor of H . Hence, $k[H] : S = t^{c(H)}S$. Unless otherwise specified, we throughout assume that $1 \notin H$. Therefore, $c(H) \geq 2$, and $\text{Sing}(k[H]) = \{P_0\}$, where $P_0 = (t^{a_1}, t^{a_2}, \dots, t^{a_v}) = tS \cap k[H]$

Let R be a k -subalgebra of S . Then we say that R is a *core* of S , if $t^{c_0}S \subseteq R$ for some integer $c_0 > 0$. If R is a core of S , then

$$k[t^{c_0}, t^{c_0+1}, \dots, t^{2c_0-1}] \subseteq R \subseteq S,$$

and a given k -subalgebra R of S is a core of S if and only if $R \supseteq k[H]$ for some numerical semigroup H . Therefore, once R is a core of S , R is a finitely generated k -algebra of dimension one, and S is a birational module-finite extension of R with $t^{c_0}S \subseteq R : S$.

Typical examples of cores are, of course, the semigroup rings $k[H]$ of numerical semigroups H . However, cores of S do not necessarily arise as semigroup rings for some numerical semigroups. Let us note one of the simplest examples.

Example 2.1. Let $R = k[t^2 + t^3] + t^4S$. Then $R \neq k[H]$ for any numerical semigroup H .

We summarize a few basic facts about cores. For $P \in \text{Spec } R$, we say that P is a *k-rational closed point* of $\text{Spec } R$, if $k = R/P$.

Proposition 2.2. *Let R be a core of S and set $P_0 = tS \cap R$. Let S_P denote, for each $P \in \text{Spec } R$, the localization of S by the multiplicative system $R \setminus P$. Then the following assertions hold true.*

- (1) *Let $Q \in \text{Spec } S$ and set $P = Q \cap R$. If $P \neq P_0$, then $R_P = S_P = S_Q$.*
- (2) *The map $\text{Spec } S \rightarrow \text{Spec } R, Q \mapsto Q \cap R$ is a bijection.*
- (3) *Let $Q \in \text{Spec } S$ and set $P = Q \cap R$. Then Q is a *k-rational closed point* of $\text{Spec } S$ if and only if so is P in $\text{Spec } R$.*

Proof. (1) If $t^c S \subseteq P$, then $t \in Q$, so that $P = P_0$. Therefore, $t^c S \not\subseteq P$, whence $R_P = S_P$, because $R : S \not\subseteq P$. Consequently, $S_P = S_Q$, since S_P is a local ring and $QS_P \in \text{Max } S_P$.

(2) Since S is integral over R , the map $\text{Spec } S \rightarrow \text{Spec } R, Q \mapsto Q \cap R$ is surjective. Let $Q_1, Q_2 \in \text{Spec } S$ and assume that $Q_1 \cap R = Q_2 \cap R = P$. We will show that $Q_1 = Q_2$. To do this, we may assume that $P \neq (0)$. If $P \neq P_0$, then $Q_1 = Q_2$, because by assertion (1) S_P is a local ring and $Q_1 S_P, Q_2 S_P \in \text{Max } S_P$. If $P = P_0$, then $t^c \in Q_1 \cap Q_2$, so that $Q_1 = Q_2 = tS$, which proves assertion (2).

(3) We may assume that $P \in \text{Max } R, Q \in \text{Max } S$, and by assertion (2) that $P \neq P_0$. Therefore, $R/P = R_P/PR_P$ and $S/Q = S_Q/QS_Q$, whence $R/P = S/Q$, because $R_P = S_Q$ by assertion (1). Thus, $k = R/P$ if and only if $k = S/Q$. \square

In what follows, we fix a core R of $S = k[t]$. Let $f \in S = k[t]$ such that $f(0) = 1$. Choose integers $\ell, c \geq c_0$ so that $\ell \geq 2$. We write $f = \sum_{i \geq 0} a_i t^i$ with $a_i \in k$ (hence $a_0 = 1$) and consider the following $(\ell - 1) \times \ell$ matrix

$$A = \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ a_2 & a_1 & 1 & 0 & \cdots & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{\ell-2} & a_{\ell-3} & a_{\ell-4} & \cdots & a_1 & 1 & 0 \\ a_{\ell-1} & a_{\ell-2} & a_{\ell-3} & \cdots & a_2 & a_1 & 1 \end{bmatrix}.$$

