WHEN ARE THE RINGS $I : I$ GORENSTEIN?

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Abstract. Let I ($\neq A$) be an ideal of a d-dimensional Noetherian local ring A with $\mathrm{ht}_A I \geq 2$, containing a non-zerodivisor. The problem of when the ring $I : I = \mathrm{End}_A I$ is Gorenstein is studied, in connection with the problem of the Gorensteinness in Rees algebras $\mathcal{R}_A(Q^d)$ for certain parameter ideals Q of A, that was closely explored by the preceding paper [8] of the authors. Examples are given.

1. Introduction

Let $(A, \mathfrak{m})$ be a Noetherian local ring with $d = \dim A \geq 2$ and $t = \operatorname{depth} A \geq 1$. For each ideal I of A we set $\mathcal{R}_A(I) = \bigoplus_{n \geq 0} I^n$ and call it the Rees algebra of I. In the previous paper [8], the authors are particularly interested in the question of when the Rees algebra $\mathcal{R}_A(Q^N)$ is a Gorenstein ring, where Q is a parameter ideal of A and $N \geq 1$. This question is already settled, when A is a Cohen-Macaulay ring, and it is known that $N = d$ or $d-1$ ([6, Theorem (1.2)], [11, Lemma 2.4]). Nevertheless, even though A is not a Cohen-Macaulay ring and $N = d \geq 2$, the Rees algebra $\mathcal{R}_A(Q^d)$ can be a Gorenstein ring. As far as we know, the first result was reported by Y. Shimoda around 30 years ago in his seminar talk at Meiji University, who showed that provided $d = 2$ and $t = 1$, $\mathcal{R}_A((a, b)^2)$ is a Gorenstein ring, under certain specific conditions on the system a, b of parameters of A. His argument was recently rediscovered and motivated the researches [8]. Together with many other results, the authors succeeded in providing a complete generalization of Shimoda's theorem in the following way.

Theorem 1.1 ([8, Theorem 1.1])**.** *Assume that $\mathrm{H}^i_\mathfrak{m}(A) = (0)$ for all $i \notin \{1, d\}$ and $\mathrm{H}^1_\mathfrak{m}(A)$ is a finitely generated A-module. Let $Q = (a_1, a_2, \ldots, a_d)$ be a parameter ideal of A. Then the following conditions are equivalent.*

(1) *$\mathcal{R}_A(Q^d)$ is a Gorenstein ring.*
(2) *$\mathrm{H}^1_\mathfrak{m}(A) \neq (0)$, $\mathrm{r}_A(\mathrm{H}^1_\mathfrak{m}(A)) = 1$, and $(0) :_A \mathrm{H}^1_\mathfrak{m}(A) = \sum_{i=1}^d \mathrm{U}(a_i A)$.*

When this is the case, the (S_2)-ification $\widetilde{A}$ of A is a Gorenstein ring.

Here, $\mathrm{H}^i_\mathfrak{m}(*)$ stands for the local cohomology functor of A with respect to $\mathfrak{m}$, $\mathrm{r}_A(M)$ denotes, for each Cohen-Macaulay A-module M, the Cohen-Macaulay type of M, and $\mathrm{U}(a_i A)$ denotes the unmixed component of the ideal $a_i A$ in A. Added to it, they showed

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that once $\mathcal{R}_A(Q^d)$ is a Gorenstein ring for some parameter ideal Q of A and the (S_2)-ification $\widetilde{A}$ is a Cohen-Macaulay ring, the basic hypothesis that $\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ for all $i \notin \{1, d\}$ and the A-module $\mathrm{H}^1_{\mathfrak{m}}(A)$ is finitely generated should be naturally satisfied.

The theorems in [8], especially the above theorem 1.1, give a clear characterization for $\mathcal{R}_A(Q^d)$ to be a Gorenstein ring, establishing a generalization of Shimoda's result. Nevertheless, it should be noticed here that it seems not so easy to provide ample examples of A and Q, which satisfy Condition (2) of Theorem 1.1. Because the (S_2)-ification $\widetilde{A}$ of A is a Gorenstein ring once $\mathcal{R}_A(Q^d)$ is Gorenstein, there might be a different approach towards the construction of new examples, based on the Gorensteinness of (S_2)-ifications, and if we can do this, it could provide a further viewpoint, not only for the theory of Gorenstein Rees algebras, but also for the study of (S_2)-ifications. This expectation has strongly motivated the present researches.

Let us now state our own results, explaining how this paper is organized. To develop our arguments, we need some basic results on (S_2)-ifications and those on trace ideals as well, which we shall briefly summarize in Section 2. In Section 3, we shall discuss the problem of when the rings $I : I$ are Gorenstein rings, where I $(\neq A)$ is an ideal of A with $\mathrm{ht}_A I \geq 2$, containing a non-zerodivisor of A. We always consider the colon $I : I$ inside the total ring $\mathrm{Q}(A)$ of fractions of A, whence $I : I \cong \mathrm{End}_A I$ as an A-algebra. The A-algebra $I : I$ is closely related to the (S_2)-ification $\widetilde{A}$ of A, which we will explain in Section 3, and eventually we shall give ample concrete examples of the local rings A, which possess Gorenstein Rees algebras $\mathcal{R}_A(Q^d)$ for some parameter ideals Q.

Section 4 is devoted to more constructions of the ideals I, for which $I : I = \widetilde{A}$ and $I : I$ is a regular ring. We will leave in this section the problem on the Gorensteinness in Rees algebras. Instead, we are interested in finding what values the number $t = \operatorname{depth} A$ can take, when $I : I$ is a Gorenstein ring. In general, we have $0 < t < d$, and for $t = 1, 2$ we can construct the rings A and ideals I $(\neq A)$ such that $\operatorname{depth} A = t$, $\mathrm{ht}_A I \geq 2$, and $I : I$ is a regular ring. Although we can show the case where $t = 3$ is also possible, we are not able to cover all the cases of $0 < t < d$. We would like to leave the remainder cases to the interested readers.

2. Preliminaries

2.1. **Some basic results on (S_2)-ifications.** Let R be an arbitrary commutative ring and let $\mathrm{Q}(R)$ denote the total ring of fractions of R. We set

- $\mathrm{Ht}_{\geq 2}(R)=\{I \mid I \text{ is an ideal of } R, \mathrm{ht}_R I \geq 2\}$ and
- $\mathrm{W}(R) = \{a \in R \mid a \text{ is a non-zerodivisor of } R\}$.

Throughout, let us fix a $\mathrm{Q}(R)$-module V and an R-submodule M of V.

Definition 2.1. $\widetilde{M} = \{f \in V \mid If \subseteq M \text{ for some } I \in \mathrm{Ht}_{\geq 2}(R)\}$.

Hence, if L is an R-submodule of V and $M \subseteq L$, then $\widetilde{M} \subseteq \widetilde{L}$. We call $\widetilde{M}$ the (S_2)-ification of M. In fact, $\widetilde{M}$ is an R-submodule of V containing M. In particular, $\widetilde{R}$ considered inside $\mathrm{Q}(R)$ is an intermediate ring $R \subseteq \widetilde{R} \subseteq \mathrm{Q}(R)$. Notice that

$$\widetilde{R} = \{f \in \mathrm{Q}(R) \mid \mathrm{ht}_R[R :_R f] \geq 2\}$$

and that $\widetilde{M}$ is an $\widetilde{R}$-submodule of V also. For $a \in \mathrm{W}(R)$, $x \in V$, and an R-submodule X of V, let us denote $\frac{x}{a} = a^{-1}x$ and $\frac{X}{a} = a^{-1}X$ in V. We then have, for all $a, b \in \mathrm{W}(R)$, that
$$\frac{aM : b}{a} = \frac{M}{a} \cap \frac{M}{b}.$$

Lemma 2.2. *Suppose that R is a Noetherian ring, $\mathfrak{p} \in \operatorname{Spec} R$, and $a \in \mathrm{W}(R)$. If $a \in \mathfrak{p}$ and $\mathrm{ht}_R\mathfrak{p} \geq 2$, then $\mathrm{ht}_R(a, b) \geq 2$ for some $b \in \mathrm{W}(R) \cap \mathfrak{p}$.*

Proof. Notice that $\mathfrak{p} \not\subseteq \bigcup_{P \in \mathrm{Min}_R R/aR} P \cup \bigcup_{P \in \mathrm{Ass}\, R} P$. $\square$

Let $a, b \in R$ and N an R-module. We say that the sequence a, b is N-regular, if a is N-nonzerodivisor and b is N/aN-nonzerodivisor. So, we don't require that $N/(a, b)N \neq (0)$.

