Ulrich ideals in numerical semigroup rings

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based on the recent works jointly with

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1. Introduction

This talk is based on the recent researches below.

- N. Endo and S. Goto, *Ulrich ideals in numerical semigroup rings of small multiplicity*, arXiv:2111.00498
- N. Endo, S. Goto, S.-i. Iai, and N. Matsuoka, *Ulrich ideals in the ring* $k[[t^5, t^{11}]]$, arXiv:2111.01085

Problem 1.1

Determine all the Ulrich ideals in a given CM local ring.

What is an Ulrich ideal?

- In 1971, J. Lipman investigated **stable maximal ideal** in a CM local ring.
- In 2014, S. Goto, K. Ozeki, R. Takahashi, K.-i. Watanabe, K.-i. Yoshida modified the notion of stable maximal ideal, which they call an **Ulrich ideal**.

Let

- $(A, \mathfrak{m})$ be a CM local ring with $d = \dim A$.
- $\sqrt{I} = \mathfrak{m}$, I contains a parameter ideal Q of A as a reduction
  (i.e. $I^{n+1} = QI^n$ for some $n \geq 0$)

Definition 1.2 (Goto-Ozeki-Takahashi-Watanabe-Yoshida, 2014)

We say that I is an *Ulrich ideal of A*, if

(1) $I \supsetneq Q$, $I^2 = QI$, and

(2) I/I^2 is A/I-free.

Note that

- (1) $\iff$ $\mathrm{gr}_I(A) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ is a CM ring with $\mathrm{a}(\mathrm{gr}_I(A)) = 1 - d$.
- If $I = \mathfrak{m}$, then (1) $\iff$ A has minimal multiplicity $\mathrm{e}(A) > 1$.
- (2) and $I \supsetneq Q$ $\implies$ $\mathrm{pd}_A I = \infty$ (Ferrand, Vasconcelos, 1967)

Assume that $I^2 = QI$. Then the exact sequence

$$0 \to Q/QI \to I/I^2 \to I/Q \to 0$$

of A/I-modules shows

$$I/I^2 \text{ is } A/I\text{-free} \iff I/Q \text{ is } A/I\text{-free.}$$

Therefore, if I is an Ulrich ideal of A, then

- $I/Q \cong (A/I)^{\oplus(\mu_A(I)-d)}$,
- $Q :_A I = I$ (i.e., I is a good ideal of A),
- $\mathrm{r}_A(I/Q) = (\mu_A(I) - d) \cdot \mathrm{r}(A/I) = \mathrm{r}(A)$

so that

$$d + 1 \leq \mu_A(I) \leq d + \mathrm{r}(A).$$

Hence, when A is a Gorenstein ring,

every Ulrich ideal I is generated by $d + 1$ elements (if it exists).

For every Ulrich ideal I of A, we have

Theorem 1.3 (Goto-Takahashi-T, 2015)

$$\operatorname{Ext}_A^i(A/I, A) \text{ is } A/I\text{-free for } \forall i \in \mathbb{Z}.$$

Hence

$$\mu_A(I) = d + 1 \iff \text{G-dim}_A\, A/I < \infty.$$

This shows if A is G-regular, then $\mu_A(I) \geq d + 2$.

Consequently, if I is an Ulrich ideal of A with $\mu_A(I) = d + 1$, then

- A/I is Gorenstein $\iff$ A is Gorenstein,
- I is a totally reflexive A-module,
- $\operatorname{pd}_A I = \infty$, and

the minimal free resolution of I has a very restricted form.

In what follows, assume $d = 1$ and I is an Ulrich ideal of A with $\mu_A(I) = 2$.

Write $I = (a, b)$, where $a, b \in A$ and $Q = (a)$ is a reduction of I.

By taking $c \in I$ with $b^2 = ac$, the minimal free resolution of I has the form

$$\cdots \longrightarrow A^{\oplus 2} \xrightarrow{\begin{pmatrix} -b & -c \\ a & b \end{pmatrix}} A^{\oplus 2} \xrightarrow{\begin{pmatrix} -b & -c \\ a & b \end{pmatrix}} A^{\oplus 2} \xrightarrow{\begin{pmatrix} a & b \end{pmatrix}} I \longrightarrow 0$$

We then have $I = J$, once

$$\mathrm{Syz}_A^i(I) \cong \mathrm{Syz}_A^i(J) \quad \text{for some } i \geq 0$$

provided I, J are Ulrich ideals of A. (GOTWY, 2014)

Corollary 1.4 (GOTWY, 2014)

Suppose that A is a Gorenstein ring. If I, J are Ulrich ideals of A with $\mathfrak{m}J \subseteq I \subsetneq J$, then A is a hypersurface.