Then, $\text{rank } A = \ell - 1$, so that there exists a unique element $\mathbf{v} = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_{\ell-1} \end{bmatrix} \in k^{(\ell)}$ such that

$A\mathbf{v} = \mathbf{0}$ and $v_0 = 1$. Set $g = \sum_{i=0}^{\ell-1} v_i t^i \in S$. Hence, $g(0) = 1$, and we have the following.

Proposition 2.3. *The following assertions hold true.*

- (1) *$fg - 1 \in t^\ell S$, whence $fg \in R$.*
- (2) *$f \in R$ if and only if $g \in R$.*
- (3) *Let $\varphi \in \sum_{i=0}^{\ell-1} kt^i$. If $f\varphi - 1 \in t^\ell S$, then $\varphi = g$.*

Proof. (1) The coefficient b_n of t^n in fg is given by $b_n = \sum_{i=0}^n a_i v_{n-i}$, so that $b_n = 0$ for all $1 \leq n \leq \ell - 1$, while $(fg)(0) = f(0) \cdot v_0 = 1$. Therefore

$$fg - 1 \in t^\ell S.$$

(2) Let $\bar{R} = R/J$ and $\bar{S} = S/J$, where $J = t^\ell S$. Then, \bar{R} is a subring of \bar{S} , and \bar{S} is a module-finite extension of \bar{R} . Therefore, \bar{R} is a local ring, since so is \bar{S} , and $\mathfrak{m}_{\bar{R}} = \mathfrak{m}_{\bar{S}} \cap \bar{R}$, where $\mathfrak{m}_{\bar{R}}$ and $\mathfrak{m}_{\bar{S}}$ denote respectively the maximal ideals of \bar{R} and \bar{S} . Then, since $fg \equiv 1 \pmod{t^\ell S}$ in S and $t^\ell S = J$, we have $\bar{f}\bar{g} = 1$ in \bar{S} , where \bar{f}, \bar{g} denote respectively the images of f, g in \bar{S} . Therefore, if $f \in R$, $\bar{f} \in \bar{R}$ and it is a unit of \bar{R} , since it is a unit of \bar{S} . Because \bar{g} is the inverse of \bar{f} in the ring \bar{S} , which should belong to the ring \bar{R} , just thanks to the uniqueness of the inverse. Therefore, $g \in R$, since $g \equiv r \pmod{J}$ in S for some $r \in R$. The converse is similarly proved.

(3) This is clear, since $\bar{\varphi} = \bar{g}$ in \bar{S} . □

We set $I = fS \cap R$. Then, $fg \in I$ since $fg \in R$, and $t^c f \in I$ for every integer $c \geq c_0$ since $t^c S \subseteq R$. Therefore, $(t^c f, fg) \subseteq I$. We furthermore have the following.

Theorem 2.4. *The following assertions hold true.*

- (1) $I = (t^c f, fg)$ for every integer $c \geq c_0$, and $IS = fS$.
- (2) I is a principal ideal of R if and only if $f \in R$. When this is the case, $I = fR$.

Let us divide the proof of Theorem 2.4 into several steps. We may assume that $f \notin k$. We fix an irreducible decomposition $f = \prod_{i=1}^n f_i^{e_i}$ of f . Hence, f_i 's are irreducible polynomials such that $f_i S = f_j S$ only if $i = j$, and e_i 's are positive integers. We set $\Lambda = \{1, 2, \dots, n\}$. Let

$$Q_i = f_i S, \quad P_i = Q_i \cap R \text{ for each } i \in \Lambda, \text{ and } P_0 = tS \cap R.$$

Then, $fg \notin P_0$, since $(fg)(0) = f(0)g(0) = 1$, but $fg \in P_i$ for every $i \in \Lambda$, since $f \in Q_i$. Let $P \in \text{Spec } R$ and write $P = Q \cap R$ for some $Q \in \text{Spec } S$. Let $c \geq c_0$ be an integer. Then, $t^c f \in P$ if and only if $t^c f \in Q$, and the latter condition is equivalent to saying that either $t \in Q$ or $f_i \in Q$ for some $i \in \Lambda$. Therefore, setting $V(t^c f) = \{P \in \text{Spec } R \mid t^c f \in P\}$, we have the following.

