Generalization of Gorenstein rings – from the past to the future –

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by and for young mathematicians

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§1. Introduction

Question 1.1

Why are there so many Cohen-Macaulay rings which are not Gorenstein?

- Hierarchy of local rings (in terms of homological algebra)

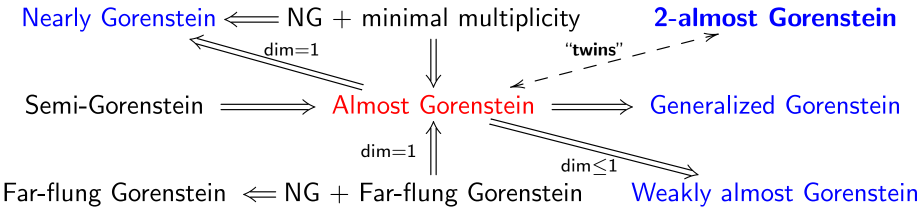
Regular $\Rightarrow$ Complete Intersection $\Rightarrow$ Gorenstein $\Rightarrow$ Cohen-Macaulay
$\Rightarrow$ Buchsbaum $\Rightarrow$ Generalized Cohen-Macaulay (FLC)

Problem 1.2

Find new and interesting classes of rings which fill in a gap between Gorenstein and Cohen-Macaulay rings so as to stratify Cohen-Macaulay rings.

Preceding researches

- Almost Gorenstein rings ··· Barucci-Fröberg, Goto-Matsuoka-Phuong, Goto-Takahashi-Taniguchi
- Semi-Gorenstein rings ··· Goto-Takahashi-Taniguchi
- Generalized Gorenstein rings ··· Goto-Kumashiro
- **2-almost Gorenstein rings** ··· Chau-Goto-Kumashiro-Matsuoka
- Weakly almost Gorenstein rings ··· Dao-Kobayashi-Takahashi
- Nearly Gorenstein rings ··· Herzog-Hibi-Stamate
- Far-flung Gorenstein rings ··· Herzog-Kumashiro-Stamate



§2. Preliminaries

Let

- $(A, \mathfrak{m})$ a CM local ring with $d = \dim A > 0$
- I an $\mathfrak{m}$-primary ideal of A.

Then $\exists\ \mathrm{e}_i(I) \in \mathbb{Z}\ (0 \le i \le d)$ s.t.

$$\ell_A(A/I^{n+1}) = \mathrm{e}_0(I)\binom{n+d}{d} - \mathrm{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d\,\mathrm{e}_d(I) \quad (n \gg 0).$$

Note that

- $\mathrm{e}_0(I) > 0$ and $\mathrm{e}_1(I) \ge 0$
- A is a RLR $\iff$ $\mathrm{e}_0(\mathfrak{m}) = 1$, if A is unmixed. (Samuel, Nagata)

Theorem 2.1 (Koura-Taniguchi)

Set $\Lambda(A) = \{\mathrm{e}_1(I) \mid \sqrt{I} = \mathfrak{m}\}$. Then

$$\#\Lambda(A) < \infty \iff d = 1 \text{ and } A \text{ is analytically unramified.}$$

When this is the case, $\sup \Lambda(A) = \ell_A(\overline{A}/A)$.

In what follows, let

- $(R, \mathfrak{m})$ a CM local ring with $\dim R = 1$
- $I \subsetneq R$ an ideal of R s.t. $I \cong \mathrm{K}_R$
- $\mathrm{r}(R) = \ell_R(\mathrm{Ext}_R^1(R/\mathfrak{m}, R))$.

Definition 2.2 (Goto-Matsuoka-Phuong)

We say that R is an almost Gorenstein ring (abbr. AGL ring), if $\mathrm{e}_1(I) \le \mathrm{r}(R)$.

Suppose I contains a parameter ideal $Q = (a)$ as a reduction, i.e.

$$I^{r+1} = QI^r \quad \text{for} \quad \exists\, r \ge 0.$$

For $\forall\, n \ge 0$, since $I^{n+1}/Q^{n+1} \cong [\frac{I^{n+1}}{a^{n+1}}]/R \subseteq R^I/R$, we have

$$\begin{aligned} \ell_R(R/I^{n+1}) &= \ell_R(R/Q^{n+1}) - \ell_R(I^{n+1}/Q^{n+1}) \\ &\ge \ell_R(R/Q^{n+1}) - \ell_R(R^I/R) \\ &= \ell_R(R/Q)\binom{n+1}{1} - \ell_R(R^I/R) \end{aligned}$$

where $R^I = R[\frac{I}{a}] \subseteq \overline{R}$.