Lemma 2.3. *Let $a, b \in \mathrm{W}(R)$. If $\mathrm{ht}_R(a, b) \geq 2$, then the sequence a, b is $\widetilde{M}$-regular.*

Proof. Let $f \in \widetilde{M}$ and assume that $bf = ag$ for some $g \in \widetilde{M}$. We set $x = \frac{f}{a} = \frac{g}{b}$, and choose $I, J \in \mathrm{Ht}_{\geq 2}(R)$ so that $If + Jg \subseteq M$. Then, since $Iax + Jbx \subseteq M$, we get $(Ia + Jb)x \subseteq M$, whence $x \in \widetilde{M}$ because $Ia + Jb \in \mathrm{Ht}_{\geq 2}(R)$. $\square$

Proposition 2.4. *Let R be a Noetherian ring and suppose that one of the following conditions is satisfied.*

(1) $\mathrm{Q}(R)M = V$.
(2) $\mathrm{ht}_R\mathfrak{p} \leq 1$ *for every* $\mathfrak{p} \in \operatorname{Ass} R$.

Then $M = \widetilde{M}$ if and only if every pair $a, b \in \mathrm{W}(R)$ with $\mathrm{ht}_R(a, b) \geq 2$ is M-regular.

Proof. It suffices to prove the *if* part. Assume that $M \neq \widetilde{M}$ and consider the exact sequence
$$\text{(E)} \qquad 0 \to M \to \widetilde{M} \to Z \to 0$$
of R-modules, where $Z = \widetilde{M}/M$. Let $\mathfrak{p} \in \mathrm{Ass}_R Z$. Hence, $\mathfrak{p} = M :_R f$ for some $f \in \widetilde{M}$. Choose $I \in \mathrm{Ht}_{\geq 2}(R)$ so that $If \subseteq M$. We then have $I \subseteq \mathfrak{p}$. Notice that, if Condition (1) is satisfied, $af \in M$ for some $a \in \mathrm{W}(R)$, and if Condition (2) is satisfied, $I \not\subseteq \bigcup_{P \in \mathrm{Ass}\, R} P$. In any case, we get $af \in M$ for some $a \in \mathrm{W}(R)$. Therefore, by Lemma 2.2, $\mathrm{ht}_R(a, b) \geq 2$ for some $b \in \mathrm{W}(R) \cap \mathfrak{p}$, whence by Lemma 2.3, the sequence a, b is $\widetilde{M}$-regular. Therefore, by Sequence (E), we get the long exact sequence
$$0 \to (0) :_Z a \xrightarrow{\sigma} M/aM \to \widetilde{M}/a\widetilde{M} \to Z/aZ \to 0$$
where
$$b\sigma(f \bmod M) = \sigma(bf \bmod M) = 0$$
because $bf \in M$. Therefore, $\sigma(f \bmod M) = 0$ since b is a non-zerodivisor for M/aM, whence $f \in M$. This is impossible. Thus $M = \widetilde{M}$. $\square$

Corollary 2.5. *If R is a Noetherian ring and satisfies the condition (S_2) of Serre, then $R = \widetilde{R}$.*

Corollary 2.6. *With the same assumption as is in Proposition 2.4, the following assertions hold true.*

(1) $\widetilde{\widetilde{M}} = \widetilde{M}$.
(2) *Let $M \subseteq L \subseteq V$ be an R-submodule of V. If every pair $a, b \in \mathrm{W}(R)$ with $\mathrm{ht}_R(a,b) \geq 2$ is L-regular, then $\widetilde{M} \subseteq \widetilde{L} = L$.*

In what follows, unless otherwise specified, let R be a Noetherian ring. We assume that the following additional conditions are also satisfied.

(1) M is a finitely generated R-module.
(2) $\mathrm{Q}(R)M = V$.
(3) $(0) :_{\mathrm{Q}(R)} V = (0)$.

Hence, if $a \in \mathrm{W}(R)$ and $M/aM \neq (0)$, then $\mathrm{ht}_R\mathfrak{p} = 1$ for every $\mathfrak{p} \in \mathrm{Min}_R\, M/aM$ since $(0) :_R M = (0)$. Thanks to Condition (2), every $f \in V$ has an expression of the form $f = \frac{m}{a}$ with $a \in \mathrm{W}(R)$ and $m \in M$.

Let $a \in \mathrm{W}(R)$ and let

$$aM = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_R\, M/aM} Q(\mathfrak{p})$$

be a primary decomposition of aM in M. We set

$$\mathrm{U}(aM) = \begin{cases} M, & \text{if } \ aM = M, \\ \bigcap_{\mathfrak{p} \in \mathrm{Min}_R\, M/aM} Q(\mathfrak{p}), & \text{if } \ aM \neq M. \end{cases}$$

Theorem 2.7. *Let $a \in \mathrm{W}(R)$ and $m \in M$. Then $\frac{m}{a} \in \widetilde{M}$ if and only if $m \in \mathrm{U}(aM)$.*

Proof. We may assume $aM \neq M$. Suppose $\frac{m}{a} \in \widetilde{M}$ and choose $I \in \mathrm{Ht}_{\geq 2}(R)$ so that $I \subseteq aM :_R m$. Let $\mathfrak{p} \in \mathrm{Min}_R\, M/aM$. Then, since $\mathrm{ht}_R\mathfrak{p} = 1$, $aM :_R m \not\subseteq \mathfrak{p}$, so that $m \in [aM]_\mathfrak{p} \cap M = Q(\mathfrak{p})$. Hence, $m \in \mathrm{U}(aM)$.

Conversely, suppose $m \in \mathrm{U}(aM)$. If $aM = \mathrm{U}(aM)$, then $m \in aM$, so that $\frac{m}{a} \in M \subseteq \widetilde{M}$. Suppose $aM \neq \mathrm{U}(aM)$ and set $\mathcal{F} = \mathrm{Ass}_R\, M/aM \setminus \mathrm{Min}_R\, M/aM$. Then, $\mathcal{F} \neq \emptyset$ and for each $\mathfrak{p} \in \mathcal{F}$, there is an integer $\ell = \ell(\mathfrak{p}) \gg 0$ such that $\mathfrak{p}^\ell M \subseteq Q(\mathfrak{p})$, whence $\mathfrak{p}^\ell(aM) \subseteq Q(\mathfrak{p})$. Therefore, setting $\mathfrak{a} = \prod_{\mathfrak{p} \in \mathcal{F}} \mathfrak{p}^{\ell(\mathfrak{p})}$, we have

$$\mathfrak{a}\mathrm{U}(aM) \subseteq \bigcap_{\mathfrak{p} \in \mathcal{F}} Q(\mathfrak{p}) \cap \mathrm{U}(aM) = aM.$$

Hence, $\frac{\mathrm{U}(aM)}{a} \subseteq \widetilde{M}$, because $\mathfrak{a} \in \mathrm{Ht}_{\geq 2}(R)$. $\square$