Let $\mathcal{X}_A$ be the set of Ulrich ideals in A.

On the other hand

- If A has finite CM representation type, then $\mathcal{X}_A$ is finite. (GOTWY, 2014)
- Suppose that $\exists$ a fractional canonical ideal K. Set $\mathfrak{c} = A : A[K]$.

  If A is a non-Gorenstein almost Gorenstein ring, then

  $$\mathcal{X}_A \subseteq \{\mathfrak{m}\} \qquad \text{(GTT, 2015)}$$

  If A is a 2-almost Gorenstein ring with minimal multiplicity, then

  $$\{\mathfrak{m}\} \subseteq \mathcal{X}_A \subseteq \{\mathfrak{m}, \mathfrak{c}\} \qquad \text{(Goto-Isobe-T, 2020)}$$

We expect that there is a strong connection between

the behavior of Ulrich ideals and the structure of base rings.

Problem 1.1

Determine all the Ulrich ideals in a given CM local ring.

Question 1.5

How many two-generated Ulrich ideals are contained in a given numerical semigroup ring?

Let

- $0 < a_1, a_2, \ldots, a_\ell \in \mathbb{Z}$ s.t. $\gcd(a_1, a_2, \ldots, a_\ell) = 1$
- $H = \langle a_1, a_2, \ldots, a_\ell \rangle = \left\{ \sum_{i=1}^{\ell} c_i a_i \;\middle|\; 0 \le c_i \in \mathbb{Z} \text{ for all } 1 \le i \le \ell \right\}$
- $A = k[[H]] = k[[t^{a_1}, t^{a_2}, \ldots, t^{a_\ell}]] \subseteq V = k[[t]] = \overline{A}$, where k is a field
- $\mathrm{c}(H) = \min\{n \in \mathbb{Z} \mid m \in H \text{ for all } m \in \mathbb{Z} \text{ s.t. } m \ge n\}$

Note that $t^{\mathrm{c}(H)} V \subseteq A$.

2. Method of computation

Previous Method

Let

- $(A, \mathfrak{m})$ be a Gorenstein local ring with $\dim A = 1$,
- $\mathcal{X}_A$ be the set of Ulrich ideals in A,
- $\mathcal{Y}_A$ be the set of birational module-finite extensions B of A

  (i.e., $A \subseteq B \subseteq \mathrm{Q}(A)$ and B is a finitely generated A-module)

  s.t. B is a Gorenstein ring and $\mu_A(B) = 2$.

Then, there exist bijective correspondences

$$\mathcal{X}_A \to \mathcal{Y}_A, \quad I \mapsto A^I \quad \text{and} \quad \mathcal{Y}_A \to \mathcal{X}_A, \quad B \mapsto A : B$$

where

$$A^I = \bigcup_{n \geq 0} [I^n : I^n] = I : I.$$

Example 2.1

Let $A = k[[t^2, t^{2\ell+1}]]$ $(\ell \geq 1)$. Then

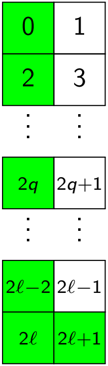
$$\mathcal{X}_A = \{(t^{2q}, t^{2\ell+1}) \mid 1 \leq q \leq \ell\}.$$

(Proof) Note that $\mathcal{Y}_A = \{k[[t^2, t^{2(\ell-q)+1}]] \mid 1 \leq q \leq \ell\}$.

For $1 \leq \forall q \leq \ell$, we have

$$\begin{aligned} A : k[[t^2, t^{2(\ell-q)+1}]] &= A : (A + At^{2(\ell-q)+1}) \\ &= A : At^{2(\ell-q)+1} \\ &= (t^{2q}, t^{2\ell+1}). \end{aligned}$$

This shows $\mathcal{X}_A = \{(t^{2q}, t^{2\ell+1}) \mid 1 \leq q \leq \ell\}$. ■



Let

- $V = k[[t]]$ be the formal power series ring over a field k
- A be a k-subalgebra of V.

