Huneke–Wiegand conjecture and change of rings

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Introduction

1. R an integral domain
2. M, N finitely generated **torsionfree** R-modules

Question

When is the tensor product $M \otimes_R N$ torsionfree?

Auslander–Reiten conjecture ([2])

Let R be a commutative Noetherian ring, M a finitely generated R–module. If $\mathrm{Ext}_R^i(M, M \oplus R) = (0)$ for $i > 0$, then M is projective.

Huneke–Wiegand conjecture ([5])

Let R be a Gorenstein local domain. Let M be a maximal C–M R-module. If $M \otimes_R \mathrm{Hom}_R(M, R)$ is torsionfree, then M is free.

Theorem 1.1 ([3, 4, 5])

Consider the following assertions.

(1) (HWC) holds for Gorenstein local domains.

(2) (HWC) holds for one-dimensional Gorenstein local domains.

(3) (ARC) holds for Gorenstein local domains.

Then the implications (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3) hold.

Conjecture 1.2

Let R be a Gorenstein local domain with $\dim R = 1$ and I an ideal of R. If $I \otimes_R \mathrm{Hom}_R(I, R)$ is torsionfree, then I is principal.

In my lecture we are interested in the question of what happens if I replace $\mathrm{Hom}_R(I, R)$ by $\mathrm{Hom}_R(I, \mathrm{K}_R)$.

Conjecture 1.3

Let R be a C–M local ring with $\dim R = 1$ and assume $\exists\, \mathrm{K}_R$. Let I be a faithful ideal of R. If $I \otimes_R \mathrm{Hom}_R(I, \mathrm{K}_R)$ is torsionfree, then $I \cong R$ or K_R as an R-module.

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Contents

Notation

In what follows, unless other specified, we assume

1. $(R, \mathfrak{m})$ a C–M local ring, $\dim R = 1$
2. $\exists$ a canonical module K_R of R
3. $M^\vee = \mathrm{Hom}_R(M, \mathrm{K}_R)$ for each R-module M
4. $\mu_R(M) = \ell_R(M/\mathfrak{m}M)$ for each R-module M

Main results

Theorem 2.1 (Main Theorem)

Let I be a faithful ideal of R.

(1) Assume that the canonical map

$$t : I \otimes_R I^\vee \to \mathrm{K}_R, \ x \otimes f \mapsto f(x)$$

is an isomorphism. If $r, s \geq 2$, then $\mathrm{e}(R) > (r+1)s \geq 6$, where $r = \mu_R(I)$ and $s = \mu_R(I^\vee)$.

(2) Suppose that $I \otimes_R I^\vee$ is torsionfree. If $\mathrm{e}(R) \leq 6$, then $I \cong R$ or K_R.

Corollary 2.2

Let R be a C–M local ring with $\dim R \geq 1$. Assume that $R_\mathfrak{p}$ is Gorenstein and $\mathrm{e}(R_\mathfrak{p}) \leq 6$ for every height one prime $\mathfrak{p}$. Let I be a faithful ideal of R. If $I \otimes_R \mathrm{Hom}_R(I, R)$ is reflexive, then I is principal.

Corollary 2.3

Let R be a Gorenstein local ring with $\dim R = 1$ and $\mathrm{e}(R) \leq 6$. Let I be a faithful ideal of R. If $I \otimes_R \mathrm{Hom}_R(I, R)$ is torsionfree, then I is principal.

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Theorem 2.4

Let $(R, \mathfrak{m})$ be a C–M local ring with $\dim R = 1$ and assume that $\mathfrak{m}\overline{R} \subseteq R$. Let I be a faithful fractional ideal of R. If $I \otimes_R I^\vee$ is torsionfree, then $I \cong R$ or K_R.

Theorem 2.5

Let R be a C–M local ring with $\dim R = 1$. Assume $\exists\, \mathrm{K}_R$ and $\mathrm{v}(R) = \mathrm{e}(R)$. Let I be a faithful ideal of R. If $I \otimes_R I^\vee \cong \mathrm{K}_R$, then $I \cong R$ or K_R.

Let k be a field.

Proposition 2.6

Let $R = k[[t^a, t^{a+1}, \ldots, t^{2a-1}]]$ $(a \geq 1)$ be the semigroup ring and let $I \neq (0)$ be an ideal of R. If $I \otimes_R I^\vee$ is torsionfree, then $I \cong R$ or K_R.

Corollary 2.7

Let $R = k[[t^a, t^{a+1}, \ldots, t^{2a-2}]]$ $(a \geq 3)$ be the semigroup ring and let I be an ideal of R. If $I \otimes_R \mathrm{Hom}_R(I, R)$ is torsionfree, then I is principal.

Numerical semigroup rings

Setting 3.1

Let $0 < a_1 < a_2 < \cdots < a_\ell \in \mathbb{Z}$ such that $\gcd(a_1, a_2, \dots, a_\ell) = 1$.
We set $H = \langle a_1, a_2, \dots, a_\ell \rangle = \{\sum_{i=1}^{\ell} c_i a_i \mid 0 \le c_i \in \mathbb{Z}\}$ and

$$R = k[[t^{a_1}, t^{a_2}, \dots, t^{a_\ell}]] \quad \subseteq \quad V = k[[t]].$$

Notice that $\mathrm{e}(R) = a_1 = \mu_R(V)$.

Definition 3.2

Let I be a fractional ideal of R. Then I is said to be a monomial ideal, if $I = \sum_{n \in \Lambda} Rt^n$ for some $\Lambda \subseteq \mathbb{Z}$.

Theorem 3.3

Let R be a numerical semigroup ring with $\mathrm{e}(R) \leq 7$. Let I be a monomial ideal of R. If $I \otimes_R I^\vee$ is torsionfree, then $I \cong R$ or K_R.

Corollary 3.4

Let R be a Gorenstein numerical semigroup ring with $\mathrm{e}(R) \leq 7$ and let I be a monomial ideal of R. If $I \otimes_R \mathrm{Hom}_R(I, R)$ is torsionfree, then I is principal.

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Examples

Let I be a monomial ideal of R. Set $J = \mathrm{K}_R : I$.
Condition: $IJ = \mathrm{K}_R$ and $\mu_R(\mathrm{K}_R) = 4$.

Example 4.1

Let $R = k[[t^8, t^{11}, t^{14}, t^{15}]]$. Then $\mathrm{K}_R = (1, t, t^3, t^4)$. We take $I = (1, t)$ and set $J = \mathrm{K}_R : I$. Then $J = (1, t^3)$, $IJ = \mathrm{K}_R$, $\mu_R(\mathrm{K}_R) = 4$, but

$$\mathrm{T}(I \otimes_R J) = R(t \otimes t^{16} - 1 \otimes t^{17}) \cong R/\mathfrak{m}.$$

Remark 4.2

In the ring R of Example 4.1 $\nexists$ monomial ideals I such that $I \not\cong R, I \not\cong \mathrm{K}_R$, and $I \otimes_R I^\vee$ is torsionfree.

If $\mathrm{e}(R) \geq 9$, then **Conjecture 1.3 is not true** in general.

Example 4.3

Let $R = k[[t^9, t^{10}, t^{11}, t^{12}, t^{15}]]$. Then $\mathrm{K}_R = (1, t, t^3, t^4)$. Let $I = (1, t)$ and put $J = \mathrm{K}_R : I$. Then

$$J = (1, t^3), \quad IJ = \mathrm{K}_R, \text{ and } \mu_R(\mathrm{K}_R) = 4,$$

but $I \otimes_R I^\vee$ is torsionfree.

Thank you very much for your attention!

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