Sally modules of extended canonical ideals and Goto rings

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1. Introduction

Question 1.1

Why are there so many Cohen-Macaulay rings which are not Gorenstein?

Regular ⇒ Complete Intersection ⇒ Gorenstein ⇒ Cohen-Macaulay
⇒ Buchsbaum ⇒ Generalized Cohen-Macaulay (FLC)

Problem 1.2

Find new and interesting classes of rings which fill in a gap between Gorenstein and Cohen-Macaulay rings so as to stratify Cohen-Macaulay rings.

We introduce a new concept of CM rings, called Goto rings.

Slogan

Goto rings is a CM local rings admitting "good" canonical ideals.

2. Extended canonical ideals

- $(A, \mathfrak{m})$ a CM local ring with $d = \dim A > 0$, $\exists\, \mathrm{K}_A$, and $|A/\mathfrak{m}| = \infty$
- $I \subsetneq A$ an ideal of A s.t. $I \cong \mathrm{K}_A$ (canonical ideal)

Recall that $\exists$ a canonical ideal $\iff A_\mathfrak{p}$ is Gorenstein for $\forall\, \mathfrak{p} \in \operatorname{Min} A$

$\iff \mathrm{Q}(A)$ is Gorenstein.

For ideals J and Q with $Q \subseteq J$,

- Q is a reduction of J if $J^{r+1} = QJ^r$ for $\exists\, r \geq 0$
- $\mathrm{red}_Q(J) = \min\{r \geq 0 \mid J^{r+1} = QJ^r\}$.

An ideal J is called an extended canonical ideal of A, if $J = I + Q$ for some parameter ideal $Q = (a_1, a_2, \ldots, a_d)$ s.t. $a_1 \in I$ and Q is a reduction of J.

- When $d = 1$, extended canonical ideals = canonical ideals.
- When $d \geq 2$, $J\overline{A}$ is a canonical ideal of $\overline{A} = A/(a_2, a_3, \ldots, a_d)$.
- An extended canonical ideal exists.

3. Goto rings

- $(A, \mathfrak{m})$ a CM local ring with $d = \dim A > 0$, $\exists\, \mathrm{K}_A$, and $|A/\mathfrak{m}| = \infty$
- $I \subsetneq A$ an ideal of A s.t. $I \cong \mathrm{K}_A$, and $n \geq 0$ an integer

Definition 3.1 (My proposal)

The ring A is called n-Goto, if $\exists\, Q = (a_1, a_2, \ldots, a_d)$ a parameter ideal of A s.t.

(1) $a_1 \in I$

(2) $J^3 = QJ^2$, i.e., $\mathrm{red}_Q(J) \leq 2$ (hence, J is an extended canonical ideal)

(3) $\ell_A(J^2/QJ) = n$

where $J = I + Q$.

- A is 0-Goto $\iff$ A is Gorenstein
- A is 1-Goto $\iff$ A is non-Gorenstein almost Gorenstein
- A is 2-Goto $\iff$ A is 2-almost Gorenstein, provided $d = 1$
- A is $\ell_A(A/\mathfrak{a})$-Goto $\Longleftarrow$ A is generalized Gorenstein with respect to $\mathfrak{a}$.

As $J^3 = QJ^2$, the sequence $a_2, a_3, \ldots, a_d$ is super-regular with respect to J.

Example 3.2

Let k be a field. For any $\ell \geq 3$, $m \geq n \geq 2$,

$$A = k[[X_1, X_2, \ldots, X_\ell, V_1, V_2, \ldots, V_{\ell-1}]]/\mathrm{I}_2 \begin{pmatrix} X_1^n & X_2+V_1 & \cdots & X_{\ell-1}+V_{\ell-2} & X_\ell+V_{\ell-1} \\ X_2 & X_3 & \cdots & X_\ell & X_1^m \end{pmatrix}$$

is an n-Goto ring with $\dim A = \ell$ and $\mathrm{r}(A) = \ell - 1$.

Example 3.3

(1) The semigroup ring $k[[t^3, t^{3n+1}, t^{3n+2}]]$ is n-Goto and is an integral domain.

(2) The fiber product $k[[t^3, t^{3n+1}, t^{3n+2}]] \times_k k[[t]]$ is n-Goto and reduced, but not an integral domain.

(3) The idealization $k[[t^3, t^{3n+1}, t^{3n+2}]] \ltimes k[[t]]$ is n-Goto and is not reduced.

When $d = 1$, with suitable assumption,

- A is n-Goto $\iff$ $\mathrm{Bl}_A(\mathfrak{m}) = \bigcup_{n \geq 0} [\mathfrak{m}^n : \mathfrak{m}^n]$ is $(n-1)$-Goto.
- If R is n-Goto and S is 2-Goto, then $A = R \times_k S$ is $(n+1)$-Goto.