We say that

$$A \text{ is a } \textit{core of } V \quad \stackrel{def}{\Longleftrightarrow} \quad t^c V \subseteq A \text{ for some } c \gg 0.$$

Example 2.2

- $k[[H]]$ is a core of V,
- $A = k[t^2 + t^3] + t^4 V$ is core, but $A \neq k[[H]]$ for any numerical semigroup H.

Let A be a core of V and suppose $t^c V \subseteq A$ with $c \gg 0$. Then

$$k[[t^c, t^{c+1}, \dots, t^{2c-1}]] \subseteq A \subseteq V$$

so that V is a birational module-finite extension of A.

Hence, for every core A of V,

- $V = \overline{A}$
- A is a CM complete local domain with $\dim A = 1$
- $V/\mathfrak{n} \cong A/\mathfrak{m}$

where $\mathfrak{m}$ (resp. $\mathfrak{n} = tV$) stands for the maximal ideal of A (resp. V).

Let $\mathrm{o}(-)$ denote the $\mathfrak{n}$-adic valuation of V, and set

$$H = v(A) = \{\mathrm{o}(f) \mid 0 \neq f \in A\}.$$

Note that

$H = v(A)$ is symmetric $\iff$ A is Gorenstein (Kunz, 1970)

Let I be an Ulrich ideal of A with $\mu_A(I) = 2$. Choose $f, g \in I$ s.t. $I = (f, g)$ and $I^2 = fI$. Then

$$A^I = I : I = \frac{I}{f} = A + A \cdot \frac{g}{f}$$

is a core of V, and $v(A^I)$ is symmetric if A is Gorenstein.

Lemma 2.3 (Key Lemma)

Let I be an Ulrich ideal in A with $\mu_A(I) = 2$. Then one can choose $f, g \in I$ satisfying the following conditions, where $a = \mathrm{o}(f)$ and $b = \mathrm{o}(g)$.

(1) $I = (f, g)$ and $I^2 = fI$.

(2) $a, b \in H$ and $0 < a < b < a + \mathrm{c}(H)$.

(3) $b - a \notin H$, $2b - a \in H$, $a = 2 \cdot \ell_A(A/I)$, and $I \supseteq A : V$.

(4) If $a \geq \mathrm{c}(H)$, then $\mathrm{e}(A) = 2$ and $I = A : V$.

■ Method of computation

- Step 1 ··· Let $I \in \mathcal{X}_A$ with $\mu_A(I) = 2$. Choose $f, g \in I$ which satisfy the conditions in Lemma 2.3.
- Step 2 ··· Consider $A^I = A + A \cdot \frac{g}{f}$ and determine $v(A^I)$.
- Step 3 ··· Determine the possible pair $(\mathrm{o}(f), \mathrm{o}(g))$.
- Step 4 ··· Determine the form of generators of I.
- Step 5 ··· Conversely, the ideal of the form as in Step 4 is an Ulrich ideal.

3. Main theorem

Example 3.1

Let $A = k[[t^3, t^7]]$. Then

$$\mathcal{X}_A = \{(t^6 + \alpha t^7, t^{10}) \mid 0 \neq \alpha \in k\}.$$

(Proof) Set $H = \langle 3, 7 \rangle$. Note that $\mathrm{c}(H) = 12$. As A is Gorenstein, every $I \in \mathcal{X}_A$ is generated by two elements. Choose $f, g \in I$ which satisfy the conditions in Lemma 2.3, i.e.,

- $I = (f, g)$ and $I^2 = fI$
- $a, b \in H$ and $0 < a < b < a + \mathrm{c}(H) = a + 12$
- $b - a \notin H$, $a = 2 \cdot \ell_A(A/I)$, and $I \supseteq A : V = t^{12}V$
- $a < \mathrm{c}(H) = 12$

where $a = \mathrm{o}(f)$ and $b = \mathrm{o}(g)$.

Then $a = 6, 10$ and $b - a = 1, 2, 4, 5, 8, 11$.