Corollary 2.8. *If $\widetilde{M} \subseteq \frac{M}{a}$ for some $a \in \mathrm{W}(R)$, then $\widetilde{M} = \frac{\mathrm{U}(aM)}{a}$. Consequently*

$$\widetilde{M} = \bigcup_{a \in \mathrm{W}(R)} \frac{\mathrm{U}(aM)}{a}.$$

Proof. Let us check the second assertion. Let $f \in \widetilde{M}$ and write $f = \frac{m}{a}$ with $a \in \mathrm{W}(R)$ and $m \in M$. We take $I \in \mathrm{Ht}_{\geq 2}(R)$ so that $Im \subseteq aM$. If $aM = M$, then $f \in \frac{\mathrm{U}(aM)}{a}$. Suppose $aM \neq M$. We then have $I \not\subseteq \mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{Min}_R\, M/aM$, so that $m \in [aM]_\mathfrak{p} \cap M = Q(\mathfrak{p})$ for every $\mathfrak{p} \in \mathrm{Min}_R\, M/aM$. Thus, $f = \frac{m}{a} \in \frac{\mathrm{U}(aM)}{a}$. $\square$

Corollary 2.9. *$\widetilde{M}$ is a finitely generated R-module if and only if $a\widetilde{M} = \mathrm{U}(aM)$ for some $a \in \mathrm{W}(R)$.*

2.2. **Trace ideals.** Let R be an arbitrary commutative ring and let M, X be R-modules. Let

$$\tau : \mathrm{Hom}_R(M, X) \otimes_R M \to X$$

denote the homomorphism defined by $\tau(f \otimes m) = f(m)$ for each $f \in \mathrm{Hom}_R(M, X)$ and $m \in M$. We set $\mathrm{Tr}_X(M) = \mathrm{Im}\,\tau$ and call it the trace module of M in X. In fact, $\mathrm{Tr}_X(M)$ is an R-submodule of X, and a given R-submodule Y of X is called a *trace submodule* in X, if $Y = \mathrm{Tr}_X(M)$ for some R-module M.

The following result is due to H. Lindo [13].

Proposition 2.10 ([13, Lemma 2.3])**.** *Let I be an ideal of R. Then the following conditions are equivalent.*

(1) *I is a trace ideal in R, that is $I = \mathrm{Tr}_R(M)$ for some R-module M.*
(2) *$I = \mathrm{Tr}_R(I)$.*
(3) *For each homomorphism $f : I \to R$ of R-modules, there is an endomorphism $g : I \to I$ such that $f = \iota{\cdot}g$, where $\iota : I \to R$ denotes the embedding.*

When I contains a non-zerodivisor, one can add the following, where the colons are considered inside the total ring $\mathrm{Q}(R)$ of fractions of R.

(4) *$I : I = R : I$.*

3. When are the rings $I : I$ Gorenstein?

We are now in a position to discuss the question of when the (S_2)-ifications are Gorenstein rings. Let us begin with the following.

Lemma 3.1. *Let A be a Noetherian ring. Let $A \subseteq B \subseteq \mathrm{Q}(A)$ be a subring of $\mathrm{Q}(A)$ and assume that B is a finitely generated A-module. Let I be an ideal of B such that $I \subseteq A$ and $\mathrm{ht}_A I \geq 2$. If A is locally quasi-unmixed and B satisfies (S_2), then I is a trace ideal in A and $B = I : I$.*

Proof. Since A is locally quasi-unmixed, we have $\mathrm{ht}_B P = \mathrm{ht}_A(P \cap A)$ for every $P \in \mathrm{Spec}\, B$ ([14, THEOREM 3.8]), so that $\mathrm{ht}_B I \geq 2$, whence $\mathrm{grade}_B\, I \geq 2$ because B satisfies (S_2). Therefore, $B = I : I$, so that

$$A : I \subseteq B : I = B = I : I \subseteq A : I,$$

which implies, by Proposition 2.10, that I is a trace ideal in A. □

Example 3.2. Let k be a field and $S = k[[X, Y, Z, W]]$ be the formal power series ring over k. We consider the ring $A = S/[(X, Y) \cap (Z, W)]$ and let $\mathfrak{m}$ denote the maximal ideal of A. Then, $\mathfrak{m} = A : B$, where $B = S/(X, Y) \times S/(Z, W)$, and Lemma 3.1 tells us that for every $\ell \geq 1$, $\mathfrak{m}^\ell$ is a trace ideal in A and $B = \mathfrak{m}^\ell : \mathfrak{m}^\ell$.

Proposition 3.3. *Let A be a Noetherian local ring possessing the canonical module K_A and I $(\neq A)$ an ideal of A with $\mathrm{ht}_A I \geq 2$. Assume that there exists an exact sequence*

$$0 \to A \to \mathrm{K}_A \to C \to 0$$

of A-modules such that $IC = (0)$. Then the following assertions hold true, where $R = \widetilde{A}$.

(1) $R \cong \mathrm{K}_A$ *as an* A*-module.*
(2) $\mathrm{Hom}_A(R, \mathrm{K}_A) \cong R$ *as an* R*-module.*
(3) *If* K_A *is a Cohen-Macaulay* A*-module, then* R *is a Gorenstein ring.*
(4) *If* I *is a trace ideal in* A*, then* $R = I : I$*.*

Proof. (1), (2) Let $\mathfrak{p} \in \operatorname{Ass} A$. Then $I \not\subseteq \mathfrak{p}$ because $\mathfrak{p} \in \operatorname{Ass}_A \mathrm{K}_A = \operatorname{Assh} A$, so that $C_\mathfrak{p} = (0)$ and

$$A_\mathfrak{p} \cong (\mathrm{K}_A)_\mathfrak{p} = \mathrm{K}_{A_\mathfrak{p}}$$

([2, Theorem 4.2], see also [10, Satz 5.22]). Hence, $A_\mathfrak{p}$ is a Gorenstein ring for every $\mathfrak{p} \in \operatorname{Ass} A$, that is $\mathrm{Q}(A)$ is a Gorenstein ring. We choose an A-submodule K of $\mathrm{Q}(A)$ so that $K \cong \mathrm{K}_A$ as an A-module. Hence, $\mathrm{Q}(A)K = \mathrm{Q}(A)$, because $\mathrm{Q}(A)$ is self-injective. Then, since I contains a subsystem of parameters of A of length 2, we get, taking the K-dual of the sequence $0 \to A \xrightarrow{\varphi} K \to C \to 0$, the natural isomorphism

$$\psi : K : K = \mathrm{Hom}_A(K, K) \xrightarrow{\varphi^*} \mathrm{Hom}_A(A, K) = K$$

of A-modules, where $\psi(1) = \varphi(1)$. Hence, $R \cong \mathrm{K}_A$ as an A-module since $R = K : K$ by [3, Theorem 1.6], which shows Assertion (1). On the other hand, since $\psi(1) = \varphi(1)$, $(K : K)/A \cong C$ as an A-module, so that

$$IR = I(K : K) \subseteq A.$$

We now notice that $R \cong K$ as an R-module, because $R \cong K$ as an A-module and K is an R-submodule of $\mathrm{Q}(A)$ such that $\mathrm{Q}(A)K = \mathrm{Q}(A)$. Let $K = \alpha R$ for some α of $\mathrm{Q}(A)$. Then, α is invertible in $\mathrm{Q}(A)$, so that

$$K : R = \alpha R : R = \alpha[R : R] = \alpha R = K,$$

which shows Assertion (2).

(3) Since $\operatorname{depth}_A R = \dim A$ by Assertion (1), every system of parameters of A forms a regular sequence for R, whence $\mathrm{ht}_R M = \dim A$ for every $M \in \operatorname{Max} R$, so that R_M is a Gorenstein ring by Assertion (2) ([10, Korollar 5.14]).

(4) Suppose that I is a trace ideal in A and set $B = I : I$. Then, since $IR \subseteq A$, we have $R \subseteq A : I = B$ where the equality follows from Proposition 2.10, while $B \subseteq \widetilde{A} = R$ by Definition 2.1, since $IB = I \subseteq A$ and $\mathrm{ht}_A I \geq 2$. Consequently, $R = B$, whence $\widetilde{A} = I : I$. $\square$

Corollary 3.4. *Let* A *be a Noetherian local ring with* $\dim A = d$ *and* $t = \operatorname{depth} A$*. Let* $A \subseteq B \subseteq \mathrm{Q}(A)$ *be a subring of* $\mathrm{Q}(A)$ *such that* B *is a finitely generated* A*-module. We set* $\mathfrak{a} = A : B$ *and assume the following three conditions are satisfied.*

(1) A *is a quasi-unmixed ring.*
(2) $\mathrm{ht}_A \mathfrak{a} \geq 2$.
(3) B *is a Gorenstein ring.*

Then the following assertions hold true.