Proposition 2.5. *The following assertions hold true.*

- (1) $fg \in P_i$ for each $i \in \Lambda$, but $fg \notin P_0$. Consequently, $IR_{P_0} = R_{P_0}$.
- (2) $V(t^c f) = \{P_0, P_1, \dots, P_n\}$.

Proposition 2.6. *For $i \in \Lambda$ the following assertions hold true.*

- (1) $R_{P_i} = S_{Q_i}$ and $IR_{P_i} = fR_{P_i}$, whence $I \subseteq P_i$.
- (2) Let $h \in S$. If $h \notin Q_i$, then $IR_{P_i} = fhR_{P_i}$.

Therefore, for each $P \in \text{Spec } R$, $I \subseteq P$ if and only if $P = P_j$ for some $j \in \Lambda$.

Proof. By Proposition 2.2 (1) $R_{P_i} = S_{P_i} = S_{Q_i}$, since $P_i \neq P_0$, while

$$IR_{P_i} = [fS \cap R]R_{P_i} = fS_{P_i} \cap R_{P_i}.$$

Therefore $IR_{P_i} = fS_{Q_i} \cap S_{Q_i} = fS_{Q_i} = fR_{P_i}$, so that $I \subseteq P_i$ because $fS_{Q_i} \neq S_{Q_i}$. If $h \in S$ but $h \notin Q_i$, then $IR_{P_i} = fS_{Q_i} = fhS_{Q_i} = fhR_{P_i}$. Let $P \in \text{Spec } R$. If $I \subseteq P$, then

$t^c f \in P$, so that by Proposition 2.5 $P = P_j$ for some $j \in \Lambda$. Since by assertion (1) $I \subseteq P_i$ for every $i \in \Lambda$, the last assertion follows. \square

The following is a direct consequence of Proposition 2.6.

Corollary 2.7. *I is an invertible ideal of R .*

We set $W = R \setminus \bigcup_{i=0}^n P_i$. Then, $W^{-1}I$ is a principal ideal of $W^{-1}R$, because it is an invertible ideal of $W^{-1}R$ (Corollary 2.7) and the ring $W^{-1}R$ is a one-dimensional semi-local ring whose maximal ideals are precisely $\{W^{-1}P_i\}_{0 \leq i \leq n}$. We now notice that $R/(t^c f) = W^{-1}[R/(t^c f)]$, since for every $w \in W$ the image of w in $R/(t^c f)$ is invertible in $R/(t^c f)$ (Proposition 2.5 (2)). Therefore, in order to prove Theorem 2.4 (1), it suffices to show that $W^{-1}I = fg \cdot W^{-1}R$, or equivalently

$$\text{Supp}_{W^{-1}R} W^{-1}I / fg \cdot W^{-1}R = \emptyset.$$

Because $\text{Max } W^{-1}R = \{W^{-1}P_i \mid 0 \leq i \leq n\}$, by Proposition 2.5 (1) and Proposition 2.6 (2) this is certainly the case, once $g \notin Q_i$ for any $i \in \Lambda$. Suppose that $g \in Q_i$ for some $i \in \Lambda$, and set $\xi = \prod_{j \in \Gamma} f_j$, where $\Gamma = \{j \in \Lambda \mid g \notin Q_j\}$. Choose an integer q so that $q \geq c + c_0$ and set $h = g + t^q \xi$. We then have the following.