0	1	2
3	4	5
6	**7**	8
9	**10**	11
12	**13**	**14**

Consider

$$A^I = I : I = \frac{I}{f} = A + A\xi$$

where $\xi = \frac{g}{f}$. Then $\mu_A(A^I) = 2$ and $A^I = k[[t^3, t^7, \xi]]$ is Gorenstein. We have $\mathrm{o}(\xi) = b - a$, whence $b - a \in v(A^I) \setminus H$.

- If $1 \in v(A^I)$, then $A^I = V$. This is absurd, because $\mu_A(V) = 3$.
- If $2 \in v(A^I)$, then $v(A^I) = \langle 2, 3 \rangle$, so that $A^I = k[[t^2, t^3]]$. As $t^4 \notin \mathfrak{m}A^I$, $\mu_A(A^I) = \ell_A(A^I/\mathfrak{m}A^I) = \dim_k(k[\overline{t^2}]) > 2$. This makes a contradiction.

Hence, $\mathrm{e}(v(A^I)) = 3$, so that $v(A^I) = \langle 3, \alpha \rangle$ for $\exists\ \alpha \not\equiv 0 \mod 3$.

Then, one can show that $\alpha = b - a$ and $\alpha \equiv 1 \mod 3$. Thus

$$\alpha = 4 \quad \text{and} \quad v(A^I) = \langle 3, 4 \rangle.$$

Suppose $a = 10$. Since $\ell_A(V/A) = 6, \ell_A(A/I) = \frac{a}{2} = 5, \ell_A(V/A : V) = 12$ and

$$I \supseteq (f) + A : V \supsetneq A : V = t^{12}V,$$

we get, $I = (f) + A : V = (f, t^{12}, t^{13}, t^{14}) = (t^{10}, t^{12}, t^{14})$. This is impossible.

Therefore, $a = 6$ and $b = 10$.

Hence

$$I = (t^6 + \alpha t^7 + \beta t^9, t^{10}) + t^{12} V = (t^6 + \alpha t^7 + \beta t^9, t^{10}, t^{12}, t^{13}, t^{14})$$

where $\alpha, \beta \in k$.

Since $t^9 = t^3(t^6 + \alpha t^7 + \beta t^9) - \alpha t^{10} - \beta t^{12}$ and $t^9 = t^3(t^6 + \alpha t^7) - \alpha t^{10}$, we get

$$\begin{aligned} I &= (t^6 + \alpha t^7 + \beta t^9, t^{10}, t^{12}, t^{13}, t^{14}) \\ &= (t^6 + \alpha t^7 + \beta t^9, t^{10}, t^{12}, t^{14}) \\ &= (t^6 + \alpha t^7 + \beta t^9, t^9, t^{10}, t^{12}, t^{14}) \\ &= (t^6 + \alpha t^7, t^9, t^{10}, t^{12}, t^{14}) \\ &= (t^6 + \alpha t^7, t^9, t^{10}, t^{14}) \\ &= (t^6 + \alpha t^7, t^{10}, t^{14}). \end{aligned}$$

If $\alpha = 0$, then $I = (t^6, t^{10}, t^{14})$, which is a contradiction. Thus $\alpha \neq 0$. Since

$$t^{14} = \frac{1}{\alpha} t^7 (t^6 + \alpha t^7) - \frac{1}{\alpha} t^3 \cdot t^{10},$$

we finally get $I = (t^6 + \alpha t^7, t^{10})$.

Theorem 3.2 (Main theorem)

Let $\ell \geq 7$ be an integer such that $\gcd(3, \ell) = 1$ and set $A = k[[t^3, t^\ell]]$.