(a) $B = \widetilde{A}$, $\operatorname{depth}_A B = d$, $\mathfrak{a}$ *is a trace ideal in* A*, and* $B = \mathfrak{a} : \mathfrak{a}$*.*
(b) $\operatorname{Ass} A = \operatorname{Assh} A$.
(c) A *possesses the canonical module* K_A*, and* $\mathrm{K}_A \cong B$ *as an* A*-module.*

(d) *$A = B$ if and only if $t = d$.*

Proof. We have, by Lemma 3.1, $B = \mathfrak{a} : \mathfrak{a}$ and $\mathfrak{a}$ is a trace ideal in A. Because $\mathrm{ht}_B M = \mathrm{ht}_A \mathfrak{m} = d$ for all $M \in \operatorname{Max} B$, every system of parameters of A forms a regular sequence in B_M, so that it forms a regular sequence for the A-module B. Hence, $\operatorname{depth}_A B = d$. Let $C = B/A$. Then $\dim_A C \le d-2$ since $\mathfrak{a}C = (0)$, so that $\mathrm{H}^d_\mathfrak{m}(A) \cong \mathrm{H}^d_\mathfrak{m}(B)$ as an A-module (here $\mathfrak{m}$ denotes the maximal ideal of A). Therefore, $\mathrm{K}_{\widehat{A}} \cong \widehat{A} \otimes_A B$ as an $\widehat{A}$-module where $\widehat{A}$ denotes the $\mathfrak{m}$-adic completion of A, whence A possesses the canonical module K_A and $\mathrm{K}_A \cong B$ as an A-module. We have $B \subseteq \widetilde{A}$ since $\mathrm{ht}_A \mathfrak{a} \ge 2$, while $\widetilde{A} \subseteq \widetilde{B} = B$ by Corollary 2.6 (2). Hence, $B = \widetilde{A}$. Notice that $\operatorname{Ass} A \subseteq \operatorname{Ass}_A B = \operatorname{Ass}_A \mathrm{K}_A = \operatorname{Assh} A$, and we have $\operatorname{Ass} A = \operatorname{Assh} A$. If A is a Cohen-Macaulay ring, then the depth lemma tells us that $\operatorname{depth}_A C \ge d-1$, which forces $C = (0)$ because $\dim_A C \le d-2$. Hence, $A = B$ if $t = d$, which completes the proof. $\square$

Theorem 3.5. *Let A be a Noetherian local ring with $d = \dim A \ge 2$ and $\operatorname{depth} A \ge 1$. Assume that A is a quasi-unmixed ring. Let I ($\ne A$) be an ideal of A with $\mathrm{ht}_A I \ge 2$ and assume that I contains a non-zerodivisor of A. We set $B = I : I$. Then the following conditions are equivalent.*

(1) *B is a Gorenstein ring.*
(2) *A possesses the canonical module K_A and I is a trace ideal in A such that* (i) *K_A is a Cohen-Macaulay A-module and* (ii) *there exists an exact sequence*
$$0 \to A \to \mathrm{K}_A \to C \to 0$$
of A-modules such that $IC = (0)$.
(3) *$\operatorname{depth}_A B = d$, A possesses the canonical module K_A, and $B \cong \mathrm{K}_A$ as an A-module.*

When this is the case, A is an unmixed ring with $B = \widetilde{A}$, and $\mathfrak{a} = A : B$ is a trace ideal in A with $B = \mathfrak{a} : \mathfrak{a}$.

Proof. (1) $\Rightarrow$ (3) See Corollary 3.4.

(3) $\Rightarrow$ (2) Notice that $\operatorname{grade}_B I = \mathrm{ht}_B I \ge 2$, since B is a Cohen-Macaulay ring. We have the exact sequence
$$0 \to A \to \mathrm{K}_A \to C \to 0$$
of A-modules such that $IC = (0)$, while by Lemma 3.1 I is a trace ideal in A.

(2) $\Rightarrow$ (1) Since $\operatorname{depth}_A \mathrm{K}_A = d$, by Proposition 3.3 B is a Gorenstein ring.

See Lemma 3.1 and Corollary 3.4 for the last assertion. $\square$

Corollary 3.6. *Let A be a Noetherian local ring and let I ($\ne A$) be an ideal of A with $\mathrm{ht}_A I \ge 2$. Assume that I contains a non-zerodivisor of A. We set $B = I : I$. Then the following conditions are equivalent.*

(1) *B is a Gorenstein ring, A is a homomorphic image of a Cohen-Macaulay ring, and $\operatorname{Min} A = \operatorname{Assh} A$.*
(2) *A possesses the canonical module K_A and I is a trace ideal in A such that* (i) *K_A is a Cohen-Macaulay A-module and* (ii) *there exists an exact sequence*
$$0 \to A \to \mathrm{K}_A \to C \to 0$$

of A-modules such that $IC = (0)$.
(3) $\operatorname{depth}_A B = d$, *$A$ possesses the canonical module* K_A, *and* $B \cong \mathrm{K}_A$ *as an A-module.*

When this is the case, A is an unmixed ring with $B = \widetilde{A}$, and $\mathfrak{a} = A : B$ is a trace ideal in A with $B = \mathfrak{a} : \mathfrak{a}$.

Proof. Suppose that Condition (2) or (3) is satisfied. Then, $(0) :_A \mathrm{K}_A = (0)$ and K_A is a Cohen-Macaulay A-module, so that all the formal fibers of A are Cohen-Macaulay, while A is unmixed since A is a submodule of K_A. Therefore, thanks to Kawasaki's theorem [12, Theorem 1.1], A is a homomorphic image of a Cohen-Macaulay ring. On the other hand, once A is a homomorphic image of a Cohen-Macaulay ring with $\operatorname{Min} A = \operatorname{Assh} A$, A is quasi-unmixed. These observations enable us to assume, from the beginning, that A is quasi-unmixed, whence the assertion follows from Theorem 3.5. □

4. Relationship with Gorenstein Rees algebras of powers of parameters

Theorem 4.1 (cf. [8]). *Let $(A, \mathfrak{m})$ be a Noetherian complete local ring such that $d = \dim A \geq 2$, $\operatorname{depth} A = 1$, and $\operatorname{Min} A = \operatorname{Assh} A$. Let I be an $\mathfrak{m}$-primary ideal of A. We set $B = I : I$ and $\mathfrak{a} = A : B$, and suppose that the following three conditions are satisfied.*

(1) *B is a Gorenstein ring.*
(2) *$A \neq B$ and $\mathrm{r}_A(B/A) = 1$, that is the socle of the A-module B/A has length one.*
(3) *$\mathfrak{a} = (a_1, a_2, \ldots, a_d)B$ for some $a_1, a_2, \ldots, a_d \in \mathfrak{m}$.*

Then, $B = \mathfrak{a} : \mathfrak{a}$, and the Rees algebra $\mathcal{R}_A(Q^d)$ of Q^d is a Gorenstein ring, where $Q = (a_1, a_2, \ldots, a_d)$ in A.

