Ulrich ideals of dimension one

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S. Goto, K. Ozeki, R. Takahashi, K.-i. Watanabe, and K.-i. Yoshida [GOTWY] have recently generalized the notions of Ulrich ideal and Ulrich module, and closely studied their relations and structures as well. In the present lecture we shall continue the research [GOTWY] of dimension one, giving some practical methods for counting Ulrich ideals.

Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring and $d = \dim A \geq 0$. Assume that the residue class field $A/\mathfrak{m}$ of A is infinite. Let M be a finitely generated A–module. In [BHU], B. Ulrich and other authors gave structure theorems of Maximally Generated Maximal Cohen–Macaulay modules, i.e., those Cohen–Macaulay A–modules M such that $\dim_A M = d$ and $\mathrm{e}^0_{\mathfrak{m}}(M) = \mu_A(M)$, where $\mathrm{e}^0_{\mathfrak{m}}(M)$ (resp. $\mu_A(M)$) denotes the multiplicity of M with respect to $\mathfrak{m}$ (resp. the number of elements in a minimal system of generators for M). This notion of MGMCM module has been generalized by [GOTWY] in the following way.

Definition 1 ([GOTWY])**.** Let I be an $\mathfrak{m}$–primary ideal in A and let M be a finitely generated A–module. Then M is called an Ulrich A–module with respect to I, if

(1) M is a Cohen–Macaulay A–module with $\dim_A M = d$,
(2) $\mathrm{e}^0_I(M) = \ell_A(M/IM)$, and
(3) M/IM is A/I–free,

where $\ell_A(*)$ stands for the length.

Similarly, Ulrich ideals are defined as follows.

Definition 2 ([GOTWY])**.** Let I be an $\mathfrak{m}$–primary ideal in A. Then we say that I is an Ulrich ideal of A, if I/I^2 is A/I–free, I is not a parameter ideal of A, but contains a minimal reduction Q such that $I^2 = QI$.

One of the fundamental results in [GOTWY] is the following. Let $\mathcal{X}_A$ be the set of Ulrich ideals in A. Then, giving the complete lists of Ulrich ideals in certain one–dimensional Cohen–Macaulay local rings A of finite C–M representation type, they proved the following.

Theorem 3 ([GOTWY])**.** *Suppose that A is of finite C–M representation type. Then $\mathcal{X}_A$ is a finite set.*

Based on this result, they intensively explored Ulrich ideals also in two–dimensional rational singularities A, and gave the complete lists of Ulrich ideals in A, using McKay correspondences.

In our lecture we shall study the more basic question of how many Ulrich ideals are contained in a given Cohen–Macaulay local ring A with $\dim A = 1$. This was partially done by [GOTWY] for numerical semigroup rings

$$A = k[[t^{a_1}, t^{a_2}, \ldots, t^{a_\ell}]] \subseteq k[[t]]$$

over a field k, where $0 < a_1, a_2, \ldots, a_\ell \in \mathbb{Z}$ such that $\mathrm{GCD}(a_1, a_2, \ldots, a_\ell) = 1$. In fact, they showed that the set $\mathcal{X}_A^g$ of Ulrich ideals in A which are generated by powers of t is finite, exploring several concrete examples, say:

(1) $\mathcal{X}^g_{k[[t^3,t^4,t^5]]} = \{\mathfrak{m}\}$.
(2) $\mathcal{X}^g_{k[[t^4,t^5,t^6]]} = \{(t^4, t^6)\}$.
(3) $\mathcal{X}^g_{k[[t^a,t^{a+1},\ldots,t^{2a-2}]]} = \emptyset$, if $a \geq 5$.
(4) Let $1 < a < b$ be integers such that $\mathrm{GCD}(a, b) = 1$. Then $\mathcal{X}^g_{k[[t^a,t^b]]} \neq \emptyset$ if and only if a or b is even.
(5) Let $A = k[[t^4, t^6, t^{4\ell-1}]]$ $(\ell \geq 2)$. Then $\sharp\mathcal{X}_A^g = 2\ell - 2$.

We continue the research of one–dimensional case, and provide certain practical methods which are different from those given by [GOTWY] for counting Ulrich ideals, proving the following as applications.

Theorem 4. *(1) $\mathcal{X}_{k[[t^3,t^5]]} = \emptyset$.*
(2) $\mathcal{X}_{k[[t^3,t^7]]} = \{(t^6 - ct^7, t^{10}) \mid 0 \neq c \in k\}$.
(3) $\mathcal{X}_{k[[t^{2q+i} \ \mid \ 1 \ \leq \ i \ \leq \ 2q]]} = \emptyset$ for $\forall q \geq 2$.
(4) $\mathcal{X}_{k[[t^{2q+i} \ \mid \ 0 \ \leq \ i \ \leq \ 2q-2]]} = \emptyset$ for $\forall q \geq 3$.
(5) $\sharp(\mathcal{X}_{k[[X,Y]]/(Y^n)}) = \infty$ for $\forall n \geq 2$.

References

[BHU] J. Brennan, J. Herzog, and B. Ulrich, *Maximally generated Cohen–Macaulay modules*, Math. Scand. **61**, 1987, 181–203.

[GOTWY] S. Goto, K. Ozeki, R. Takahashi, K.-i. Watanabe, and K.-i. Yoshida, *Ulrich ideals and modules*, Preprint 2012.

[HK] J. Herzog and M. Kühl, *Maximal Cohen–Macaulay modules over Gorenstein rings and Bourbaki sequences. Commutative Algebra and Combinatorics*, Adv. Stud. Pure Math., **11**, 1987, 65–92.

[S] J. Sally, *Cohen–Macaulay local rings of maximal embedding dimension*, J. Algebra, **56** (1979), 168–183.

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