(1) *Suppose that $\ell = 3n + 1$ where $n \geq 3$ is odd. Let $q = \frac{n-1}{2}$. Then*

$$\begin{aligned}\mathcal{X}_A &= \left\{\left(t^\ell + \sum_{j=1}^{q} \alpha_j t^{\ell+3j-1}, t^{\ell+3q+2}\right) \;\middle|\; \alpha_1, \alpha_2, \dots, \alpha_q \in k\right\} \\ &\quad \bigcup \left\{\left(t^{6i} + \sum_{s=0}^{i-1} \alpha_s t^{\ell+3s}, t^{\ell+3i}\right) \;\middle|\; 1 \leq i \leq q, \alpha_0, \dots, \alpha_{i-1} \in k, \alpha_0 \neq 0\right\}.\end{aligned}$$

(2) *Suppose that $\ell = 3n + 1$ where $n \geq 2$ is even. Let $q = \frac{n}{2}$. Then*

$$\mathcal{X}_A = \left\{\left(t^{6i} + \sum_{s=0}^{i-1} \alpha_s t^{\ell+3s}, t^{\ell+3i}\right) \;\middle|\; 1 \leq i \leq q, \alpha_0, \dots, \alpha_{i-1} \in k, \alpha_0 \neq 0\right\}.$$

Theorem 3.1 (continued)

(3) Suppose that $\ell = 3n+2$ where $n \geq 1$ is odd. Let $q = \frac{n-1}{2}$. Then

$$\mathcal{X}_A = \left\{ \left(t^{6i} + \sum_{s=0}^{i-1} \alpha_s t^{\ell+3s}, t^{\ell+3i} \right) \,\middle|\, 1 \leq i \leq q, \alpha_0, \ldots, \alpha_{i-1} \in k, \alpha_0 \neq 0 \right\}.$$

(4) Suppose that $\ell = 3n+2$ where $n \geq 2$ is even. Let $q = \frac{n}{2}$. Then

$$\begin{aligned} \mathcal{X}_A &= \left\{ \left(t^{\ell} + \sum_{j=1}^{q} \alpha_j t^{\ell+3j-2}, t^{\ell+3q+1} \right) \,\middle|\, \alpha_1, \alpha_2, \ldots, \alpha_q \in k \right\} \\ &\quad \bigcup \left\{ \left(t^{6i} + \sum_{s=0}^{i-1} \alpha_s t^{\ell+3s}, t^{\ell+3i} \right) \,\middle|\, 1 \leq i \leq q, \alpha_0, \ldots, \alpha_{i-1} \in k, \alpha_0 \neq 0 \right\}. \end{aligned}$$

Moreover, the coefficients α_i's in the system of generators of $I \in \mathcal{X}_A$ are uniquely determined for I.

We denote by $\mathcal{X}_A^g$ the set of Ulrich ideals in A generated by monomials in t.

Then $\mathcal{X}_A^g$ is a finite set (GOTWY, 2014).

Corollary 3.3

Let $\ell \geq 7$ be an integer s.t. $\gcd(3, \ell) = 1$ and set $A = k[[t^3, t^\ell]]$. Then

(1) $\mathcal{X}_A \neq \emptyset$.

(2) $\mathcal{X}_A$ *is finite* $\iff$ k *is a finite field.*

(3) $\mathcal{X}_A^g = \emptyset$ $\iff$ $\ell = 3n+1$ *or* $\ell = 3n+2$ *for some even integer* $n \geq 2$

Example 3.4

Let $A = k[[t^3, t^7]]$. Then $\mathcal{X}_A = \{(t^6 + \alpha t^7, t^{10}) \mid 0 \neq \alpha \in k\}$.

Hence, ${}^\#\mathcal{X}_A = {}^\# k - 1$ and A does not contain monomial Ulrich ideals.