Hence

$$\ell_R(R/I^{n+1}) = \ell_R(R/Q)\binom{n+1}{1} - \ell_R(R^I/R) \quad (\forall\, n \geq r-1).$$

This shows

- $\mathrm{e}_0(I) = \ell_R(R/Q)$
- $\mathrm{e}_1(I) = \ell_R(R^I/R) \leq \ell_R(\overline{R}/R)$.

The embeddings

$$I/Q \overset{a}{\hookrightarrow} I^2/Q^2 \overset{a}{\hookrightarrow} \cdots \overset{a}{\hookrightarrow} I^{r-1}/Q^{r-1} \overset{a}{\hookrightarrow} I^r/Q^r \overset{\simeq}{\to} I^{r+1}/Q^{r+1} \overset{\simeq}{\to} \cdots \overset{\simeq}{\to} R^I/R$$

yield that

$$\mathrm{r}(R) - 1 = \mu_R(I/Q) \leq \ell_R(I/Q) \leq \ell_R(I^r/Q^r) = \mathrm{e}_1(I).$$

Therefore

- $\mu_R(I/Q) = \ell_R(I/Q) \iff \mathfrak{m}I \subseteq Q$
- $\ell_R(I/Q) = \mathrm{e}_1(I) \iff I^2 = QI$. (Huneke, Ooishi)

We set

$$K = \frac{I}{a} = \left\{ \frac{x}{a} \mid x \in I \right\} \subseteq \mathrm{Q}(R).$$

Then K is a fractional ideal of R s.t. $R \subseteq K \subseteq \overline{R}$ and $K \cong \mathrm{K}_R$.

Theorem 2.3 (Goto-Matsuoka-Phuong)

R is an almost Gorenstein local ring $\iff$ $\mathfrak{m}K \subseteq R$ (*i.e.* $\mathfrak{m}I \subseteq Q$).

Example 2.4 (AGL rings)

Let k be a field.

(1) $k[[t^3, t^4, t^5]]$

(2) $k[[t^3, t^4, t^5]] \times_k k[[t]]$

(3) $k[[t^3, t^4, t^5]] \ltimes k[[t]]$

(4) $k[[X, Y, Z]]/\mathrm{I}_2 \begin{pmatrix} X & Y & Z \\ Y^4 & Z & X^3 \end{pmatrix}$

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Example 2.5 (non-AGL rings)

(1) $k[[t^3, t^{3n+1}, t^{3n+2}]]$ $(n \geq 2)$; in particular, $k[[t^3, t^7, t^8]]$

(2) $k[[t^3, t^7, t^8]] \times_k k[[t]]$

(3) $k[[t^3, t^7, t^8]] \ltimes k[[t]]$

(4) $k[[X, Y, Z]]/\mathrm{I}_2 \begin{pmatrix} X^2 & Y^2 & Z \\ Y^4 & Z & X^3 \end{pmatrix}$

Question 2.6

How can we classify these non-almost Gorenstein rings ?

Recall that

- $R \subseteq K \subseteq \overline{R}$ s.t. $K \cong \mathrm{K}_R$
- $\mathrm{e}_0(I) - \ell_R(R/I) \le \mathrm{e}_1(I)$ (Northcott's inequality)
- $\mathrm{e}_0(I) - \ell_R(R/I) = \mathrm{e}_1(I) \iff I^2 = QI \iff R$ is Gorenstein.

Theorem 2.7 (Goto-Matsuoka-Phuong)

TFAE.

(1) *R is a non-Gorenstein AGL ring.*

(2) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_R(R/I) + 1$, *i.e., Sally's equality holds true.*

(3) $\ell_R(K^2/K) = 1$.

When this is the case, one has $K^2 = K^3$ and

$$\ell_R(R/I^{n+1}) = (\mathrm{r}(R) + \ell_R(R/I) - 1)\binom{n+1}{1} - \mathrm{r}(R) \quad \text{for } \forall\, n \ge 1.$$

We set

$$\mathcal{R} = \mathcal{R}(I) = R[It] \cong \bigoplus_{i \geq 0} I^i \quad \text{and} \quad \mathcal{T} = \mathcal{R}(Q) = R[Qt] \cong \bigoplus_{i \geq 0} Q^i$$

where t is an indeterminate. We define

$$\mathcal{S}_Q(I) = I\mathcal{R}/I\mathcal{T} \cong \bigoplus_{i \geq 1} I^{i+1}/IQ^i.$$

Then

- $\mathcal{S}_Q(I) = (0) \iff I^2 = QI$
- $\mathcal{S}_Q(I) = \mathcal{T} \cdot [\mathcal{S}_Q(I)]_1 \iff I^3 = QI^2$.