Let $\mathrm{e}_i(J)$ be the i-th Hilbert coefficients of A with respect to J. Then

- $\mathrm{e}_1(J) \geq \mathrm{e}_0(J) - \ell_A(A/J)$
- $\mathrm{e}_1(J) = \mathrm{e}_0(J) - \ell_A(A/J) \iff J^2 = QJ$, i.e., $\mathrm{red}_Q(J) \leq 1$.

  When this is the case,
  - $\mathrm{gr}_J(A) = \bigoplus_{i\geq 0} J^i/J^{i+1}$ and $\mathcal{F}(J) = \bigoplus_{i\geq 0} J^i/\mathfrak{m}J^i$ are CM
  - $\mathcal{R}(J) = \bigoplus_{i\geq 0} J^i$ is CM, provided $d \geq 2$.

As next border,

- Sally characterized the ideal J with $\mathrm{e}_1(J) = \mathrm{e}_0(J) - \ell_A(A/J) + 1$ and $\mathrm{e}_2(J) \neq 0$.
- Vasconcelos introduced Sally modules $\mathcal{S}_Q(J) = \bigoplus_{i\geq 1} J^{i+1}/JQ^i$, recovered Sally's results, and made further progress in this direction, e.g.,

$$\mathrm{rank}\, \mathcal{S}_Q(J) = \mathrm{e}_1(J) - \mathrm{e}_0(J) + \ell_A(A/J).$$

- Goto, Nishida, and Ozeki brought fruit to fruition for the theory of Sally modules of rank one.

Whereas they considered general $\mathfrak{m}$-primary ideals, we concentrate on extended canonical ideals and raise the rank of the Sally modules.

When $J^3 = QJ^2$, since $\mathrm{rank}\, \mathcal{S}_Q(J) = \mathrm{e}_1(J) - \mathrm{e}_0(J) + \ell_A(A/J) = \ell_A(J^2/QJ)$,

- A is n-Goto $\iff$ $a_1 \in I$, $\mathcal{S}_Q(J) = \mathcal{R}(Q)\,[\mathcal{S}_Q(J)]_1$, and $\mathrm{rank}\, \mathcal{S}_Q(J) = n$.

We set $\mathcal{B} = \mathcal{F}(Q) = \mathcal{R}(Q)/\mathfrak{m}\mathcal{R}(Q) \cong (A/\mathfrak{m})[X_1, X_2, \ldots, X_d]$. Then

- A is Gorenstein $\iff$ $\mathcal{S}_Q(J) = (0)$
- A is non-Gorenstein almost Gorenstein if and only if

  $\mathcal{S}_Q(J) \cong \mathcal{B}(-1)$ (Goto-Takahashi-Taniguchi)

- When $d = 1$, the ring A is 2-almost Gorenstein if and only if

  $\exists\, 0 \to \mathcal{B}(-1) \to \mathcal{S}_Q(J) \to \mathcal{B}(-1) \to 0$ (Chau-Goto-Kumashiro-Matsuoka)

- If A is generalized Gorenstein with respect to $\mathfrak{a}$, then

  $\mathcal{S}_Q(J) \cong [\mathcal{R}(Q)/\mathfrak{a}\mathcal{R}(Q)]\,(-1).$ (Goto-Isobe-Kumashiro-Taniguchi)

Theorem 3.4

Suppose that $n \geq 1$. Then TFAE.

(1) *A is an n-Goto ring (with respect to Q).*

(2) *There exist integers $0 \leq \ell < n$ and $s_i \geq 1$ $(0 \leq i \leq \ell)$ s.t. $n = \sum_{i=0}^{\ell} s_i$ and $\mathfrak{m}^{\ell}\mathcal{S}_Q(J) \cong \mathcal{B}(-1)^{\oplus s_0}$, and if $\ell > 0$, there exist exact sequences*

$$
\begin{array}{c}
0 \to \mathfrak{m}^{\ell}\mathcal{S}_Q(J) \to \mathfrak{m}^{\ell-1}\mathcal{S}_Q(J) \to \mathcal{B}(-1)^{\oplus s_1} \to 0 \\
0 \to \mathfrak{m}^{\ell-1}\mathcal{S}_Q(J) \to \mathfrak{m}^{\ell-2}\mathcal{S}_Q(J) \to \mathcal{B}(-1)^{\oplus s_2} \to 0 \\
\vdots \\
0 \to \mathfrak{m}\mathcal{S}_Q(J) \to \mathcal{S}_Q(J) \to \mathcal{B}(-1)^{\oplus s_{\ell}} \to 0.
\end{array}
$$

Corollary 3.5

Suppose that $n \geq 1$ and A is an n-Goto ring (with respect to Q). Then

- $\mathrm{e}_2(J) = n$ *if $d \geq 2$*
- $\mathrm{e}_i(J) = 0$ *for all $3 \leq i \leq d$, if $d \geq 3$.*

Thank you for your attention.