On the stratification of one-dimensional Cohen-Macaulay rings

Naoki Endo

Meiji University

日本数学会 2023 年度年会

March 16, 2023

1. Introduction

Question 1.1

Why are there so many Cohen-Macaulay rings which are not Gorenstein?

- Hierarchy of local rings (in terms of homological algebra)

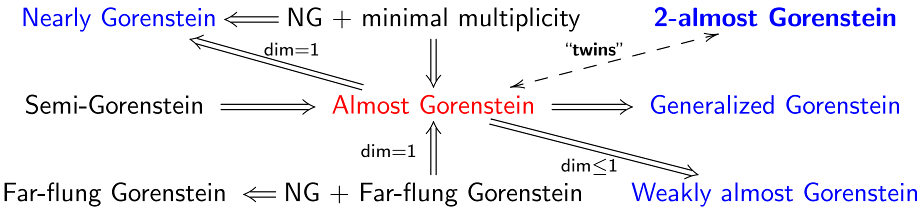
Regular ⇒ Complete Intersection ⇒ Gorenstein ⇒ Cohen-Macaulay
⇒ Buchsbaum ⇒ Generalized Cohen-Macaulay (FLC)

Problem 1.2

Find new and interesting classes of rings which fill in a gap between Gorenstein and Cohen-Macaulay rings so as to stratify Cohen-Macaulay rings.

Preceding researches

- Almost Gorenstein rings ··· Barucci-Fröberg, Goto-Matsuoka-Phuong Goto-Takahashi-Taniguchi
- Semi-Gorenstein rings ··· Goto-Takahashi-Taniguchi
- Generalized Gorenstein rings ··· Goto-Kumashiro
- **2-almost Gorenstein rings** ··· Chau-Goto-Kumashiro-Matsuoka
- Weakly almost Gorenstein rings ··· Dao-Kobayashi-Takahashi
- Nearly Gorenstein rings ··· Herzog-Hibi-Stamate
- Far-flung Gorenstein rings ··· Herzog-Kumashiro-Stamate



In what follows, let

- $(R, \mathfrak{m})$ a CM local ring with $\dim R = 1$ and $\exists\, \mathrm{K}_R$
- $I \subsetneq R$ an ideal of R s.t. $I \cong \mathrm{K}_R$, $Q = (a) \subseteq I$ a reduction of I

By setting $K = \frac{I}{a}$, we have $R \subseteq K \subseteq \overline{R}$ and $K \cong \mathrm{K}_R$ (fractional canonical ideal).

Definition 1.3 (Goto-Takahashi-Taniguchi)

We say that R is an almost Gorenstein ring, if $\mathfrak{m}K \subseteq R$.

- $\mathcal{S}_Q(I) = \bigoplus_{i \geq 1} I^{i+1}/IQ^i$, $\mathcal{T} = R[Qt] \subseteq R[t]$, and $\mathfrak{p} = \mathfrak{m}\mathcal{T} \in \operatorname{Spec} \mathcal{T}$
- $\operatorname{rank} \mathcal{S}_Q(I) = \ell_{\mathcal{T}_\mathfrak{p}}([\mathcal{S}_Q(I)]_\mathfrak{p}) = \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_R(R/I)$

Then

R is an almost Gorenstein ring $\iff$ $\operatorname{rank} \mathcal{S}_Q(I) \leq 1$ (GMP)

R is a 2-almost Gorenstein ring $\overset{def}{\iff}$ $\operatorname{rank} \mathcal{S}_Q(I) = 2$. (CGKM)

Question 1.4

For a given integer $n \geq 0$, what kind of rings satisfy $\operatorname{rank} \mathcal{S}_Q(I) = n$?

2. One-dimensional Goto rings

Let $n \geq 0$ be an integer.

Definition 2.1 (My proposal)

We say that R is an *n-Goto ring*, if $\operatorname{rank} \mathcal{S}_Q(I) = n$ and $\mathcal{S}_Q(I) = \mathcal{T}\cdot[\mathcal{S}_Q(I)]_1$.

Note that R is n-Goto $\iff \ell_R(K^2/K) = n$ and $K^2 = K^3$. Then

- R is 0-Goto $\iff$ R is Gorenstein
- R is 1-Goto $\iff$ R is non-Gorenstein almost Gorenstein
- R is 2-Goto $\iff$ R is 2-almost Gorenstein
- R is $\ell_R(R/\mathfrak{c})$-Goto $\Longleftarrow$ R is generalized Gorenstein, where $\mathfrak{c} = R : R[K]$.

Example 2.2

The ring $R = k[[H]] = k[[t^h \mid h \in H]]$ $(\subseteq k[[t]])$ is an n-Goto ring, where

- $H = \langle 3, 3n+1, 3n+2 \rangle$ $(n \geq 1)$
- $H = \langle e, \{en - e + i\}_{3 \leq i \leq e-1}, en+1, en+2 \rangle$ $(n \geq 2,\ e \geq 4)$.

3. Minimal free resolutions

Let

- $(T, \mathfrak{n})$ a RLR with $\dim T = \ell \geq 3$, $\mathfrak{a} \subsetneq T$ and ideal of T s.t. $\mathfrak{a} \subseteq \mathfrak{n}^2$, $n \geq 2$
- $R = T/\mathfrak{a}$ is a CM local ring with $\dim R = 1$, $\mathfrak{m} = \mathfrak{n}/\mathfrak{a}$
- K a fractional canonical ideal of R, $\mathfrak{c} = R : R[K]$.