Theorem 2.8 (Goto-Nishida-Ozeki)

Set $\mathfrak{p} = \mathfrak{m}\mathcal{T} \in \operatorname{Spec} \mathcal{T}$. *The following assertions hold true.*

(1) $\mathfrak{m}^\ell \cdot \mathcal{S}_Q(I) = (0)$ *for* $\ell \gg 0$.

(2) $\operatorname{Ass}_{\mathcal{T}} \mathcal{S}_Q(I) \subseteq \{\mathfrak{p}\}$; *hence* $\dim_{\mathcal{T}} \mathcal{S}_Q(I) = \dim R$, *if* $\mathcal{S}_Q(I) \neq (0)$.

(3) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_R(R/I) + \ell_{\mathcal{T}_\mathfrak{p}}([\mathcal{S}_Q(I)]_\mathfrak{p})$.

We consider

$$\operatorname{rank} \mathcal{S}_Q(I) := \ell_{\mathcal{T}_\mathfrak{p}}([\mathcal{S}_Q(I)]_\mathfrak{p}) = \mathrm{e}_1(I) - [\mathrm{e}_0(I) - \ell_R(R/I)]$$

which is an invariant of R. Then

$$\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_R(R/I) + \operatorname{rank} \mathcal{S}_Q(I).$$

Therefore

- R is a Gorenstein ring $\iff$ $\operatorname{rank} \mathcal{S}_Q(I) = 0$
- R is a non-Gorenstein AGL ring $\iff$ $\operatorname{rank} \mathcal{S}_Q(I) = 1$ (GMP)
- R is a 2-almost Gorenstein ring $\overset{def}{\iff}$ $\operatorname{rank} \mathcal{S}_Q(I) = 2.$ (CGKM)

Question 2.9

For a given integer $n \geq 0$, what kind of rings satisfy $\operatorname{rank} \mathcal{S}_Q(I) = n$?

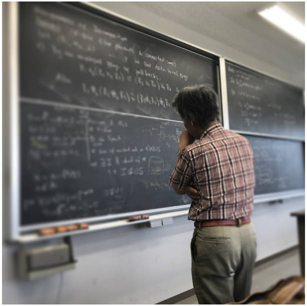
§3. One-dimensional Goto rings

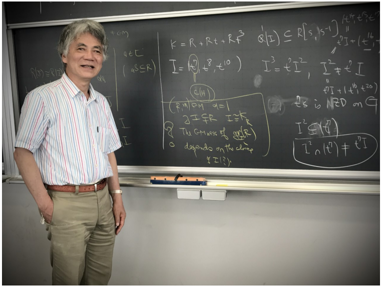
Let $n \geq 0$ be an integer.

Definition 3.1 (My proposal)

We say that R is an *n-Goto ring*, if $\operatorname{rank} \mathcal{S}_Q(I) = n$ and $\mathcal{S}_Q(I) = \mathcal{T} \cdot [\mathcal{S}_Q(I)]_1$.

Note that R is n-Goto $\iff$ $K^2 = K^3$ and $\ell_R(K^2/K) = n$.





Note that R is n-Goto $\iff$ $K^2 = K^3$ and $\ell_R(K^2/K) = n$. Moreover

- R is 0-Goto $\iff$ R is Gorenstein
- R is 1-Goto $\iff$ R is non-Gorenstein almost Gorenstein
- R is 2-Goto $\iff$ R is 2-almost Gorenstein
- R is $\ell_R(R/\mathfrak{c})$-Goto $\Longleftarrow$ R is generalized Gorenstein.

Remark 3.2

(1) $\operatorname{rank} \mathcal{S}_Q(I) \le 2 \implies K^2 = K^3$.

(2) There is an example s.t. $\operatorname{rank} \mathcal{S}_Q(I) \ge 3$ and $K^2 \ne K^3$.