4. More examples

Example 4.1

We have

$$\begin{aligned}
\mathcal{X}_{k[[t^4,t^{13}]]} &= \{(t^{12}+2\beta t^{17}+\alpha t^{26}, t^{21}+\beta t^{26}) \mid \alpha,\beta\in k,\ \beta\neq 0\}\\
&\bigcup \{(t^{16}+2\beta t^{17}+\alpha_2 t^{21}+\alpha_3 t^{26}, t^{25}+\beta t^{26}) \mid \alpha_2,\alpha_3,\beta\in k,\ \beta\neq 0\}\\
&\bigcup \{(t^{4}+\alpha t^{13}, t^{26}) \mid \alpha\in k\}\\
&\bigcup \left\{(t^{8}+\alpha_1 t^{13}+\alpha_2 t^{17}, t^{26}) \mid \alpha_1,\alpha_2\in k\right\}\\
&\bigcup \{(t^{12}+\alpha_1 t^{13}+\alpha_2 t^{17}+\alpha_3 t^{21}, t^{26}) \mid \alpha_1,\alpha_2,\alpha_3\in k\}\\
&\bigcup \{(t^{16}+\alpha_1 t^{17}+\alpha_2 t^{21}+\alpha_3 t^{25}, t^{26}) \mid \alpha_1,\alpha_2,\alpha_3\in k\}\\
&\bigcup \{(t^{20}+\alpha_1 t^{21}+\alpha_2 t^{25}+\alpha_3 t^{29}, t^{26}+\beta t^{29}) \mid \alpha_1,\alpha_2,\alpha_3,\beta\in k,\ \alpha_1^3=2\beta\}\\
&\bigcup \{(t^{24}+\alpha_1 t^{25}+\alpha_2 t^{29}+\alpha_3 t^{33}, t^{26}+\beta_1 t^{29}+\beta_2 t^{33}) \mid \alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2\in k,\\
&\qquad \alpha_1=0 \text{ if } \operatorname{ch} k=2;\ \alpha_1=\alpha_2=\beta_1=\beta_2=0 \text{ if } \operatorname{ch} k\neq 2\}.
\end{aligned}$$

For each $I\in\mathcal{X}_{k[[t^4,t^{13}]]}$, the elements of k which appear in the listed expression are uniquely determined by I.

5. three-generated numerical semigroup rings

- $0 < a, b, c \in \mathbb{Z}$ s.t. $\gcd(a, b, c) = 1$ and set $H = \langle a, b, c \rangle$
- $A = k[[H]] = k[[t^a, t^b, t^c]] \subseteq V = k[[t]]$
- $\mathfrak{m} = (t^a, t^b, t^c)$

For a finitely generated A-module M, let

$$P_M^A(t) = \sum_{n=0}^{\infty} \beta_n^A(M) t^n \in \mathbb{Z}[[t]]$$

where $\beta_n^A(M)$ denotes the n-th Betti number of M.

Theorem 5.1

Suppose that $A = k[[H]]$ is not a Gorenstein ring. Then

$$\beta_n^A(A/\mathfrak{m}) = \begin{cases} 1 & (n = 0) \\ 3 \cdot 2^{n-1} & (n > 0) \end{cases} \quad \text{and} \quad P_{A/\mathfrak{m}}^A(t) = \frac{1+t}{1-2t}.$$

(Proof) As A is not Gorenstein, we have

$$A \cong k[[X,Y,Z]]/\mathrm{I}_2\begin{pmatrix} X^\alpha & Y^\beta & Z^\gamma \\ Y^{\beta'} & Z^{\gamma'} & X^{\alpha'} \end{pmatrix}$$

for $\exists \alpha, \beta, \gamma, \alpha', \beta', \gamma' > 0$. Hence

$$A/(t^a) \cong k[Y,Z]/\mathrm{I}_2\begin{pmatrix} 0 & Y^\beta & Z^\gamma \\ Y^{\beta'} & Z^{\gamma'} & 0 \end{pmatrix} = k[Y,Z]/(Y^{\beta+\beta'}, Y^{\beta'}Z^\gamma, Z^{\gamma+\gamma'}).$$

Let

$$B = k[Y,Z]/(Y^{\beta+\beta'}, Y^{\beta'}Z^\gamma, Z^{\gamma+\gamma'})$$

and let y, z denote the images of Y, Z in B, respectively. Then, because

$$P^B_{B/(y,z)}(t) = \frac{P^A_{A/\mathfrak{m}}(t)}{1+t},$$

we get $P^A_{A/\mathfrak{m}}(t) = \frac{1+t}{1-2t}$, once we have

$$P^B_{B/(y,z)}(t) = \frac{1}{1-2t} = 1 + 2t + 4t^2 + \cdots + 2^n t^n + \cdots .$$

To see this, we consider the minimal B-free resolution of $B/(y,z)$.

