THE ALMOST GORENSTEIN PROPERTY OF ASSOCIATED GRADED RINGS

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1. Abstract

The purpose of my lecture is to prove the following theorem, which shows that the almost Gorenstein property of base local rings is inherited from that of the associated graded rings. Remember that a Noetherian graded ring $R = \bigoplus_{n\geq 0} R_n$ with R_0 a local ring is called an almost Gorenstein graded ring, if R is a Cohen-Macaulay ring, possessing the graded canonical module K_R, such that there exists an exact sequence

$$0 \to R \to \mathrm{K}_R(-a) \to C \to 0$$

of graded R-modules with $\mu_R(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$, where $a = \mathrm{a}(R)$ is the a–invariant of R and $\mathfrak{M}$ is the graded maximal ideal of R (see [GT]).

Theorem 1.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with infinite residue class field. Suppose that R is a homomorphic image of a Gorenstein local ring. Let I be an $\mathfrak{m}$-primary ideal of R and let $\mathrm{gr}_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ be the associated graded ring of I. If $\mathrm{gr}_I(R)$ is an almost Gorenstein graded ring with $\mathrm{r}(\mathrm{gr}_I(R))) = \mathrm{r}(R)$, then R is an almost Gorenstein local ring.*

Theorem 1.1 is reduced, by induction on $\dim R$, to the case where $\dim R = 1$. Let me explain the key result of dimension one (Theorem 1.2).

Let R be a Cohen-Macaulay local ring of dimension one. We consider a filtration $\mathscr{I} = \{I_n\}_{n\in\mathbb{Z}}$ of ideals of R. Let t be an indeterminate and we set

$$\mathscr{R} = \mathscr{R}(\mathscr{I}) = \sum_{n\geq 0} I_n t^n \subseteq R[t],$$

$$\mathscr{R}' = \mathscr{R}'(\mathscr{I}) = \mathscr{R}[t^{-1}] = \sum_{n\in\mathbb{Z}} I_n t^n \subseteq R[t, t^{-1}],$$

and

$$G = G(\mathscr{I}) = \mathscr{R}'(\mathscr{I})/t^{-1}\mathscr{R}'(\mathscr{I}).$$

For these algebras we assume the following three conditions are satisfied:

(1) R is a homomorphic image of a Gorenstein local ring,
(2) $\mathscr{R}$ is a Noetherian ring, and
(3) G is a Cohen-Macaulay ring.

With this notation we have the following.

Theorem 1.2. *Assume that G is an almost Gorenstein graded ring with $\mathrm{r}(G) = \mathrm{r}(R)$. Then R is an almost Gorenstein local ring.*

The converse of Theorem 1.2 is also true when G satisfies some additional conditions.

This is a joint work with Shiro Goto.

Theorem 1.3. *Suppose that R is an almost Gorenstein local ring and the field $R/\mathfrak{m}$ is infinite. Assume that one of the following conditions is satisfied:*

(i) *G is an integral domain;*

(ii) *$\mathrm{Q}(G)$ is a Gorenstein ring and G is a level ring.*

Then G is an almost Gorenstein graded ring with $\mathrm{r}(G) = \mathrm{r}(R)$.

When I is generated by a subsystem $a_1, a_2, \dots, a_r$ of parameters of a Cohen-Macaulay local ring R, the associated graded ring $\mathrm{gr}_I(R) = (R/I)[X_1, X_2, \dots, X_r]$ of I is the polynomial ring over R/I, and the almost Gorenstein property of $\mathrm{gr}_I(R)$ is equivalent to that of R/I. To see this, we shall explore how the almost Gorenstein property is inherited under flat local base changes. Let us summarize our results below.

Theorem 1.4. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim R$ and infinite residue class field, possessing the canonical module K_R. Let $(S, \mathfrak{n})$ be a Noetehrian local ring and let $\varphi : R \to S$ be a flat local homomorphism such that $S/\mathfrak{m}S$ is a regular local ring. Then the following conditions are equivalent.*

(1) *S is an almost Gorenstein local ring.*

(2) *R is an almost Gorenstein local ring.*

As consequences of Theorem 1.4, we get the following.

Corollary 1.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with infinite residue class field. Let $S = R[X_1, X_2, \dots, X_n]$ be the polynomial ring and consider S as a $\mathbb{Z}$-graded ring with $S_0 = R$ and $\deg X_i = 1$ for every $1 \le i \le n$. Then the following conditions are equivalent.*

(1) *R is an almost Gorenstein local ring.*

(2) *S is an almost Gorenstein graded ring.*

The goal of my talk is the following.

Corollary 1.6 (cf. [I, Theorem 1.1]**).** *Let $(R, \mathfrak{m})$ be a Noetherian local ring with $d = \dim R > 0$ and infinite residue class field. Assume that R is a homomorphic image of a Gorenstein local ring. We choose a system $a_1, a_2, \dots, a_d$ of parameters of R. Let $1 \le r \le d$ be an integer and set $I = (a_1, a_2, \dots, a_r)$. If $\mathrm{gr}_I(R)$ is an almost Gorenstein graded ring, then R is an almost Gorenstein local ring. In particular, R is a Gorenstein local ring, if $r = d$ and $\mathrm{gr}_I(R)$ is an almost Gorenstein graded ring.*

References

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