Example 3.3

The ring $R = k[[H]] = k[[t^h \mid h \in H]]$ ($\subseteq k[[t]]$) is an n-Goto ring, where

- $H = \langle 3, 3n+1, 3n+2 \rangle$ $(n \ge 1)$
- $H = \langle e, \{en - e + i\}_{3 \le i \le e-1}, en+1, en+2 \rangle$ $(n \ge 2,\ e \ge 4)$.

Example 3.3 (continued)

Let $H = \langle 3, 3n+1, 3n+2 \rangle$. Consider $R = k[[H]]$ and set $K = R + Rt$. Then

$$R \subseteq K \subseteq \overline{R} = k[[t]] \ \text{ and } \ K \cong \mathrm{K}_R\,.$$

Since $K^2 = R + Rt + Rt^2 = \overline{R}$, we have $K^2 = K^3$ and $\ell_R(K^2/K) = n$. Hence R is an n-Goto ring and $\mu_R(K^2/K) = 1$.

$R =$

0	1	2
3	4	5
6	7	8
⋮	⋮	⋮
$3n-3$	$3n-2$	$3n-1$
$3n$	$3n+1$	$3n+2$
$3n+3$	$3n+4$	$3n+5$

$K =$

0	1	2
3	4	5
6	7	8
⋮	⋮	⋮
$3n-3$	$3n-2$	$3n-1$
$3n$	$3n+1$	$3n+2$
$3n+3$	$3n+4$	$3n+5$

$K^2 =$

0	1	2
3	4	5
6	7	8
⋮	⋮	⋮
$3n-3$	$3n-2$	$3n-1$
$3n$	$3n+1$	$3n+2$
$3n$	$3n+1$	$3n+2$

§4. Flat base changes

- $(R_1, \mathfrak{m}_1)$ a CM local ring with $\dim R_1 = 1$
- $\varphi : R \to R_1$ a flat local homomorphism s.t. $R_1/\mathfrak{m}R_1$ is Gorenstein.

Then $\dim R_1/\mathfrak{m}R_1 = 0$, $K_1 := R_1 \otimes_R K \cong \mathrm{K}_{R_1}$ and

$$R_1 \subseteq K_1 \subseteq R_1 \otimes_R \overline{R} \subseteq \overline{R_1}.$$

Theorem 4.1

For each $n > 0$, we have

$$R_1 \text{ is } n\text{-Goto} \iff \exists\, m > 0 \text{ s.t. } m \mid n,\ R \text{ is } m\text{-Goto, and } \ell_{R_1}(R_1/\mathfrak{m}R_1) = \frac{n}{m}.$$

Corollary 4.2

Let $n \geq 2$ be a prime number. Then R_1 is an n-Goto ring if and only if one of the following conditions hold:

(1) *R is a non-Gorenstein AGL ring and $\ell_{R_1}(R_1/\mathfrak{m}R_1) = n$.*

(2) *R is an n-Goto ring and $\mathfrak{m}R_1 = \mathfrak{m}_1$.*

Corollary 4.3

For each $n > 0$, we have

$$R \text{ is an } n\text{-Goto ring} \iff \widehat{R} \text{ is an } n\text{-Goto ring.}$$

Example 4.4 (cf. Chau-Goto-Kumashiro-Matsuoka)

Let $R_1 = R[X]/(X^n + \alpha_1 X^{n-1} + \cdots + \alpha_n)$ $(n \geq 1,\ \alpha_i \in \mathfrak{m})$. Then

- R_1 is a flat local R-algebra with $\mathfrak{m}_1 = \mathfrak{m}R_1 + (x)$, where $x = \overline{X}$ in R_1
- $R_1/\mathfrak{m}R_1 = (R/\mathfrak{m})[X]/(X^n)$ is an Artinian Gorenstein ring
- $\ell_{R_1}(R_1/\mathfrak{m}R_1) = n$.