Proposition 2.8. *We have $h \notin Q_i$ for any $i \in \Lambda$. Hence, $IR_{P_i} = fhR_{P_i}$ for every $i \in \Lambda$.*

Proof. Assume the contrary and let $h \in Q_i$ for some $i \in \Lambda$. If $g \in Q_i$, we then have $t^q \xi \in Q_i$, so that either $t \in Q_i$ or $\xi \in Q_i$. However, if $t \in Q_i$, then $Q_i = tS$, which forces $f(0) = 0$ because $f \in Q_i$. Therefore, $\xi \in Q_i$, so that $f_j \in Q_i$ for some $j \in \Gamma$. Hence, $i = j \in \Gamma$, whence $g \notin Q_i$. This is a contradiction. Thus, $g \notin Q_i$, that is $i \in \Gamma$, whence $t^q \xi \in Q_i$ so that $h = g + t^q \xi \notin Q_i$. This is also a contradiction. Hence, $h \notin Q_i$ for any $i \in \Lambda$. The second assertion is a direct consequence of Proposition 2.6 (2). \square

Since $t^{q-c} \xi \in R$ (remember that $q \geq c + c_0$), we have $fh = fg + t^c f \cdot t^{q-c} \xi \equiv fg \pmod{t^c f R}$, so that $(t^c f, fh) = (t^c f, fg)$ in R . On the other hand, because $fh \notin P_0$ (notice that $(fh)(0) = (fg)(0) = 1$) and by Proposition 2.8 $IR_{P_i} = fhR_{P_i}$ for every $i \in \Lambda$, we get $\text{Supp}_{W^{-1}R} W^{-1}I / fh \cdot W^{-1}R = \emptyset$, so that $I = (t^c f, fh)$. Therefore, $I = (t^c f, fg)$. Because $(t^c, g)S = S$ and $IS = (t^c f, fg)S = f \cdot (t^c, g)S$, we readily get $IS = fS$, which proves assertion (1) of Theorem 2.4.

Let us consider assertion (2). Suppose that $I = (t^c f, fg)$ is a principal ideal of R and let $I = \varphi R$ for some $\varphi \in R$. We write $\varphi = f\psi$ with $\psi \in S$. Then, since $\psi R = t^c R + gR$ and $t^c S + gS = S$, we have $\psi S = S$, so that $0 \neq \psi \in k$. Therefore, $I = \varphi R = f\psi R = fR$, whence $f \in R$. If $f \in R$, then $fR \subseteq I = fS \cap R$, while $g \in R$ by Proposition 2.3. Consequently, because $t^c, g \in R$, we get $fR \subseteq I = (t^c f, fg) \subseteq fR$. Hence, $I = fR$, which completes the proof of Theorem 2.4.

Example 2.9. Let $f = 1 - t$. Then, $g = \sum_{i=0}^{\ell-1} t^i$, where $\ell \geq \max\{2, c_0\}$. We set $I = (1 - t)S \cap R$. Then, I is a maximal ideal of R , and $I = (t^c - t^{c+1}, 1 - t^\ell)$ for every $c \geq c_0$. The ideal I is a principal ideal of R if and only if $R = S$.

Example 2.10. Let $k = \mathbb{Z}/(2)$ and $f = 1 + t^2 + t^3 + t^5 + t^6 \in S = k[t]$. Then f is an irreducible polynomial in S . Choose a k -subalgebra R of S so that $t^{10}S \subseteq R$. Let

$I = fS \cap R$ and set $\ell = c = 10$. We then have $g = 1 + t^2 + t^3 + t^4 + t^5 + t^6 + t^7$ and $I = (t^{10}f, 1 + t^{10} + t^{13})$. The maximal ideal I is a principal ideal of R if and only if $f \in R$.

We consider semigroup rings $R = k[H]$ of numerical semigroups H .

Example 2.11. Let $e \geq 2$ be an integer and set $H = \langle e, e+1, \dots, 2e-1 \rangle$. Hence, $R = k[t^{e+i} \mid 0 \leq i \leq e-1]$ and $c(H) = e$. Let $0 \neq \alpha \in k$ and set $f = 1 - \alpha t \in S$, $M = fS \cap R$. Then, taking $\ell = c = e$, we have $g = \sum_{i=0}^{e-1} \alpha^i t^i$, whence

$$M = (t^e - \alpha t^{e+1}, 1 - \alpha^e t^e) = \left(\left(\frac{1}{\alpha}\right)^{e+1} - t^{e+1}, \left(\frac{1}{\alpha}\right)^e - t^e \right).$$

A similar result holds true for k -rational closed points except the origin of arbitrary monomial curves $\text{Spec } k[H]$, which we shall discuss in Section 4.

