On weakly Arf rings

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based on the works jointly with

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1 Introduction

In 1971, J. Lipman proved:

For a one-dimensional complete Noetherian local domain A with an algebraically closed residue field of characteristic 0, if A is saturated, then A has minimal multiplicity.

The proof based on the fact that

if A is saturated, then A is **an Arf ring**.

Definition 1.1 (Lipman)

Let A be a CM semi-local ring with $\dim A = 1$. Then A is called *an Arf ring*, if the following hold:

(1) Every integrally closed *open* ideal has a principal reduction.

(2) If $x, y, z \in A$ s.t.

$$x \text{ is a NZD on } A \text{ and } \frac{y}{x}, \ \frac{z}{x} \in \overline{A},$$

then $yz/x \in A$.

Question 1.2

What happens if we remove the condition (1)?

Definition 1.3

A commutative ring A is said to be *weakly Arf*, provided

$$yz/x \in A, \text{ whenever } x, y, z \in A \text{ s.t. } x \in A \text{ is a NZD, } y/x, z/x \in \overline{A}.$$

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2 Basic properties

Throughout this talk

- A a Noetherian ring
- $W(A)$ the set of NZDs on A
- $\mathcal{F}_A$ the set of ideals in A which contain a NZD on A.

For $I \in \mathcal{F}_A$, there is a filtration:

$$A \subseteq I : I \subseteq I^2 : I^2 \subseteq \cdots \subseteq I^n : I^n \subseteq \cdots \subseteq \overline{A}.$$

Define

$$A^I = \bigcup_{n \geq 0} [I^n : I^n]$$

which is a module-finite extension over A and $A \subseteq A^I \subseteq \overline{A}$.

- If $a \in I$ is a reduction of I, i.e., $I^{r+1} = aI^r$ for $\exists\ r \geq 0$, then

$$A^I = A\left[\frac{I}{a}\right] = \frac{I^r}{a^r} \quad \text{where } \frac{I}{a} = \left\{\frac{x}{a} \;\middle|\; x \in I\right\} \subseteq \mathrm{Q}(A).$$

  Hence $A^I = I^n : I^n$ for $\forall n \geq r$.

- $\mathrm{red}_{(a)}(I) = \min\{r \geq 0 \mid I^{r+1} = aI^r\} = \min\{n \geq 0 \mid A^I = I^n : I^n\}$

- $I \in \mathcal{F}_A$ is *stable* in $A \iff A^I = I : I$

$$\iff I^2 = aI \text{ for } \exists\ a \in I.$$

Theorem 2.1 (Lipman)

Let A be a CM semi-local ring with $\dim A = 1$. Then TFAE.

(1) *A is an Arf ring.*

(2) *Every integrally closed ideal $I \in \mathcal{F}_A$ is stable.*

When A is a CM local ring with $\dim A = 1$,

if A is an Arf ring, then A has minimal multiplicity.

Set $\Lambda(A) = \{\overline{(x)} \mid x \in W(A)\}$.

Theorem 2.2

A is a weakly Arf ring if and only if every $I \in \Lambda(A)$ is stable.

Proposition 2.3

Let $\varphi : A \to B$ be a homomorphism of rings. Suppose $aB \cap A = aA$ and $\varphi(a) \in W(B)$ for $\forall a \in W(A)$. If B is weakly Arf, then so is A.

Corollary 2.4

(1) *Let B be an integral domain, $A \subseteq B$ a subring of B s.t. A is a direct summand of B. If B is a weakly Arf ring, then so is A.*

(2) *If $B = A[X_1, X_2, \dots, X_n]$ $(n > 0)$ is weakly Arf, then so is A.*

(3) *Let $\varphi : A \to B$ be the faithfully flat homomorphism of rings. If B is a weakly Arf ring, then so is A.*

Proposition 2.5

Let $(A, \mathfrak{m})$ be a Noetherian local ring with $\dim A = 1$. *Then A is a weakly Arf ring if and only if so is $\widehat{A}$.*

Let $R = \mathbb{C}[[t^4, t^5, t^6, s]] \subseteq \mathbb{C}[[t, s]]$. Choose a UFD A s.t. $R \cong \widehat{A}$. Then A is a weakly Arf ring. If $\widehat{A}$ is weakly Arf, then

$$S = \mathbb{C}[[t^4, t^5, t^6]] \to R \cong \widehat{A}$$

ensures that S is weakly Arf, whence S is Arf. This is impossible. Hence $\widehat{A}$ is not weakly Arf.