Hence, if $n \geq 2$ is a prime integer, then

$$R_1 \text{ is an } n\text{-Goto ring} \iff R \text{ is a non-Gorenstein AGL ring.}$$

Example 4.5

Let K/k be a finite extension of fields with $[K:k]=n \geq 2$. Set $\omega_1 = 1$ and choose a k-basis $\{\omega_1, \omega_2, \ldots, \omega_n\}$ of K. For a numerical semigroup H and $0 < a \in H$, we consider

$$R = k[[H]] \subseteq R_1 = k[[H, \{\omega_i t^a\}_{1 \leq i \leq n}]] \subseteq K[[H]] \subseteq K[[t]].$$

Suppose $\mathrm{r}(T) \geq 2$. Then R_1 is a free R-module of rank n and $\ell_{R_1}(R_1/\mathfrak{m}R_1) = n$. Hence, if $n \geq 2$ is a prime integer, then

$$R_1 \text{ is an } n\text{-Goto ring} \iff R \text{ is a non-Gorenstein AGL ring.}$$

Example 4.6

Let $a_1, a_2, \ldots, a_\ell \in \mathbb{Z}$ $(\ell > 0)$ s.t. $\gcd(a_1, \cdots, a_\ell) = 1$. Set $H = \langle a_1, a_2, \ldots, a_\ell \rangle$. For an odd integer $0 < \alpha \in H$ s.t. $\alpha \neq a_i$ $(1 \leq i \leq \ell)$, we consider

$$H_1 = \langle 2a_1, 2a_2, \ldots, 2a_\ell, \alpha \rangle \quad (\text{the gluing of } H \text{ and } \mathbb{N}).$$

Then $R_1 = k[[H_1]]$ is a free module of rank 2 and $\ell_{R_1}(R_1/\mathfrak{m}R_1) = 2$. Hence

$$R_1 \text{ is a 2-Goto ring} \iff R = k[[H]] \text{ is a non-Gorenstein AGL ring.}$$

§5. Quasi-trivial extension

- T a birational module-finite extension of R s.t. $K \subseteq T$ and $T \neq R$
- $J = R : T$.

For each $\alpha \in R$, we set $A(\alpha) = R \oplus J$ as an additive group and define

$$(a, x) \cdot (b, y) := (ab, ay + bx + \alpha\cdot(xy)) \quad \text{for} \quad (a, x), (b, y) \in A(\alpha).$$

Then $A(\alpha)$ is a CM local ring with $\dim A(\alpha) = 1$.

- If $\alpha = 0$, then $A(0) = R \ltimes J$.
- If $\alpha = 1$, then $A(1) \cong R \times_{R/J} R$, $(a, j) \mapsto (a, a + j)$.

Note that

- $L = T \times K$ is a fractional canonical ideal of $A(\alpha)$.
- $\mathrm{r}\,(A(\alpha)) = \mu_R(T) + \mathrm{r}(R) = \mathrm{r}_R(J) + \mu_R(K/J)$.

Theorem 5.1

Let $n \geq 1$. Then TFAE.

(1) $A(\alpha)$ *is an* n*-Goto ring for* $\forall\, \alpha \in R$.

(2) $A(\alpha)$ *is an* n*-Goto ring for* $\exists\, \alpha \in R$.

(3) $R \times_{R/J} R$ *is an* n*-Goto ring.*

(4) $R \ltimes J$ *is an* n*-Goto ring.*

(5) $\ell_R(R/J) = n$.

We choose $T = R[K]\ (= R^I)$ and set $\mathfrak{c} = R : R[K]\ (= J)$.

Corollary 5.2

Let $n \geq 1$. Then TFAE.

(1) R *is an* n*-Goto ring and* $\mu_R(K^2/K) = 1$.

(2) $A = R \times_{R/\mathfrak{c}} R$ *is an* n*-Goto ring and* $\mu_A(L^2/L) = 1$.

(3) $A = R \ltimes \mathfrak{c}$ *is an* n*-Goto ring and* $\mu_A(L^2/L) = 1$.

Recall that

- $R = k[[t^3, t^{3n+1}, t^{3n+2}]]$ $(n \geq 1)$ is n-Goto and $\mu_R(K^2/K) = 1$.

Example 5.3 (cf. Chau-Goto-Kumashiro-Matsuoka)

Let $n \geq 1$. Suppose R is n-Goto and $\mu_R(K^2/K) = 1$. Consider

$$A_\ell = \begin{cases} R & (\ell = 0) \\ A_{\ell-1} \ltimes \mathfrak{c}_{\ell-1} & (\ell \geq 1) \end{cases}$$

where $\mathfrak{c} = A_{\ell-1} : A_{\ell-1}[K_{\ell-1}]$ and $K_{\ell-1}$ is the fractional canonical ideal of $A_{\ell-1}$.

We have an infinite family $\{A_\ell\}_{\ell \geq 0}$ of n-Goto rings with $\mu_{A_\ell}(K_\ell^2/K_\ell) = 1$ and $\mathrm{e}(A_\ell) = 2^\ell \cdot \mathrm{e}(R)$ for $\forall \ell \geq 0$.