Proof. We have $B = \mathfrak{a} : \mathfrak{a}$ by Theorem 3.5. Since $I{\cdot}(B/A) = (0)$, applying the functor $\mathrm{H}^i_{\mathfrak{m}}(*)$ to the exact sequence

$$0 \to A \to B \to B/A \to 0$$

of A-modules, we get $\mathrm{H}^1_{\mathfrak{m}}(A) \cong B/A$ and $\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ for all $i \notin \{1, d\}$. Hence

$$(0) :_A \mathrm{H}^1_{\mathfrak{m}}(A) = \mathfrak{a} \quad \text{and} \quad \mathrm{r}_A(\mathrm{H}^1_{\mathfrak{m}}(A)) = 1.$$

On the other hand, since $\operatorname{depth}_A B = d$ and $a_1, a_2, \ldots, a_d$ forms a system of parameters in A, the sequence $a_1, a_2, \ldots, a_d$ is B-regular. Hence, each a_i is a non-zerodivisor of A and $a_i B \subseteq A$, so that $B = \widetilde{A} \subseteq a_i^{-1}A$, whence $B = a_i^{-1}\mathrm{U}(a_iA)$ by Corollary 2.8, where $\mathrm{U}(a_iA)$ denotes the unmixed component of the ideal a_iA. Hence

$$\mathfrak{a} = \sum_{i=1}^{d} a_iB = \sum_{i=1}^{d} \mathrm{U}(a_iA).$$

Consequently, $\mathcal{R}_A(Q^d)$ is a Gorenstein ring, thanks to Theorem 1.1. □

4.1. **The simplest examples given by specific Buchsbaum rings.** Let $(S, \mathfrak{n})$ be a Gorenstein complete local ring with $d = \dim S \geq 2$ and assume that S contains a coefficient field k. Let $\mathfrak{q} = (a_1, a_2, \ldots, a_d)S$ be a parameter ideal of S such that $\mathfrak{q} \neq \mathfrak{n}$. We set $A = k + \mathfrak{q}$. Then, A is a subring of S, and $\mathfrak{q}$ is a maximal ideal in A, since $k \cong A/\mathfrak{q}$. We have $\ell_A(S/A) = \ell_A(S/\mathfrak{q}) - 1 < \infty$, where $\ell_A(*)$ denotes the length.

Therefore, S is a finitely generated A-module, so that $(A, \mathfrak{q})$ is a Noetherian complete local ring with $\dim A = d$. Let $\mathfrak{m}$ $(= \mathfrak{q})$ stand for the maximal ideal of A. We then have $\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ for all $i \notin \{1, d\}$ and $\mathrm{H}^1_{\mathfrak{m}}(A) = S/A$, because $\operatorname{depth}_A S = d$ and $\ell_A(S/A) < \infty$. Hence, $\operatorname{depth} A = 1$, and A is a Buchsbaum ring ([4, 5]). We have $S = \mathfrak{m} : \mathfrak{m}$, and $\operatorname{Ass} A = \operatorname{Ass}_A S = \operatorname{Assh} A$, since $\operatorname{depth}_A S = d$. Thus, we get the following.

Theorem 4.2. *Suppose that $\ell_S(S/\mathfrak{q}) = 2$. Then, $\mathcal{R}_A(Q^d)$ is a Gorenstein ring, where $Q = (a_1, a_2, \ldots, a_d)A$.*

Corollary 4.3. *Let $S = k[[X_1, X_2, \ldots, X_d]]$ $(d \geq 2)$ be the formal power series ring over a field k and let $\mathfrak{q} = (X_1^2, X_2, \ldots, X_d)S$. Then, $\mathcal{R}_A(Q^d)$ is a Gorenstein ring, where $A = k + \mathfrak{q}$ and $Q = (X_1^2, X_2, \ldots, X_d)A$.*

4.2. **The case where A is an integral domain.** Let $B = k[[t_1, t_2, \ldots, t_n, s]]$ $(n \geq 1)$ be the formal power series ring over a field k, and set $V = k[[t_1, t_2, \ldots, t_n]]$, $d = n+1$. Hence, $B = V[[s]]$. We choose a subring P of V so that V is a finitely generated P-module and $P : V \neq (0)$. Therefore, P is a Noetherian local ring and $V = \overline{P}$. We set $\mathfrak{c} = P : V$ and assume that $\sqrt{\mathfrak{c}} = \mathfrak{n}$, where $\mathfrak{n}$ denotes the maximal ideal of P.

We set $A = P + sB$. Hence, A is a subring of B and B is a finitely generated A-module, because B/A is a homomorphic image of B/sB and B/sB is a finitely generated P-module. Therefore, A is a Noetherian complete local ring with $\dim A = d \geq 2$, and $B = \overline{A}$. We set $\mathfrak{a} = A : B$. Let $\mathfrak{m}$ denote the maximal ideal of A.

Lemma 4.4. *$\mathfrak{a} = \mathfrak{c} + sB$ and $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal of A. Hence, $B = \mathfrak{a} : \mathfrak{a}$ and* $\operatorname{depth} A = 1$.

Proof. Let $b \in B$ and write $b = v + x$ with $v \in V$ and $x \in sB$. We then have, for each $c \in \mathfrak{c}$, $cb = cv + cx \in P + sB = A$, whence $\mathfrak{c} + sB \subseteq \mathfrak{a}$. Conversely, let $\varphi \in \mathfrak{a}$ and write $\varphi = a + x$ with $a \in P$ and $x \in sB$. Then $a \in A : B$, so that $aV \subseteq A \cap V = P + (sB \cap V) = P$, whence $a \in \mathfrak{c}$ and $\varphi \in \mathfrak{c} + sB$. Therefore, $\mathfrak{a} = \mathfrak{c} + sB$, so that $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal of A. Consequently, $B = \mathfrak{a} : \mathfrak{a}$ by Lemma 3.1. The last assertion follows from the fact that $\operatorname{depth}_A B = d$ and $\ell_A(B/A) < \infty$. $\square$

Proposition 4.5. $\mathrm{r}_A(B/A) = \mathrm{r}_P(V/P)$.

Proof. Notice that $V/P \cong B/A$ as a P-module, since $V = B/sB$ and $P = A/sB$. We then have
$$\mathrm{r}_A(B/A) = \mathrm{r}_{A/sB}(B/A) = \mathrm{r}_P(B/A) = \mathrm{r}_P(V/P)$$
as claimed. $\square$

Corollary 4.6. *If $\mathfrak{c} = (a_1, a_2, \ldots, a_n)V$ for some $a_1, a_2, \ldots, a_n \in P$ and $\mathrm{r}_P(V/P) = 1$, then $\mathcal{R}_A(Q^d)$ is a Gorenstein ring, where $Q = (a_1, a_2, \ldots, a_n, s)$.*

Proof. We get $\mathfrak{a} = (a_1, a_2, \ldots, a_n, s)B$ by Lemma 4.4. Hence, the assertion follows from Theorem 4.1. $\square$

Theorem 4.7. *For each one of the following cases, we have $B = \mathfrak{a} : \mathfrak{a}$ and the assumptions of Corollary 4.6 are satisfied. Therefore, $\mathcal{R}_A(Q^d)$ is a Gorenstein ring for the appropriate parameter ideal Q of A.*

(1) *$n = 1$ and $P = k[[H]]$, where H is a symmetric numerical semigroup such that $1 \notin H$.*
(2) *$n = 1$ and $P = k[[t^2 + t^3, t^4, t^6]]$, where $t = t_1$.*
(3) *Let k/k_0 be an extension of fields with $[k : k_0] = 2$. Choose $\alpha \in k \setminus k_0$ and set $P = k_0[[t_1, t_2, \ldots, t_n, \alpha t_1, \alpha t_2, \ldots, \alpha t_n]]$ in $V = k[[t_1, t_2, \ldots, t_n]]$.*

Notice that for Case (2), P is not the semigroup ring for any numerical semigroup.

Proof. See Lemma 4.4 for the equality $B = \mathfrak{a} : \mathfrak{a}$.

(1) We have $\mathfrak{c} = t^c V$ $(t = t_1)$ for some $c > 0$, while $\mathrm{r}_P(V/P) = 1$, because P is a Gorenstein ring and V/P is a P-submodule of $\mathrm{H}^1_{\mathfrak{n}}(P)$.