Suppose R is an n-Goto ring and $v(R/\mathfrak{c}) = 1$. Since $\ell_R(R/\mathfrak{c}) = n$, we can choose

$$x_1, x_2, \ldots, x_\ell \in \mathfrak{m} \ \text{ s.t. } \ \mathfrak{m} = (x_1, x_2, \ldots, x_\ell) \quad \text{and} \quad \mathfrak{c} = (x_1^n, x_2, \ldots, x_\ell).$$

By setting $I_i = (x_1^i, x_2, \ldots, x_\ell)$ $(1 \leq i \leq n)$, we have

$$R : K = \mathfrak{c} = I_n \subsetneq I_{n-1} \subsetneq \cdots \subsetneq I_1 = \mathfrak{m} \quad \text{and}$$

$$K/R \cong \bigoplus_{i=1}^{n} (R/I_i)^{\oplus \ell_i} \text{ for } \exists\, \ell_n > 0, \exists\, \ell_i \geq 0 \ (1 \leq i \leq n-1).$$

Write $K = R + \sum_{i=1}^{n} \sum_{j=1}^{\ell_i} R \cdot f_{ij}$ s.t. $(R/I_i)^{\oplus \ell_i} \cong \sum_{j=1}^{\ell_i} (R/\mathfrak{c}) \cdot \overline{f_{ij}}$ in K/R.

Choose $X_i \in \mathfrak{n}$ s.t. $x_i = \overline{X_i}$ in R.

Theorem 3.1

If $R = T/\mathfrak{a}$ is an n-Goto ring and $v(R/\mathfrak{c}) = 1$, then $F_1 \xrightarrow{\mathbb{M}} F_0 \xrightarrow{\mathbb{N}} K \to 0$ gives a minimal free presentation of K, where $\mathbb{N} = \begin{bmatrix} -1 & f_{n1} \cdots f_{n\ell_n} & f_{n-1,1} \cdots f_{n-1,\ell_{n-1}} & \cdots & f_{11} \cdots f_{1\ell_1} \end{bmatrix}$ and

$$\mathbb{M} = \begin{bmatrix} a_{11}^{(n)} a_{12}^{(n)} \cdots a_{1\ell}^{(n)} & \cdots & a_{\ell_n 1}^{(n)} a_{\ell_n 2}^{(n)} \cdots a_{\ell_n \ell}^{(n)} & & \cdots & & a_{11}^{(1)} a_{12}^{(1)} \cdots a_{1\ell}^{(1)} & \cdots & a_{\ell_n 1}^{(1)} a_{\ell_n 2}^{(1)} \cdots a_{\ell_n \ell}^{(1)} & c_1 c_2 \cdots c_q \\ X_1^n X_2 \cdots X_\ell & & & & & & & & & 0 \\ & \ddots & & & & & & & & 0 \\ & & X_1^n X_2 \cdots X_\ell & & & & & & & \vdots \\ & & & X_1^{n-1} X_2 \cdots X_\ell & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & X_1^{n-1} X_2 \cdots X_\ell & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & X_1 X_2 \cdots X_\ell & & \\ & & & & & & & & \ddots & 0 \\ & & & & & & & & X_1 X_2 \cdots X_\ell & 0 \end{bmatrix}$$

Moreover, one has

$$\mathfrak{a} = \sum_{i=1}^{n} \sum_{j=1}^{\ell_i} \mathrm{I}_2 \begin{pmatrix} a_{j1}^{(i)} & a_{j2}^{(i)} & \cdots & a_{j\ell}^{(i)} \\ X_1^i & X_2 & \cdots & X_\ell \end{pmatrix} + (c_1, c_2, \ldots, c_q).$$

Example 3.2

Let k be a field. For any $\ell \geq 3$, $m \geq n \geq 2$,

$$R = k[[X_1, X_2, \ldots, X_\ell]]/\mathrm{I}_2 \begin{pmatrix} X_1^n & X_2 & \cdots & X_{\ell-1} & X_\ell \\ X_2 & X_3 & \cdots & X_\ell & X_1^m \end{pmatrix}$$

is an n-Goto ring with $\dim R = 1$ and $\mathrm{r}(R) = \ell - 1$.

4. Higher-dimensional Goto rings

- $(A, \mathfrak{m})$ a CM local ring with $d = \dim A > 0$ and $\exists\, \mathrm{K}_A$
- $I \subsetneq A$ an ideal of A s.t. $I \cong \mathrm{K}_A$, and $n \geq 0$ an integer

Definition 4.1 (My proposal)

The ring A is called *n-Goto*, if $\exists\, Q = (a_1, a_2, \dots, a_d)$ a parameter ideal of A s.t.

(1) $a_1 \in I$

(2) $\mathcal{S}_Q(J) = \mathcal{T}{\cdot}[\mathcal{S}_Q(J)]_1$ (i.e., $J^3 = QJ^2$)

(3) $\operatorname{rank} \mathcal{S}_Q(J) = n$, where $J = Q + I$, $\mathcal{T} = \mathcal{R}(Q)$, and $\mathcal{S}_Q(J) = \bigoplus_{i \geq 1} J^{i+1}/JQ^i$.

Example 4.2

Let k be a field. For any $\ell \geq 3$, $m \geq n \geq 2$,

$$A = k[[X_1, X_2, \dots, X_\ell, V_1, V_2, \dots, V_{\ell-1}]]/\mathrm{I}_2 \begin{pmatrix} X_1^n & X_2+V_1 & \cdots & X_{\ell-1}+V_{\ell-2} & X_\ell+V_{\ell-1} \\ X_2 & X_3 & \cdots & X_\ell & X_1^m \end{pmatrix}$$

is an n-Goto ring with $\dim A = \ell$ and $\mathrm{r}(A) = \ell - 1$.

Thank you for your attention.