The ring $k[[t^3, t^7, t^8]] \ltimes k[[t]]$ is 2-Goto, since $\mathfrak{c} = R : k[[t]] = t^6 k[[t]] \cong k[[t]]$.

We consider

- $(S, \mathfrak{n})$ a CM local ring with $\dim S = 1$ and $k = R/\mathfrak{m} = S/\mathfrak{n}$
- $f : R \to k$, $g : S \to k$ canonical maps
- $A = R \times_k S = \{(a, b) \in R \times S \mid f(a) = g(b)\} \subseteq R \times S$.

Then A is a CM local ring with $\dim A = 1$. Note that

$$A \text{ is Gorenstein} \iff R \text{ and } S \text{ are DVRs.}$$

Theorem 5.4

Suppose $\#k = \infty$, $\exists\, \mathrm{K}_A$, and $\mathrm{Q}(A)$ is Gorenstein. Then TFAE for each $n \geq 2$.

(1) *$A = R \times_k S$ is an n-Goto ring.*

(2) *One of the following conditions hold:*

  (i) *R is Gorenstein and S is n-Goto.*
  (ii) *R is n-Goto and S is Gorenstein.*
  (iii) *R is p-Goto and S is q-Goto for $\exists\, p, q > 0$ s.t. $n + 1 = p + q$.*

Hence, if R is n-Goto and S is 2-Goto, then $A = R \times_k S$ is $(n+1)$-Goto.

§6. The case where $\mathrm{r}(R) = 2$

Recall that, for each $n \geq 0$, R is n-Goto $\iff$ $K^2 = K^3$ and $\ell_R(K^2/K) = n$.

Lemma 6.1

Suppose $\mathrm{r}(R) = 2$. For each $n \geq 1$, we have

$$R \text{ is } n\text{-Goto} \iff K^2 = K^3 \text{ and } \ell_R(K/R) = n.$$

When this is the case, $K/R \cong R/\mathfrak{c}$ and R is a generalized Gorenstein ring.

Suppose that

- $R = k[[t^{a_1}, t^{a_2}, t^{a_3}]]$, where $0 < a_1, a_2, a_3 \in \mathbb{Z}$ s.t. $\gcd(a_1, a_2, a_3) = 1$
- R is not a Gorenstein ring
- $\varphi : k[[X, Y, Z]] \to R$ the k-algebra map s.t.

$$\varphi(X) = t^{a_1}, \ \varphi(Y) = t^{a_2}, \ \text{and} \ \varphi(Z) = t^{a_3}.$$

Then $\operatorname{Ker}\varphi = \mathrm{I}_2\begin{pmatrix} X^{\alpha} & Y^{\beta} & Z^{\gamma} \\ Y^{\beta'} & Z^{\gamma'} & X^{\alpha'} \end{pmatrix}$ for $\exists\, \alpha, \beta, \gamma, \alpha', \beta', \gamma' > 0$.

Hence, $\ell_R(K/R) = \alpha\beta\gamma$ or $\ell_R(K/R) = \alpha'\beta'\gamma'$.

Example 6.2

Let $R = k[[t^7, t^{10}, t^{22}]]$. Then $K = R + Rt^8$ is a fractional canonical ideal of R. Note that $K^2 = K^3$ and

$$R \cong k[[X,Y,Z]]/\mathrm{I}_2\begin{pmatrix} X^2 & Y^2 & Z \\ Y^4 & Z & X^3 \end{pmatrix}.$$

Hence $\ell_R(K/R) = 4$, so that R is a 4-Goto ring.

Theorem 6.3

Let $R = k[[H]]$. Suppose $\mathrm{e}(R) = 3$ and R has minimal multiplicity. Then TFAE for each $n \geq 1$.