(2) Since $\mathfrak{n}V = t^2V$, we have $V = P + tP$. First, suppose ch(k) = 2. Then, $(t^2+t^3)^2 = t^4 + t^6$, so that $P = k[[t^2 + t^3, t^4]]$ and $t^7 \in P$. Therefore, $V = \overline{P}$, and P is a Gorenstein ring. Suppose that ch(k) $\neq 2$. We then have $t^5 \in P$ because $(t^2 + t^3)^2 \in P$, so that $k[[t^2 + t^3, t^5]] \subseteq P$. Hence $V = \overline{P}$. Let $\mathrm{o}(*)$ denote the tV-adic valuation (or the order function) of V. For a subring R of V, we set $v(R) = \{\mathrm{o}(f) \mid 0 \neq f \in R\}$. Because

$$\langle 2, 5\rangle \subseteq v(k[[t^2 + t^3, t^5]]) \subseteq v(P)$$

and $3 \notin v(P)$, we have $\langle 2, 5\rangle = v(k[[t^2 + t^3, t^5]]) = v(P)$, whence $k[[t^2 + t^3, t^5]] = P$. In any case, P is a Gorenstein ring but not the semigroup ring for any numerical semigroup, since $t^3 \notin P$. We have $\mathfrak{c} = t^c V$ for some $c \geq 2$, since $t^{c_0} V \subseteq P$ for all $c_0 \gg 0$.

(3) We have $V = P + \alpha P$ and $\mathfrak{n}V = \mathfrak{n}$, since $\alpha^2 \in k = k_0 + k_0\alpha$. Consequently, $\mathfrak{c} = \mathfrak{n}$ since $P \neq V$, so that $\mathfrak{c} = (t_1, t_2, \ldots, t_n)V$, while $\mathrm{r}_P(V/P) = 1$, since $V/P \cong P/\mathfrak{n}$ as a P-module. □

4.3. **The case where A is a fiber product.** Let S be a Gorenstein complete local ring with $d = \dim S \geq 2$ and let $a_1, a_2, \ldots, a_d$ be a system of parameters of S. Let $\mathfrak{q} = (a_1, a_2, \ldots, a_d)$, $T = S/\mathfrak{q}$, and $B = S \times S$. We consider the fiber product $A = S \times_T S$. By definition

$$A = \{(x, y) \in B \mid x \equiv y \bmod \mathfrak{q}\}$$

and there is the exact sequence

$$\text{(E)} \qquad 0 \to A \xrightarrow{\iota} B \xrightarrow{\varphi} T \to 0$$

of A-modules, where $\iota : A \to B$ denotes the embedding and $\varphi : B \to T$, $\varphi(x, y) = x - y \bmod \mathfrak{q}$. Because A is a subring of B and B is a finitely generated A-module, A is a Noetherian complete local ring with $\dim A = d$, while $\operatorname{depth} A = 1$, since $\dim T = 0$. Let $\alpha_i = (a_i, a_i) \in A$ for each $1 \leq i \leq d$ and set $Q = (\alpha_1, \alpha_2, \ldots, \alpha_d)$. Then, Q is a parameter ideal of A and we have the following.

Theorem 4.8. *The Rees algebra $\mathcal{R}_A(Q^d)$ is a Gorenstein ring.*

Proof. The exact sequence (E) shows $\mathrm{r}_A(B/A) = \mathrm{r}(T) = 1$. The ring B is Gorenstein with $A : B = \mathrm{Ann}_A T = QB$, so that $B = QB : QB$ by Lemma 3.1. Hence, the assertion follows from Theorem 4.1 □

Corollary 4.9. *Let $U = k[[X_1, X_2, \ldots, X_d, Y_1, Y_2, \ldots, Y_d]]$ $(d \geq 2)$ be the formal power series ring over a field k and set $A = U/[(X_1, X_2, \ldots, X_d) \cap (Y_1, Y_2, \ldots, Y_d)]$. Let z_i denote, for each $1 \leq i \leq d$, the image of $X_i + Y_i$ in A. Then, $\mathcal{R}_A(Q^d)$ is a Gorenstein ring, where $Q = (z_1, z_2, \ldots, z_d)$.*

Proof. Since $A \cong S \times_k S$ where $S = k[[X_1, X_2, \ldots, X_d]]$ and $k = S/(X_1, X_2, \ldots, X_d)$, the result readily follows from Theorem 4.8. $\square$

Remark 4.10. More generally, let $t \in S$ and let $A = S \overset{t}{\ltimes} \mathfrak{q}$ denote the amalgamated duplication of S along $\mathfrak{q}$ ([1]; see [9, Section 3] also). Let $Q = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, where $\alpha_i = (a_i, 0) \in A$. Then, for every $t \in S$, Q is a parameter ideal of A, and $\mathcal{R}_A(Q^d)$ is a Gorenstein ring. Let us emphasize that if $t = 1$, then $A = S \times_T S$, which is the fiber product, and if $t = 0$, then $A = S \ltimes \mathfrak{q}$, which is the idealization of $\mathfrak{q}$ over S.

5. More constructions

In this section we leave the problem on the Gorensteinness in Rees algebras. Instead, we are interested in finding what values the number $t = \operatorname{depth} A$ can take, when $I : I$ is a Gorenstein ring.

First of all, let R be an arbitrary commutative ring, and let $\{I_i\}_{1 \leq i \leq \ell}$ be a family of ideals of R, where $\ell \geq 2$. We set $J_i = I_1 \cap \cdots \overset{\vee}{I_i} \cap \cdots \cap I_\ell$ for each $1 \leq i \leq \ell$. We then have the following.

Lemma 5.1. $\bigcap_{i=1}^{\ell} (I_i + J_i) = \sum_{i=1}^{\ell} J_i$.

Proof. This follows, for example, by induction on ℓ. The detail is left to the reader. $\square$

Let $S = \bigoplus_{i=1}^{\ell} R/I_i$. Assume that $\bigcap_{i=1}^{\ell} I_i = (0)$ and set $I = R : S$.

Proposition 5.2. *The following assertions hold true.*
(1) $I = \sum_{i=1}^{\ell} J_i$.
(2) $I \subseteq I_i + I_j$, *if* $i \neq j$.
(3) $\mathrm{V}(I) = \bigcup_{i \neq j} \mathrm{V}(I_i + I_j)$ *in* $\operatorname{Spec} R$.

Proof. (1) We set $\mathbf{e}_i = (0, \cdots, \overset{\overset{i}{\vee}}{1}, \cdots, 0) \in S$ for each $1 \leq i \leq \ell$. Let $a \in R$ and we have

$$
\begin{aligned}
a\mathbf{e}_i \in R \text{ for all } 1 \leq i \leq \ell \quad &\Leftrightarrow \quad \text{for each } 1 \leq i \leq \ell \text{ there exists } f \in R \text{ such that} \\
&\qquad (0, \cdots, \overset{i}{\overline{a}}, \cdots, 0) = (\overline{f}, \overline{f}, \cdots, \overline{f}) \text{ in } S \\
&\Leftrightarrow \quad \text{for each } 1 \leq i \leq \ell \text{ there exists } f \in J_i \text{ such that } a - f \in I_i \\
&\Leftrightarrow \quad a \in \bigcap_{i=1}^{\ell} (I_i + J_i) = \sum_{i=1}^{\ell} J_i,
\end{aligned}
$$

where $\overline{c}$ denotes, for $c \in R$ and $1 \leq j \leq \ell$, the image of c in R/I_j. Hence $I = \sum_{i=1}^{\ell} J_i$.

(2), (3) These are straightforward. $\square$

Let us consider a more specific situation. Let k be a field and let $n, \ell \geq 2$ be integers. Let $T = k[[X_1, X_2, \cdots, X_n]]$ denote the formal power series ring over k. Let $F_1, F_2, \ldots, F_\ell$ be subsets of $\{X_1, X_2, \ldots, X_n\}$ and assume that

(1) $F_i \neq \emptyset$ for every $1 \leq i \leq n$ and
(2) $F_i \not\subseteq F_j$ if $i \neq j$.