Example 2.12. Let $k = \mathbb{R}$ and $f = 1 + at + bt^2$, where $a, b \in k$ such that $b \neq 0$ and $a^2 < 4b$. Let $H = \langle 2, 5 \rangle$ or $H = \langle 4, 5, 6, 7 \rangle$. Hence $c(H) = 4$. Let $R = k[H]$. We set $M = fS \cap R$ and choose $\ell = c = 4$. Then

$$A = \begin{bmatrix} a & 1 & 0 & 0 \\ b & a & 1 & 0 \\ 0 & b & a & 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} \frac{1}{2ab-a^3} \\ -a \\ a^2-b \\ 2ab-a^3 \end{bmatrix}.$$

We have $fg = 1 + (3a^2b - b^2 - a^4)t^4 + (2ab - a^3)bt^5$ and $M = (t^4f, fg)$. We now take $a = 0$. Then $g = 1 - bt^2$ and $M = (t^4f, 1 - b^2t^4)$. By Theorem 2.4 (2), we have $M = fR$ if $H = \langle 2, 5 \rangle$, but $\mu_R(M) = 2$ if $H = \langle 4, 5, 6, 7 \rangle$. This example shows that even though the generating system of $M = fS \cap R$ depends only on $c(H)$ and f , the whole structure of H has an influence on the minimal number $\mu_R(M)$ of generators for M .

3. INTEGRAL CLOSURES \bar{I} OF AN IDEAL I IN R

Similarly as in Section 2, we fix a core R of S . Hence, R is a k -subalgebra R of $S = k[t]$ such that $t^{c_0}S \subseteq R$ for some integer $c_0 > 0$. In this section, let $I (\neq (0))$ denote an ideal of R throughout. Let us write $IS = \varphi S$ with $\varphi \in S$. We are interested in the efficient generation of the integral closure \bar{I} of I . To do this, notice that $\bar{I} = IS \cap R$, since S is a module-finite (hence an integral) extension of R . Without loss of generality, we may assume that $\varphi = t^q f$, where $q \geq 0$ is an integer and $f \in S$ such that $f(0) = 1$. We choose integers $\ell, c \geq c_0$ so that $\ell \geq 2$ in order to obtain the polynomial $g \in S$ explored in Section 2 (see Proposition 2.3). Hence $fS \cap R = (t^c f, fg)$ by Theorem 2.4.

With this notation we have the following.

Lemma 3.1. $\bar{I} = (t^q S \cap R) \cap (fS \cap R) = (t^q S \cap R) \cdot (fS \cap R)$ and $(t^q S \cap R)S = t^q S$.

Proof. We have $t^q fS = t^q S \cap fS$, because $(t^q, f)S = S$, so that the first equality follows, since $\bar{I} = (t^q fS) \cap R$. To see the second equality, it suffices to show $(t^q S \cap R) + (fS \cap R) = R$. Assume the contrary and choose $M \in \text{Max } R$ so that $(t^q S \cap R) + (fS \cap R) \subseteq M$. We write $M = N \cap R$ for some $N \in \text{Max } S$. Then, since $t^{c_0+q}S \subseteq R$, we get $t^{c_0+q} \in M$, whence $t \in N$. On the other hand, we have $fg \in N$, since $fg \in fS \cap R$ (Proposition 2.3 (1)). Therefore, $t, fg \in N$, which is impossible, because $(fg)(0) = 1$. Hence, $(t^q S \cap R) + (fS \cap R) = R$, and the second equality follows. To see that $(t^q S \cap R)S = t^q S$, notice that $IS = \bar{I}S$, since

$$IS \subseteq \bar{I}S \subseteq \overline{IS} = \overline{\varphi S} = \varphi S = IS.$$

We then have $t^q fS = IS = \bar{I}S = [(t^q S \cap R) \cdot (fS \cap R)]S = [(t^q S \cap R)S] \cdot [(fS \cap R)S]$, so that $t^q S = (t^q S \cap R)S$, because $(fS \cap R)S = fS$ by Theorem 2.4. \square

We furthermore have the following.

