Huneke–Wiegand conjecture and change of rings

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Let M and N be finitely generated modules over an integral domain R and assume that both modules M and N are torsionfree. The final destination of this research is to get the answer for the question of when the tensor product $M \otimes_R N$ is torsionfree. Our interest dates back to the following conjecture.

Conjecture 1 (Huneke-Wiegand conjecture [3])**.** Let R be a Gorenstein local integral domain. Let M be a finitely generated R-module with $\operatorname{depth}_R M = \dim R$, that is M is a maximal Cohen-Macaulay R-module. If $M \otimes_R \operatorname{Hom}_R(M, R)$ is torsionfree, then M is a free R-module.

Conjecture 1 derives from the Auslander-Reiten conjecture and it classically holds true, when the base ring R is integrally closed ([1, Proposition 3.3]). C. Huneke and R. Wiegand [3] proved that it is true, if R is a hypersurface. They showed also that Conjecture 1 is reduced to the case where $\dim R = 1$. The problem is, however, still open in general, and no one has a complete answer to the following Conjecture 2, even in the case where R is a complete intersection, or in the case where R is a numerical semigroup ring over a filed.

Conjecture 2. Suppose that R is a Gorenstein local integral domain of dimension one and let I be an ideal of R. If $I \otimes_R \operatorname{Hom}_R(I, R)$ is torsionfree, then I is a principal ideal.

In our research we are deeply interested in the question of what happens if we replace $\operatorname{Hom}_R(I, R)$ with $\operatorname{Hom}_R(I, \mathrm{K}_R)$, where K_R stands for the canonical module of R. One of the advantages of such a modification is the usage of the symmetry between I and $\operatorname{Hom}_R(I, \mathrm{K}_R)$ and the other one is the possible change of rings, which I will explain in my lecture.

Conjecture 3. Let R be a Cohen-Macaulay local ring of dimension one and assume that R possesses the canonical module K_R. Let I be a faithful ideal of R. If $I \otimes_R \operatorname{Hom}_R(I, \mathrm{K_R})$ is torsionfree, then $I \cong R$ or $I \cong \mathrm{K_R}$ as an R-module.

The main result of my lecture is the following.

Theorem 4. *Let R be a Cohen-Macaulay local ring of dimension one and assume that R possesses the canonical module K_R. Let I be a faithful ideal of R. Suppose that* $\mathrm{e}(R) \leq 6$. *If $I \otimes_R \operatorname{Hom}_R(I, \mathrm{K_R})$ is torsionfree, then $I \cong R$ or $I \cong \mathrm{K_R}$ as an R-module.*

Theorem 4 holds true also, if the multiplicity $\mathrm{e}(R)$ of R is 7 and if we restrict our attention to monomial ideals in numerical semigroup rings. However, Theorem 4 is no longer true in general, whence so is Conjecture 3, if $\mathrm{e}(R) = 9$. We have a counterexample among monomial ideals in numerical semigroup rings, although Conjecture 3 is still open in multiplicity 8.

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