We set $P_i = (F_i) \in \operatorname{Spec} T$ and $\mathfrak{a} = \bigcap_{i=1}^{\ell} P_i$. Let $A = T/\mathfrak{a}$, $B = \bigoplus_{i=1}^{\ell} T/P_i$, and $I = A : B$. Then, $\mathfrak{a} = \bigcap_{i=1}^{\ell} P_i$ is a reduced primary decomposition of $\mathfrak{a}$ in T, and $B = \overline{A}$. Therefore, $\operatorname{Ass}_T A = \{P_1, P_2, \ldots, P_\ell\}$, and $\dim A = \max\{n - |F_i| \mid 1 \leq i \leq \ell\}$, where $|*|$ denotes the number of elements. Since $\ell \geq 2$, $A \subsetneq B$, and setting $\mathfrak{p}_i = P_i/\mathfrak{a}$ for each $1 \leq i \leq n$, we have $\mathfrak{p}_i + \mathfrak{p}_j \in \operatorname{Spec} A$ for all $1 \leq i, j \leq \ell$. Hence, by Proposition 5.2 we get the following.

Corollary 5.3. $\mathrm{V}(I) = \bigcup_{i \neq j} \mathrm{V}(\mathfrak{p}_i + \mathfrak{p}_j)$. *Consequently,* $\mathrm{ht}_A I = \min\{\mathrm{ht}_A(\mathfrak{p}_i + \mathfrak{p}_j) \mid i \neq j\}$.

Proposition 5.4. *A is an unmixed ring if and only if $|F_i|$ is independent of the choice of i. When this is the case, $|F_i \setminus F_j| = |F_j \setminus F_i|$ for all $i \neq j$ and $\mathrm{ht}_A I = \min\{|F_i \setminus F_j| \mid i \neq j\}$.*

Proof. Since $\dim A/\mathfrak{p}_i = n - |F_i|$ for every $1 \leq i \leq \ell$, the first assertion follows. Suppose A is unmixed and let $i \neq j$. We set $Q = \mathfrak{p}_i + \mathfrak{p}_j$. We then have

$$\begin{aligned} \mathrm{ht}_A Q &= \mathrm{ht}_{A/\mathfrak{p}_i} Q/\mathfrak{p}_i = \mathrm{ht}_{T/P_i}[(P_i + P_j)/P_i] \\ &= |(F_i \cup F_j) \setminus F_i| \\ &= |F_j \setminus F_i|. \end{aligned}$$

For the same reason, $\mathrm{ht}_A Q = \mathrm{ht}_{A/\mathfrak{p}_j} Q/\mathfrak{p}_j = |F_i \setminus F_j|$, whence the second assertion follows. The last assertion is now clear. □

We now obtain the following.

Theorem 5.5. *Suppose that $|F_i|$ is independent of the choice of i and that $|F_i \setminus F_j| \geq 2$ for all $i \neq j$. Let $d = \dim A$. Then the following assertions hold true.*

(1) $\mathrm{ht}_A I \geq 2$.
(2) $d = n - |F_i| \geq 2$.
(3) $B = I : I$, $\operatorname{depth}_A B = d$, *and* $\mathrm{K}_A \cong B$ *as an A-module.*
(4) $0 < \operatorname{depth} A < d$.

Proof. The assertions (1), (2) are clear. Notice that $B = \bigoplus_{i=1}^{\ell} A/\mathfrak{p}_i$ and we get the exact sequence $0 \to A \to B \to C \to 0$ of A-modules, where $IC = (0)$. Hence, Assertions (3), (4) follow from Corollary 3.4. □

In the following subsections, we shall study the question of which value among $\{1, 2, \ldots, d-1\}$ the invariant $t = \operatorname{depth} A$ can vary.

5.1. **The case where** $\operatorname{depth} A = 1$**.** Let m, n be integers such that $m \geq 4$, $n \geq 6$, and $\frac{2}{3}n \geq m \geq \frac{1}{2}n + 1$. We choose G_1, G_2, G_3 so that

$$\begin{aligned} G_1 &= \{1, 2, \dots, m\} \\ G_2 &= \{n-m+1, n-m+2, \dots, n\} \\ G_3 &= \{m+1, m+2, \dots, n\} \cup \{1, 2, \dots, 2m-n\} \end{aligned}$$

where for simplicity, we identify the number i with the indeterminate X_i. Hence, $|G_i| = m$ for all i, $G_i \not\subseteq G_j$ if $i \neq j$, and

$$|G_2 \setminus G_1| = |G_3 \setminus G_1| = n - m, \ \ |G_3 \setminus G_2| = 2m - n.$$

Therefore, $\mathrm{ht}_A I = 2m - n, \dim A = n - m$, and $\dim A/I = 2n - 3m$. Hence, $I = \mathfrak{m}$ if and only if $2n = 3m$, where $\mathfrak{m}$ denotes the maximal ideal of A. Let x_i denote the image of X_i in A. We then have

$$I = \mathfrak{p}_2 \cap \mathfrak{p}_3 + \mathfrak{p}_1 \cap \mathfrak{p}_3 + \mathfrak{p}_1 \cap \mathfrak{p}_2 = (x_1, \dots, x_{2m-n}) + (x_{n-m+1}, \dots, x_m) + (x_{m+1}, \dots, x_n),$$

so that A/I is a regular local ring. Because

$$I = IB = (I + \mathfrak{p}_1)/\mathfrak{p}_1 \oplus I/\mathfrak{p}_2 \oplus I/\mathfrak{p}_3$$

and $I + \mathfrak{p}_1 = \mathfrak{m}$, we get $B/I = A/\mathfrak{m} \oplus A/I \oplus A/I$, whence $\operatorname{depth}_A B/I = 0$ and B/I is a regular ring. Since $B/A = (B/I)/(A/I)$, we have the following.

Proposition 5.6. $\operatorname{depth}_A B/A = 0$ *and hence* $\operatorname{depth} A = 1$.

For simplicity, suppose $2n = 3m$. Then, writing $m = 2q$ and $n = 3q$ with $q \geq 2$, we get

$$\begin{aligned} G_1 &= \{1, 2, \dots, 2q\}, \\ G_2 &= \{q+1, q+2, \dots, 3q\}, \\ G_3 &= \{2q+1, 2q+2, \dots, 3q\} \cup \{1, 2, \dots, q\}, \end{aligned}$$

$I = \mathfrak{m}$, $\dim A = q$, and $B/A \cong (A/\mathfrak{m})^{\oplus 2}$ as an A-module. Therefore

$$\mathrm{H}^1_{\mathfrak{m}}(A) \cong (A/\mathfrak{m})^{\oplus 2} \ \text{ and } \ \mathrm{H}^i_{\mathfrak{m}}(A) = (0) \ \text{ for } \ i \neq 1, q,$$

so that A is a Buchsbaum ring with $\operatorname{depth} A = 1$ ([15, Proposition 2.12]).

For example, let $n = 6$, $m = 4$. Hence, $G_1 = \{1, 2, 3, 4\}$, $G_2 = \{3, 4, 5, 6\}$, and $G_3 = \{5, 6, 1, 2\}$. Let x_i denote the image of X_i in A and set $a = x_1 + x_3 + x_5$, $b = x_2 + x_4 + x_6$. Let $Q = (a, b)$. Then, $\mathfrak{m} = QB$, and a, b is a system of parameters of A. For all $N \geq 1$ the Rees algebra $\mathcal{R}_A(Q^N)$ is a Cohen-Macaulay ring of dimension 3 (see [7]). However, $\mathrm{r}(\mathcal{R}_A(Q)) = 2$ and $\mathrm{r}(\mathcal{R}_A(Q^N)) = 2N - 2$ for $N \geq 2$, so that $\mathcal{R}_A(Q^N)$ is not a Gorenstein ring for any $N \geq 1$, where $\mathrm{r}(*)$ denotes the Cohen-Macaulay type. See [8] for more details.