Proposition 3.2. *The following assertions hold true.*

- (1) *If $c \geq q$, then $\bar{I} = (t^c f) + fg \cdot (t^q S \cap R)$, whence $\mu_R(\bar{I}) \leq 1 + \mu_R(t^q S \cap R)$.*
- (2) *If $q \geq c_0$, then $\bar{I} = \varphi S$ and $\mu_R(\bar{I}) = \mu_R(S)$.*

Proof. (1) Since $t^c f \in t^q S \cap R$ and $fS \cap R = (t^c f, fg)$ by Theorem 2.4, we have by Lemma 3.1 that

$$\bar{I} = (t^q S \cap R) \cap (fS \cap R) = t^c fR + [(t^q S \cap R) \cap (fg)],$$

while $(t^q S \cap R) \cap (fg) = (fg) \cdot (t^q S \cap R)$, since $(t^q S \cap R) + (fg) = R$ (see the proof of the second equality in Lemma 3.1). Thus, $\bar{I} = (t^c f) + fg \cdot (t^q S \cap R)$.

- (2) We have $\bar{I} = \varphi S \cap R = \varphi S$, since $\varphi \in t^{c_0} S \subseteq R$. \square

We consider the case where $R = k[H]$. Let $e \geq 2$ be an integer and set $R = k[H]$, where $H = \langle e, e+1, \dots, 2e-1 \rangle$. Then, since $(t^q S \cap R)S = t^q S$ by Lemma 3.1, we have $q \in H$. Therefore, either $q = 0$, or $q \geq e = c_0$, so that Proposition 3.2 shows the following, since $\mu_R(S) = e$.

Corollary 3.3. *Let $e \geq 2$ be an integer and set $H = \langle e, e+1, \dots, 2e-1 \rangle$. Let $R = k[H]$. Then, for each ideal $I (\neq (0))$ of R , we have $\mu_R(\bar{I}) \in \{1, 2, e\}$.*

4. MAXIMAL IDEALS OF NUMERICAL SEMIGROUP RINGS

In this section we study the semigroup rings of numerical semigroups. In what follows, let $0 < a_1, a_2, \dots, a_v \in \mathbb{Z}$ be integers such that $\text{GCD}(a_1, a_2, \dots, a_v) = 1$. Let $H = \langle a_1, a_2, \dots, a_v \rangle$, and $S = k[t]$, where k is a field. We consider the ring

$$R = k[H] = k[t^{a_1}, t^{a_2}, \dots, t^{a_v}],$$

assuming that $1 \notin H$. Therefore, $c(H) \geq 2$, and $\text{Sing}(R) = \{P_0\}$, where $P_0 = (t^{a_1}, t^{a_2}, \dots, t^{a_v}) = tS \cap R$.

Let $M \in \text{Spec } R$. Recall that M is said to be a k -rational closed point of $\text{Spec } R$, if $k = R/M$. As for the rationality in $\text{Spec } R$, the following result is well-known, which allows us to naturally identify the k -rational closed points of $\text{Spec } R$ with the points of the monomial curve $\mathcal{C} = \{(\alpha^{a_1}, \alpha^{a_2}, \dots, \alpha^{a_v}) \mid \alpha \in k\}$.

Proposition 4.1. *Let $M \in \text{Max } R$. Then the following conditions are equivalent.*

- (1) *M is a k -rational closed point of $\text{Spec } R$.*
- (2) *$M = (\alpha^{a_i} - t^{a_i} \mid 1 \leq i \leq v)$ for some $\alpha \in k$.*
- (3) *$M \subseteq (\alpha - t)S$ for some $\alpha \in k$.*

When this is the case, the element $\alpha \in k$ given in conditions (2) and (3) is uniquely determined for M .