Theorem 2.6

Suppose that

- *A is an integral domain,*
- *A satisfies (S_2), and*
- *A contains an infinite field.*

Then A is weakly Arf if and only if so is $A[X_1, X_2, \ldots, X_n]$ for $\forall n \geq 1$.

Let $A = k[Y]/(Y^n)$ $(n \geq 1)$ and $B = A[X]$. Then A is weakly Arf and

$$B \text{ is a weakly Arf ring} \iff n \leq 2.$$

Theorem 2.7

Let R be a Noetherian ring, M a finitely generated torsion-free R-module. Then TFAE.

(1) $A = R \ltimes M$ *is a weakly Arf ring.*

(2) R *is a weakly Arf ring and* M *is an* $\overline{R}$*-module.*

Theorem 2.8

Let $(R, \mathfrak{m}), (S, \mathfrak{n})$ be Noetherian local rings with $k = R/\mathfrak{m} = S/\mathfrak{n}$. Suppose that depth $R > 0$ *and* depth $S > 0$*. Then TFAE.*

(1) $A = R \times_k S$ *is a weakly Arf ring.*

(2) R *and* S *are weakly Arf rings.*

3 Blow-ups

For $n \geq 0$, we set

$$A_n = \begin{cases} A & \text{if } n = 0 \\ A_{n-1}^{J(A_{n-1})} & \text{if } n \geq 1 \end{cases}$$

where $J(A_{n-1})$ stands for the Jacobson radical of A_{n-1}.

Theorem 3.1 (Lipman)

Let A be a CM semi-local ring with $\dim A = 1$. Then TFAE.

(1) *A is an Arf ring.*

(2) *$(A_n)_M$ has minimal multiplicity for $\forall n \geq 0$, $\forall M \in \operatorname{Max} A_n$.*

Recall $\Lambda(A) = \{\overline{(x)} \mid x \in W(A)\}$.

Define

- $\Gamma(A) = \{I \in \Lambda(A) \mid I \neq A\}$ and
- $\operatorname{Max} \Lambda(A)$ the set of all the maximal elements in $\Gamma(A)$ with respect to inclusion.

Then

- $A = \mathrm{Q}(A) \iff \operatorname{Max} \Lambda(A) = \emptyset$
- $A = \overline{A} \iff$ If $M \in \operatorname{Max} \Lambda(A)$, then $\mu_A(M) = 1$.

Hence, there exists $M \in \operatorname{Max} \Lambda(A)$ s.t. $\mu_A(M) \geq 2$, provided $A \neq \overline{A}$.

Definition 3.2

Define

$$\begin{aligned} A_0 &= A \\ A_1 &= \begin{cases} \overline{A} & \text{if } A = \overline{A} \\ A^M & \text{if } A \neq \overline{A},\ \exists M \in \operatorname{Max}\Lambda(A) \text{ s.t. } \mu_A(M) \geq 2. \end{cases} \\ A_n &= (A_{n-1})_1 \quad \text{for } \ n \geq 2. \end{aligned}$$

We then have a chain of rings

$$A = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots \subseteq \overline{A}.$$

Theorem 3.3

Consider the following conditions.

(1) *A is a weakly Arf ring.*

(2) *For $\forall M \in \operatorname{Max}\Lambda(A)$, $M : M$ is a weakly Arf ring and M is stable.*

(3) *For every chain $A = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots \subseteq \overline{A}$, and for $\forall n \geq 0$, A_n is a weakly Arf ring.*

(4) *For every chain $A = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots \subseteq \overline{A}$, and for $\forall n \geq 0$ and $\forall N \in \operatorname{Max}\Lambda(A_n)$, N is stable.*

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) hold. If $\dim A = 1$, or A is locally quasi-unmixed, (4) $\Rightarrow$ (1) holds.