(1) *R is an n-Goto ring.*

(2) $H = \langle 3, 2n+\alpha, n+2\alpha\rangle$ *for $\exists\,\alpha \geq n+1$ s.t. $\alpha \not\equiv n \mod 3$.*

When this is the case, one has

$$R \cong k[[X,Y,Z]]/\mathrm{I}_2\begin{pmatrix} X^n & Y & Z \\ Y & Z & X^\alpha \end{pmatrix} \quad \text{or} \quad R \cong k[[X,Y,Z]]/\mathrm{I}_2\begin{pmatrix} X^\alpha & Y & Z \\ Y & Z & X^n \end{pmatrix}.$$

§7. Minimal free resolutions

- $(T, \mathfrak{n})$ a RLR with $\dim T = \ell \geq 3$, $\mathfrak{a} \subsetneq T$ and ideal of T s.t. $\mathfrak{a} \subseteq \mathfrak{n}^2$, $n \geq 2$
- $R = T/\mathfrak{a}$ is a CM local ring with $\dim R = 1$, $\mathfrak{m} = \mathfrak{n}/\mathfrak{a}$
- K a fractional canonical ideal of R, $\mathfrak{c} = R : R[K]$.

Suppose R is an n-Goto ring and $v(R/\mathfrak{c}) = 1$. Since $\ell_R(R/\mathfrak{c}) = n$, we can choose

$$x_1, x_2, \dots, x_\ell \in \mathfrak{m} \ \text{ s.t. } \ \mathfrak{m} = (x_1, x_2, \dots, x_\ell) \quad \text{and} \quad \mathfrak{c} = (x_1^n, x_2, \dots, x_\ell).$$

By setting $I_i = (x_1^i, x_2, \dots, x_\ell)$ $(1 \leq i \leq n)$, we have

$$R : K = \mathfrak{c} = I_n \subsetneq I_{n-1} \subsetneq \cdots \subsetneq I_1 = \mathfrak{m} \quad \text{and}$$

$$K/R \cong \bigoplus_{i=1}^{n} (R/I_i)^{\oplus \ell_i} \text{ for } \exists\, \ell_n > 0,\ \exists\, \ell_i \geq 0\ (1 \leq i \leq n-1).$$

Write $K = R + \sum_{i=1}^{n} \sum_{j=1}^{\ell_i} R \cdot f_{ij}$ s.t. $(R/I_i)^{\oplus \ell_i} \cong \sum_{j=1}^{\ell_i} (R/\mathfrak{c}) \cdot \overline{f_{ij}}$ in K/R.

Choose $X_i \in \mathfrak{n}$ s.t. $x_i = \overline{X_i}$ in R.

Theorem 7.1

If $R = T/\mathfrak{a}$ is an n-Goto ring and $v(R/\mathfrak{c}) = 1$, then $F_1 \xrightarrow{\mathbb{M}} F_0 \xrightarrow{\mathbb{N}} K \to 0$ gives a minimal free presentation of K, where $\mathbb{N} = [\,-1 \ f_{n1}\cdots f_{n\ell_n} \ f_{n-1,1}\cdots f_{n-1,\ell_{n-1}} \ \cdots \ f_{11}\cdots f_{1\ell_1}\,]$ and

$$\mathbb{M} = \begin{bmatrix} a_{11}^{(n)} a_{12}^{(n)} \cdots a_{1\ell}^{(n)} & \cdots & a_{\ell_n 1}^{(n)} a_{\ell_n 2}^{(n)} \cdots a_{\ell_n \ell}^{(n)} & & \cdots & & a_{11}^{(1)} a_{12}^{(1)} \cdots a_{1\ell}^{(1)} & \cdots & a_{\ell_n 1}^{(1)} a_{\ell_n 2}^{(1)} \cdots a_{\ell_n \ell}^{(1)} & c_1 c_2 \cdots c_q \\ X_1^n X_2 \cdots X_\ell & & & & & & & & & 0 \\ & \ddots & & & & & & & & 0 \\ & & X_1^n X_2 \cdots X_\ell & & & & & & & \vdots \\ & & & X_1^{n-1} X_2 \cdots X_\ell & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & X_1^{n-1} X_2 \cdots X_\ell & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & X_1 X_2 \cdots X_\ell & & \\ & & & & & & & & \ddots & 0 \\ & & & & & & & & X_1 X_2 \cdots X_\ell & 0 \end{bmatrix}$$

Moreover, one has

$$\mathfrak{a} = \sum_{i=1}^{n} \sum_{j=1}^{\ell_i} \mathrm{I}_2 \begin{pmatrix} a_{j1}^{(i)} & a_{j2}^{(i)} & \cdots & a_{j\ell}^{(i)} \\ X_1^i & X_2 & \cdots & X_\ell \end{pmatrix} + (c_1, c_2, \ldots, c_q).$$

Example 7.2

Let $\varphi : T = k[[X, Y, Z, W]] \longrightarrow R = k[[t^4, t^{11}, t^{13}, t^{14}]]$ be the k-algebra map defined by

$$\varphi(X) = t^4,\ \varphi(Y) = t^{11},\ \varphi(Z) = t^{13},\ \text{and } \varphi(W) = t^{14}$$

Then $K = R + Rt + Rt^3$ is a fractional canonical ideal of R. Hence, $K^2 = K^3$ and $\ell_R(K^2/K) = 3$, so that R is a 3-Goto ring. Moreover, $v(R/\mathfrak{c}) = 1$.