One can show that

$$B^{\oplus 16} \xrightarrow{\mathbb{M}_3} B^{\oplus 8} \xrightarrow{\mathbb{M}_2} B^{\oplus 4} \xrightarrow{\mathbb{M}_1} B^{\oplus 2} \xrightarrow{\mathbb{M}_0} B \xrightarrow{\varepsilon} B/(y,z) \longrightarrow 0$$

forms a part of the minimal B-free resolution of $B/(y,z)$, where ε is the canonical epimorphism,

$$\begin{aligned}
\mathbb{M}_0 &= (y\ z), \quad \mathbb{M}_1 = \begin{pmatrix} y^{\beta+\beta'-1} & y^{\beta'-1}z^{\gamma} & 0 & z \\ 0 & 0 & z^{\gamma+\gamma'-1} & -y \end{pmatrix}, \\
\mathbb{M}_2 &= \begin{pmatrix} y & z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y & z & 0 & 0 \\ 0 & -y^{\beta+\beta'-1} & 0 & -y^{\beta'-1}z^{\gamma} & z^{\gamma+\gamma'-1} & 0 & y^{\beta+\beta'-1}z^{\gamma-1} & y^{\beta'-1}z^{\gamma+\gamma'-1} \end{pmatrix}, \text{ and} \\
\mathbb{M}_3 &= \begin{pmatrix} \mathbb{M}_1 & & & & \\ & \mathbb{M}_1 & & & \\ & & \mathbb{M}_1 & & \\ & & & \mathbb{M}_0 & \\ & & & & \mathbb{M}_0 \end{pmatrix}.
\end{aligned}$$

Since $\mathbb{M}_3$ consists of $\mathbb{M}_0$ and $\mathbb{M}_1$, the Poincaré series of $B/(y,z)$ has the form

$$P^{B}_{B/(y,z)}(t) = 1 + 2t + 4t^2 + \cdots + 2^n t^n + \cdots$$

as claimed. ■

Corollary 5.2 (cf. Gasharov-Peeva-Welker, 2000)

Every three-generated non-Gorenstein numerical semigroup ring is Golod.

(Proof) Let $S = k[[X, Y, Z]]$. The S-module A has a minimal free resolution

$$0 \to S^2 \xrightarrow{\begin{pmatrix} X^\alpha & Y^{\beta'} \\ Y^\beta & Z^{\gamma'} \\ Z^\gamma & X^{\alpha'} \end{pmatrix}} S^3 \to S \to A \to 0,$$

whence Theorem 5.1 tells us

$$P^A_{A/\mathfrak{m}}(t) = \frac{1+t}{1-2t} = \frac{(1+t)^3}{1-3t^2-2t^3} = \frac{P^S_{S/\mathfrak{n}}(t)}{1 - t \cdot (P^S_A(t) - 1)},$$

where $\mathfrak{n} = (X, Y, Z)$. Therefore, the natural surjection $S \to A$ is a Golod homomorphism, so that A is a Golod ring. ■

Note that

- every Golod local ring which is not a hypersurface must be G-regular. (Avramov-Martsinkovsky, 2002)

Corollary 5.3

Every three-generated non-Gorenstein numerical semigroup ring contains no Ulrich ideals generated by two elements.

Since $H = \langle a, b, c \rangle$, we have

H is symmetric $\iff$ $k[[H]]$ is a complete intersection (Herzog, 1970).

If H is symmetric, it is obtained by a gluing of a two-generated numerical semigroup H' and $\mathbb{N}$ (Herzog, 1970, Watanabe, 1973).

Let

- $0 < \alpha, \beta \in \mathbb{Z}$ s.t. $\gcd(\alpha, \beta) = 1$.
- $H' = \langle \alpha, \beta \rangle$

Choose $a \in H'$ and $b \in \mathbb{N}$ which satisfy

$$a > 0,\ b > 1,\ a \notin \{\alpha, \beta\}, \text{ and } \gcd(a, b) = 1.$$

Hence, $\gcd(b\alpha, b\beta, a) = 1$. Consider

$$H = \langle b\alpha, b\beta, a \rangle$$

and call it the *gluing* of H' and $\mathbb{N}$ with respect to $a \in H'$ and $b \in \mathbb{N}$.

Assume that $H = \langle b\alpha, b\beta, a\rangle$.

Proposition 5.4

Suppose that one of the following conditions is satisfied.

(1) *b is even and* $\ell \geq 2$.

(2) *b is even and* $m \geq 2$.

(3) *either* α *or* β *is even.*

Then $A = k[[H]]$ *admits at least one Ulrich ideal of* A.

Thank you for your attention.