Let us consider k -rational closed points of $\text{Spec } R$ which do not correspond to the origin of the curve $\mathcal{C} = \{(\alpha^{a_1}, \alpha^{a_2}, \dots, \alpha^{a_v}) \mid \alpha \in k\}$. We begin with the following.

Theorem 4.2. *Let $M = (1 - t^{a_i} \mid 1 \leq i \leq v)$. Then*

$$M = (1 - t^c \mid c \in H) = (1 - t^a, 1 - t^b)$$

for all $0 < a, b \in H$ such that $\text{GCD}(a, b) = 1$.

Proof. Let $0 < c \in H$. Without loss of generality, we may assume that $c = (-n)a + mb$ with $n, m \geq 2$. Then, since

$$[1 - t^{(q-1)a+c}] - [1 - t^{qa+c}] = -(1 - t^a)t^{(q-1)a+c}$$

for all $q = 1, 2, \dots, n$, taking the sum of both sides, we have

$$[1 - t^c] - [1 - t^{na+c}] = -(1 - t^a) \cdot \sum_{q=1}^n t^{(q-1)a+c}.$$

Consequently, because $1 - t^{na+c} = 1 - t^{mb} = (1 - t^b) \sum_{q=0}^{m-1} (t^b)^q \in (1 - t^b)$, we get $1 - t^c \in (1 - t^a, 1 - t^b)$. Therefore

$$M \subseteq (1 - t^c \mid c \in H) \subseteq (1 - t^a, 1 - t^b) \subseteq (1 - t)S \cap R,$$

whence the required equalities follow, because $M = (1 - t)S \cap R$ by Proposition 4.1. \square

Example 4.3. Let $H = \langle 3, 5, 7 \rangle$ and set $M = (1 - t^c \mid c \in H)$. Then, M is a maximal ideal of $R = k[t^3, t^5, t^7]$ and $M = (1 - t^3, 1 - t^5) = (1 - t^3, 1 - t^7) = (1 - t^5, 1 - t^7)$.

Let $0 \neq \alpha \in k$ and set $M_\alpha = (\alpha^{a_i} - t^{a_i} \mid 1 \leq i \leq v)$ in $R = k[H]$. Let $\varphi_\alpha \in \text{Aut}_k k[t]$ defined by $\varphi_\alpha(t) = t_1$, where $t_1 = \frac{t}{\alpha}$. Then, $\varphi_\alpha(k[H]) = k[H]$, so that φ_α induces the automorphism ψ_α of R . Therefore, since $\psi_\alpha(M_1) = M_\alpha$, we get the following.

Corollary 4.4. *Let $0 \neq \alpha \in k$. Then $M_\alpha = (\alpha^a - t^a, \alpha^b - t^b)$ for all $0 < a, b \in H$ such that $\text{GCD}(a, b) = 1$.*

Let $\mu(H)$ stand for the minimal number of generators of H . Hence $\mu(H) = \mu_R(P_0)$. Every $M \in \text{Max } R$ is a k -rational closed point of $\text{Spec } R$ if the base field k is algebraically closed, so that by Theorem 2.4 (2) and Corollary 4.4 we readily get the following.

Corollary 4.5 (cf. Theorem 1.1). *Suppose that k is an algebraically closed field. Then, $\mu_R(M) = 2$ if $M \neq P_0$, and $\mu_R(M) = \mu(H)$ if $M = P_0$.*

As another application of Theorem 4.2, we have an explicit system of generators for the integral closures of certain ideals in R . Let us perform the task before closing this note. In what follows, let I be an ideal of R . Similarly as in Section 3, we write $IS = \varphi S$ with $\varphi \in S$, and assume that $\varphi = t^q f$, where $0 < q \in H$ and $f \in S$ with $f(0) = 1$. Let $\mathfrak{a} = t^q S \cap R$. Hence, $\bar{I} = \mathfrak{a} \cdot (fS \cap R)$ by Lemma 3.1. We set $n = \mu_R(\mathfrak{a})$. We then have the following.

Proposition 4.6. $n \geq 2$.