The minimal free presentation of K is given by $F_1 \xrightarrow{\mathbb{M}} F_0 \longrightarrow K \longrightarrow 0$, where

$$\mathbb{M} = \begin{bmatrix} Z & -X^3 & -W & -XY & Y & W & X^4 & XZ \\ X^3 & Y & Z & W & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X^2 & Y & Z & W \end{bmatrix}.$$

Hence

$$\operatorname{Ker}\varphi = \mathrm{I}_2\begin{pmatrix} Z & -X^3 & -W & -XY \\ X^3 & Y & Z & W \end{pmatrix} + \mathrm{I}_2\begin{pmatrix} Y & W & X^4 & XZ \\ X^2 & Y & Z & W \end{pmatrix}.$$

Theorem 7.3

Let $X_1, X_2, \ldots, X_\ell \in \mathfrak{n}$ be a regular sop of T and assume K has a presentation of the form

$$F_1 \xrightarrow{\mathbb{M}} F_0 \xrightarrow{\mathbb{N}} K \longrightarrow 0$$

where $\mathbb{M}$ and $\mathbb{N}$ are the matrices of the form stated in Theorem 7.1, satisfying the conditions that

- $a_{ij}^{(n)} \in J_n \ (1 \le i \le \ell_n, \ 1 \le j \le \ell)$
- $a_{ij}^{(k)} \in J_n \ (1 \le k \le n-1, \ 1 \le i \le \ell_k, \ 2 \le j \le \ell)$
- $a_{i1}^{(k)} \in J_k \ (1 \le k \le n-1, \ 1 \le i \le \ell_k)$

where $J_i = (X_1^i, X_2, \ldots, X_\ell) \ (1 \le i \le n)$. Then R is an n-Goto ring.

Example 7.4

Let k be a field. For any $\ell \ge 3$, $m \ge n \ge 2$,

$$R = k[[X_1, X_2, \ldots, X_\ell]]/\mathrm{I}_2 \begin{pmatrix} X_1^n & X_2 & \cdots & X_{\ell-1} & X_\ell \\ X_2 & X_3 & \cdots & X_\ell & X_1^m \end{pmatrix}$$

is an n-Goto ring with $\dim R = 1$ and $\mathrm{r}(R) = \ell - 1$.

§8. Higher-dimensional Goto rings

- $(A, \mathfrak{m})$ a CM local ring with $d = \dim A > 0$
- $I \subsetneq A$ an ideal of A s.t. $I \cong \mathrm{K}_A$, and $n \geq 0$ an integer.

Definition 8.1 (My proposal)

The ring A is called n-Goto, if $\exists\, Q = (a_1, a_2, \ldots, a_d)$ a parameter ideal of A s.t.

(1) $a_1 \in I$

(2) $\mathcal{S}_Q(J) = \mathcal{T}{\cdot}[\mathcal{S}_Q(J)]_1$ (i.e., $J^3 = QJ^2$)

(3) $\operatorname{rank} \mathcal{S}_Q(J) = n$, where $J = Q + I$, $\mathcal{T} = \mathcal{R}(Q)$, and $\mathcal{S}_Q(J) = \bigoplus_{i \geq 1} J^{i+1}/JQ^i$.

Example 8.2

Let k be a field. For any $\ell \geq 3$, $m \geq n \geq 2$,

$$A = k[[X_1, X_2, \ldots, X_\ell, V_1, V_2, \ldots, V_{\ell-1}]]/\mathrm{I}_2 \begin{pmatrix} X_1^n & X_2 + V_1 & \cdots & X_{\ell-1} + V_{\ell-2} & X_\ell + V_{\ell-1} \\ X_2 & X_3 & \cdots & X_\ell & X_1^m \end{pmatrix}$$

is an n-Goto ring with $\dim A = \ell$ and $\mathrm{r}(A) = \ell - 1$.

Thank you for your attention.