For a Noetherian local ring R,

$$R \text{ is } \textit{quasi-unmixed} \overset{def}{\Longleftrightarrow} \dim \widehat{R}/Q = \dim R \text{ for } \forall Q \in \operatorname{Min} \widehat{R}.$$

Let $0 < a_1, a_2, \ldots, a_\ell \in \mathbb{Z}$ $(\ell > 0)$ s.t. $\gcd(a_1, a_2, \ldots, a_\ell) = 1$. Set

- $H = \langle a_1, a_2, \ldots, a_\ell \rangle$
- $A = k[H] = k[t^{a_1}, t^{a_2}, \ldots, t^{a_\ell}] \subseteq S = k[t] = \overline{A}$
- $e = \min(H \setminus \{0\})$
- $A_+ = tS \cap A$.

Then

- $A_+ = \overline{(t^e)} \in \operatorname{Max}\Lambda(A)$, and $\mu_A(A_+) = 1 \iff e = 1$.
- For $\forall I \in \operatorname{Max}\Lambda(A)$, $I = A_+$, or $\mu_A(I) = 1$.

Therefore, if $A \neq \overline{A}$, i.e., $\mu_A(A_+) \geq 2$, then

$$A_1 = A^{A_+} = A\left[\frac{A_+}{t^e}\right] = k[t^e, t^{a_1 - e}, t^{a_2 - e}, \ldots, t^{a_\ell - e}].$$

Example 3.4

Let $\ell \geq 2$, $A = k[t^\ell + t^{\ell+1}] + t^{\ell+2}S$ in $S = k[t]$. Then

(1) A is a weakly Arf ring.

(2) Let $I = t^{\ell+2}S$. Then $I \in \operatorname{Max}\Lambda(A)$, $\mu_A(I) \geq 2$, and

$$A_1 = A^I = S.$$

(3) Let $a = t^\ell + t^{\ell+1}$ and $I = \overline{(a)}$. Then $I \in \operatorname{Max}\Lambda(A)$, $\mu_A(I) \geq 2$, and

$$A_1 = A^I = k[t^2, t^3].$$

4 Examples

Let k be a field and set $A = k[[X, Y]]/(XY(X+Y))$. Then

- A is a CM local reduced ring with $\dim A = 1$.
- $\mathfrak{m}$ does not have a principal reduction, if $k = \mathbb{Z}/(2)$.

Theorem 4.1

$$\{\textit{integrally closed } \mathfrak{m}\textit{-primary ideals}\} = \{\mathfrak{m}\} \cup \{\textit{stable ideals}\}$$

Recall $\Lambda(A) = \left\{ \overline{(x)} \;\middle|\; x \in W(A) \right\}$.

Hence, if $k = \mathbb{Z}/(2)$, then A is a weakly Arf ring, but not an Arf ring.

Corollary 4.2

Suppose that $k = \mathbb{Z}/(2)$. *Then*

- $A \ltimes M$, *where* M *is a finitely generated* $\overline{A}$*-module s.t.* M *is torison-free as an* A*-module*
- $A \times_{A/\mathfrak{m}} A \times_{A/\mathfrak{m}} \cdots \times_{A/\mathfrak{m}} A$

are weakly Arf rings, but not Arf.

In what follows, let k be a field and

$$A = k[[X, Y, Z]]/\mathrm{I}_2\begin{pmatrix} X & Y & Z \\ Y & Z & X \end{pmatrix}.$$

Then A is a CM local ring with $\dim A = 1$.

Theorem 4.3

(1) *If* $\operatorname{ch} k = 3$*, then* A *is not an Arf ring.*

(2) *If* $\operatorname{ch} k \neq 3$ *and there is* $\alpha \in k$ *s.t.* $\alpha \neq 1$, $\alpha^3 = 1$*, then* A *is an Arf ring.*

Corollary 4.4

Suppose that k *is an algebraically closed field. Then* A *is an Arf ring if and only if* $\operatorname{ch} k \neq 3$.

Thank you for your attention.