5.2. **The case where** $\operatorname{depth} A = 2$**.** Let $\ell, m \geq 2$ be integers and set $n = \ell m$. Let k be a field and let $T = k[[X_{ij} \mid 1 \leq i \leq \ell, 1 \leq j \leq m]]$ be the formal power series ring with ℓm indeterminates $\{X_{ij}\}_{1 \leq i \leq \ell, 1 \leq j \leq m}$ over k. We set $F_i = \{X_{ij} \mid 1 \leq j \leq m\}$ for each $1 \leq i \leq \ell$. Hence, $\mathrm{ht}_A I = m$ and $\dim A = m(\ell - 1)$.

More concretely, let $\ell = 3$, $m \geq 2$, and consider the matrix $\left[\begin{smallmatrix} X_1 & X_2 & \dots & X_m \\ Y_1 & Y_2 & \dots & Y_m \\ Z_1 & Z_2 & \dots & Z_m \end{smallmatrix}\right]$ of indeterminates. Then, $\mathrm{ht}_A I = m$, $\dim A = 2m$, and $\dim A/I = m$. We have

$$\begin{aligned} I &= \mathfrak{p}_2 \cap \mathfrak{p}_3 + \mathfrak{p}_1 \cap \mathfrak{p}_3 + \mathfrak{p}_1 \cap \mathfrak{p}_2 \\ &= (\mathfrak{p}_1 + \mathfrak{p}_2 \cap \mathfrak{p}_3) + (\mathfrak{p}_2 + \mathfrak{p}_1 \cap \mathfrak{p}_3) \cap (\mathfrak{p}_3 + \mathfrak{p}_1 \cap \mathfrak{p}_2) \\ &= (I + \mathfrak{p}_1) \cap (I + \mathfrak{p}_2) \cap (I + \mathfrak{p}_3) \\ &= (\mathfrak{p}_1 + \mathfrak{p}_2) \cap (\mathfrak{p}_1 + \mathfrak{p}_3) \cap (\mathfrak{p}_2 + \mathfrak{p}_3). \end{aligned}$$

Therefore

$$A/(I+\mathfrak{p}_1) \cong T/(P_1 + P_2 \cap P_3) \cong k[[Y_1, Y_2, \dots, Y_m, Z_1, Z_2, \dots, Z_m]]/[(Y_1, Y_2, \dots, Y_m) \cap (Z_1, Z_2, \dots, Z_m)],$$

so that $\operatorname{depth} A/(I + \mathfrak{p}_1) = 1$. By symmetry, $\operatorname{depth} A/(I + \mathfrak{p}_i) = 1$ for all i, whence $\operatorname{depth}_A B/I = 1$, because $B/I = \bigoplus_{i=1}^3 A/(I + \mathfrak{p}_i)$. On the other hand, since

$$I = [(P_1 + P_2) \cap (P_1 + P_3) \cap (P_2 + P_3)] / \mathfrak{a}$$

where $\mathfrak{a} = \bigcap_{i=1}^3 P_i$, our ring A is exactly one of the special case $2n = 3m$ of the previous subsection 5.1, so that A/I is a Buchsbaum ring with $\mathrm{H}^1_m(A/I) = (A/\mathfrak{m})^{\oplus 2}$.

We set $C = B/A$ and claim the following.

Claim 1. $\mathrm{H}^1_\mathfrak{m}(C) \neq (0)$. *Hence,* $\operatorname{depth}_A C = 0$ *or* 1, *and* $\operatorname{depth} A = 1$ *or* 2.

In fact, apply the functor $\mathrm{H}^i_\mathfrak{m}(*)$ to the exact sequence $0 \to A/I \to B/I \to C \to 0$, and consider the long exact sequence

$$0 \to \mathrm{H}^0_\mathfrak{m}(C) \to \mathrm{H}^1_\mathfrak{m}(A/I) \to \mathrm{H}^1_\mathfrak{m}(B/I) \to \mathrm{H}^1_\mathfrak{m}(C)$$

of local cohomology modules. We then have $\mathrm{H}^1_\mathfrak{m}(C) \neq (0)$, because $\ell_A(\mathrm{H}^1_\mathfrak{m}(A)) = 2$ and $\ell_A(\mathrm{H}^1_\mathfrak{m}(B/I)) \geq 3$. We however eventually get the following.

Proposition 5.7. $\dim A = 2m \geq 4$, $\operatorname{depth} A = 2$, *and* $\widetilde{A} = B$ *is a regular ring.*

Proof. Since $X_1 + Y_1 \notin \bigcup_{i=1}^3 P_i$, $X_1 + Y_1$ is A-regular, and

$$A/(X_1 + Y_1)A \cong k[[\{X_i\}_{2 \leq i \leq m}, \{Y_i\}_{1 \leq i \leq m}, \{Z_i\}_{1 \leq i \leq m}]]/J,$$

where

$$J = (Y_1, X_2, \dots, X_m) \cap (Y_1, Y_2, \dots, Y_m) \cap (Y_1^2, X_2 \dots, X_m, Y_2, \dots, Y_m) \cap (Z_1, Z_2, \dots, Z_m).$$

Therefore, $\operatorname{depth} A \geq 2$, since $\operatorname{depth} A/(X_1 + Y_1)A \geq 1$. Hence, $\operatorname{depth} A = 2$ and $\operatorname{depth}_A C = 1$ by Claim 1. □

5.3. **The case where** $\operatorname{depth} A \geq 3$**.** Let us now construct the examples of A such that $\operatorname{depth} A \geq 3$. Let q, m be integers such that $3 \leq q < m$ and set $n = 2m$. We choose F_1, F_2, F_3 so that $F_1 = \{1, 2, \dots, m\}$, $F_2 = \{q, q+1, \dots, m+q-1\}$, and $F_3 = \{m+1, m+2, \dots, n\}$. Hence, $|F_i| = m$ for all i, and

$$|F_1 \setminus F_2| = q - 1, \ \ |F_2 \setminus F_3| = m - q + 1, \ \ |F_3 \setminus F_1| = m.$$

Therefore, $\mathrm{ht}_A I = \min\{m-q+1, q-1\}$ and $\dim A = m$. Let x_i denote the image of X_i in A. Then

$$\begin{aligned} I &= \mathfrak{p}_1 \cap \mathfrak{p}_2 + \mathfrak{p}_2 \cap \mathfrak{p}_3 + \mathfrak{p}_3 \cap \mathfrak{p}_1 \\ &= (x_q, x_{q+1}, \dots, x_{m+q-1}) + (x_1, x_2, \dots, x_{q-1})\cdot(x_{m+q}, x_{m+q+1}, \dots, x_n) \\ &= (x_1, x_2, \dots, x_{m+q-1}) \cap (x_q, x_{q+1}, \dots, x_n), \end{aligned}$$

while $I + \mathfrak{p}_1 = (x_1, x_2, \dots, x_{m+q-1})$, $I \supseteq \mathfrak{p}_2$, and $I + \mathfrak{p}_3 = (x_q, x_{q+1}, \dots, x_n)$, so that

$$B/I = A/[I + \mathfrak{p}_1] \oplus A/I \oplus A/[I + \mathfrak{p}_3].$$

We set $C = B/A$. Then, the canonical embedding $A/I \to B/I$ is a split monomorphism of A-modules, and we get $C = A/[I + \mathfrak{p}_1] \oplus A/[I + \mathfrak{p}_3]$. Consequently

$$\dim_A C = \max\{m-q+1, q-1\}, \quad \mathrm{depth}_A C = \min\{m-q+1, q-1\},$$

and thanks to the exact sequences $0 \to A \to B \to C \to 0$ and

$$0 \to A/I \to A/(x_1, x_2, \dots, x_{m+q-1}) \oplus A/(x_q, x_{q+1}, \dots, x_n) \to A/\mathfrak{m} \to 0,$$

we get the following.

Proposition 5.8. $\mathrm{depth}\, A = \min\{m-q+1, q-1\} + 1 \geq 3$ *and* $\mathrm{depth}\, A/I = 1$.

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