Proof. Assume that $n = 1$. Then, $\mathfrak{a} = t^q S \cap R = t^q R$, so that $\{h \in H \mid h \geq q\} = q + H$. Therefore, $\#\{h \in H \mid h < q\} = \#\{H \setminus (q + H)\} = \dim_k R/t^q R = q$, which is a contradiction, as soon as $1 \notin H$. \square

We consider the specific case where $f = 1 - t$. Let us write $\mathbf{a} = (t^{b_1}, t^{b_2}, \dots, t^{b_n})$ with $b_i \in H$ such that $b_i \geq q$ for each $1 \leq i \leq n$, and set $\mathbf{b} = (t^{b_2}, t^{b_3}, \dots, t^{b_n})$. Then, R/\mathbf{b} is an Artinian local ring with maximal ideal P_0/\mathbf{b} (see Proposition 2.2), whence $t^m S \subseteq \mathbf{b}$ for all $m \gg 0$. Therefore, we can choose an integer $b \in H$ so that $t^b \in \mathbf{b}$ and $\text{GCD}(b_1, b) = 1$. Hence $b \geq q$. Let $b_0 = b$. We then have the following.

Theorem 4.7. $\bar{I} = (t^{b_1} - t^{b_0}, \{t^{b_i} - t^{b_0+b_i}\}_{2 \leq i \leq n})$. Hence $\mu_R(\bar{I}) \leq \mu_R(\mathbf{a})$.

Proof. Let $a = b_1$. By Lemma 3.1 and Theorem 4.2, $\bar{I} = \mathbf{a} \cdot (fS \cap R) = \mathbf{a} \cdot (1 - t^a, 1 - t^b)$, while by Lemma 3.1 $t^a - t^b \in \bar{I}$, because $t^a - t^b \in t^q S \cap R = \mathbf{a}$ and $t^a - t^b \in fS \cap R$. Therefore, since $\mathbf{a} = (t^{b_0}, t^{b_1}, \dots, t^{b_n})$, we get

$$\bar{I} = (t^{b_i} - t^{a+b_i} \mid 0 \leq i \leq n) + (t^{b_i} - t^{b+b_i} \mid 0 \leq i \leq n).$$

Therefore, $\bar{I} = (t^a - t^b) + (t^{b_i} - t^{b+b_i} \mid 0 \leq i \leq n)$, since

$$t^{b_i} - t^{a+b_i} = (-t^{b_i})(t^a - t^b) + (t^{b_i} - t^{b+b_i})$$

for each $0 \leq i \leq n$. We set $J = (t^a - t^b) + (t^{b_i} - t^{b+b_i} \mid 2 \leq i \leq n)$. Then, since $t^b \in \mathbf{b}$, we have $b = b_i + h$ for some $2 \leq i \leq n$ and $h \in H$, so that $t^b - t^{b+b_0} = t^h(t^{b_i} - t^{b+b_i}) \in J$. On the other hand, because

$$t^{b_1} - t^{b+b_1} = (t^a - t^b) + (t^{b_0} - t^{a+b_0}) \quad \text{and} \quad t^{b_0} - t^{a+b_0} = (-t^{b_0})(t^a - t^b) + (t^b - t^{b+b_0}),$$

we get $t^{b_1} - t^{b+b_1} \in J$. Thus $\bar{I} = J$, as claimed. \square

Let us note one example.

Example 4.8. Let $H = \langle 4, 11, 13 \rangle$ and $R = k[H]$. Let $q = 12$ and $f = 1 - t$. Let I be an ideal of R such that $IS = (t^{12}f)S$. Hence, $\bar{I} = (t^{12}f)S \cap R$. We have $\mathbf{a} = t^{12}S \cap R = (t^{12}, t^{13}, t^{15}, t^{22})$. Set $\mathbf{b} = (t^{13}, t^{15}, t^{22})$, and take $a = 12$, $b = 13$. Then, by Theorem 4.7 $\bar{I} = (t^{12} - t^{13}, t^{13} - t^{26}, t^{15} - t^{28}, t^{22} - t^{35})$.

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