SEQUENTIALLY COHEN-MACAULAY REES MODULES

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My talk is based on the research jointly with T. N. An, N. T. Dung and T. T. Phuong ([TPDA]). The aim of this talk is to investigate the question of when the Rees modules associated to arbitrary filtration of modules are sequentially Cohen-Macaulay, which has a previous research by [CGT]. In [CGT] they gave a characterization of the sequentially Cohen-Macaulay Rees algebras of $\mathfrak{m}$-primary ideal which contains a good parameter ideal as a reduction. However their situation is quite a bit of restricted, so we are eager to try the generalization of their results.

Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$, $M \neq (0)$ a finitely generated R-module with $d = \dim_R M < \infty$. Now look at a filtration

$$D_0 := (0) \subsetneq D_1 \subsetneq D_2 \subsetneq \ldots \subsetneq D_\ell = M$$

of R-submodules of M, which we call *the dimension filtration of* M, if D_{i-1} is the largest R-submodule of D_i with $\dim_R D_{i-1} < \dim_R D_i$ for $1 \leq i \leq \ell$, here $\dim_R(0) = -\infty$ for convention. Then we say that M is *a sequentially Cohen-Macaulay R-module*, if the quotient module $C_i = D_i/D_{i-1}$ of D_i is a Cohen-Macaulay R-module for every $1 \leq i \leq \ell$. In particular, the ring R is called *a sequentially Cohen-Macaulay ring*, if $\dim R < \infty$ and R is a sequentially Cohen-Macaulay module over itself.

Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ be a filtration of ideals of R such that $F_1 \neq R$, $\mathcal{M} = \{M_n\}_{n\in\mathbb{Z}}$ an $\mathcal{F}$-filtration of R-submodules of M. Then we put

$$\mathcal{R} = \sum_{n\geq 0} F_n t^n \subseteq R[t], \quad \mathcal{R}' = \sum_{n\in\mathbb{Z}} F_n t^n \subseteq R[t, t^{-1}], \quad \mathcal{G} = \mathcal{R}'/t^{-1}\mathcal{R}'$$

and call them *the Rees algebra, the extended Rees algebra and the associated graded ring of* $\mathcal{F}$, respectively. Similarly we set

$$\mathcal{R}(\mathcal{M}) = \sum_{n\geq 0} t^n \otimes M_n \subseteq R[t] \otimes_R M, \quad \mathcal{R}'(\mathcal{M}) = \sum_{n\in\mathbb{Z}} t^n \otimes M_n \subseteq R[t, t^{-1}] \otimes_R M$$

and

$$\mathcal{G}(\mathcal{M}) = \mathcal{R}'(\mathcal{M})/t^{-1}\mathcal{R}'(\mathcal{M})$$

which we call *the Rees module, the extended Rees module and the associated graded module of* $\mathcal{M}$, respectively (here t stands for an indeterminate over R). We now assume that $\mathcal{R}$ is Noetherian and $\mathcal{R}(\mathcal{M})$ is finitely generated. We set

$$\mathcal{D}_i = \{M_n \cap D_i\}_{n\in\mathbb{Z}}, \quad \mathcal{C}_i = \{[(M_n \cap D_i) + D_{i-1}]/D_{i-1}\}_{n\in\mathbb{Z}}.$$

for all $1 \leq i \leq \ell$. Then $\mathcal{D}_i$ (resp. $\mathcal{C}_i$) is an $\mathcal{F}$-filtration of R-submodules of D_i (resp. C_i).

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With this notation the main results of my talk are the following.

Theorem 1. *The following conditions are equivalent.*

(1) $\mathcal{R}'(\mathcal{M})$ *is a sequentially Cohen-Macaulay* $\mathcal{R}'$*-module.*

(2) $\mathcal{G}(\mathcal{M})$ *is a sequentially Cohen-Macaulay* $\mathcal{G}$*-module and* $\{\mathcal{G}(\mathcal{D}_i)\}_{0\leq i\leq \ell}$ *is the dimension filtration of* $\mathcal{G}(\mathcal{M})$.

When this is the case, M *is a sequentially Cohen-Macaulay* R*-module.*

Let $\mathfrak{M}$ be a unique graded maximal ideal of $\mathcal{R}$. We set

$$\mathrm{a}(N) = \max\{n \in \mathbb{Z} \mid [\mathrm{H}^t_{\mathfrak{M}}(N)]_n \neq (0)\}$$

for a finitely generated graded $\mathcal{R}$-module N of dimension t, and call it *the a-invariant of* N (see [GW, DEFINITION (3.1.4)]). Here $\{[\mathrm{H}^t_{\mathfrak{M}}(N)]_n\}_{n\in\mathbb{Z}}$ stands for the homogeneous components of the t-th graded local cohomology module $\mathrm{H}^t_{\mathfrak{M}}(N)$ of N with respect to $\mathfrak{M}$.

Theorem 2. *Suppose that* M *is a sequentially Cohen-Macaulay* R*-module and* $F_1 \nsubseteq \mathfrak{p}$ *for every* $\mathfrak{p} \in \mathrm{Ass}_R M$. *Then the following conditions are equivalent.*

(1) $\mathcal{R}(\mathcal{M})$ *is a sequentially Cohen-Macaulay* $\mathcal{R}$*-module.*

(2) $\mathcal{G}(\mathcal{M})$ *is a sequentially Cohen-Macaulay* $\mathcal{G}$*-module,* $\{\mathcal{G}(\mathcal{D}_i)\}_{0\leq i\leq \ell}$ *is the dimension filtration of* $\mathcal{G}(\mathcal{M})$ *and* $\mathrm{a}(\mathcal{G}(\mathcal{C}_i)) < 0$ *for every* $1 \leq i \leq \ell$.

When this is the case, $\mathcal{R}'(\mathcal{M})$ *is a sequentially Cohen-Macaulay* $\mathcal{R}'$*-module.*

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