# Almost Gorenstein Rees algebras

based on the works jointly with

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# Introduction

## What is the Rees algebra?

- For a commutative ring $R$ and an ideal $I$ in $R$, set

$$\mathcal{R}(I) = R[It] = \sum_{n \geq 0} I^n t^n \subseteq R[t]$$

$$G(I) = \mathcal{R}(I)/I\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}.$$

## Example 1.1

Let $R = k[X_1, X_2, \ldots, X_d]$ $(d \geq 1)$ and $I = (X_1, X_2, \ldots, X_d)$. Then

$$\mathcal{R}(I) \cong k[X_1, X_2, \ldots, X_d, Y_1, Y_2, \ldots, Y_d]/\mathrm{I}_2 \begin{pmatrix} X_1 & X_2 & \cdots & X_d \\ Y_1 & Y_2 & \cdots & Y_d \end{pmatrix}$$

More generally, if

- $(R, \mathfrak{m})$ a CM local ring
- $Q = (a_1, a_2, \ldots, a_d)$ a parameter ideal in $R$

then

$$\mathcal{R}(Q) \cong R[Y_1, Y_2, \ldots, Y_d]/\mathrm{I}_2 \begin{pmatrix} a_1 & a_2 & \cdots & a_d \\ Y_1 & Y_2 & \cdots & Y_d \end{pmatrix}$$

is a CM ring, where $d = \dim R$.

# Preceding results

**Theorem 1.2 (Goto-Shimoda)**

*Let $(R, \mathfrak{m})$ be a CM local ring with $d = \dim R \geq 1$, $\sqrt{I} = \mathfrak{m}$. Then*

$$\mathcal{R}(I) \text{ is a CM ring} \iff G(I) \text{ is a CM ring}, \ \mathrm{a}(G(I)) < 0$$

where

$$\mathrm{a}(G(I)) = \sup\{n \in \mathbb{Z} \mid \left[\mathrm{H}^d_M(G(I))\right]_n \neq (0)\}, \ M = \mathfrak{m}\mathcal{R}(I) + \mathcal{R}(I)_+.$$

**Example 1.3**

Let $(R, \mathfrak{m})$ be a RLR with $\dim R = 2$, $I$ an ideal of $R$ s.t. $I = \overline{I}$ and $\sqrt{I} = \mathfrak{m}$. Then $\mathcal{R}(I)$ is a CM ring.

**Theorem 1.4 (Goto-Nishida, Goto-Shimoda, Ikeda)**

*Let* $(R, \mathfrak{m})$ *be a CM local ring with* $d = \dim R \geq 2$, $\sqrt{I} = \mathfrak{m}$. *Then*

$$\mathcal{R}(I) \text{ is Gorenstein} \iff G(I) \text{ is Gorenstein}, \ \mathrm{a}(G(I)) = -2.$$

*When this is the case,* $R$ *is a Gorenstein ring.*

Thus, if $R$ is a CM local ring with $\dim R \geq 2$, $Q$ is a parameter ideal, then

$$\mathcal{R}(Q) \text{ is Gorenstein} \iff R \text{ is Gorenstein}, \ \dim R = 2.$$

Moreover, if $(R, \mathfrak{m})$ is a RLR with $\dim R = 2$ and $I = \mathfrak{m}^\ell$ $(\ell \geq 1)$, then

$$\mathcal{R}(I) \text{ is Gorenstein} \iff I = \mathfrak{m}.$$

## Question 1.5

**When is the Rees algebra $\mathcal{R}(I)$ almost Gorenstein?**

- $I$ is the ideal generated by a (sub) system of parameters
- $I = \overline{I}$ in a two-dimensional RLR

# What is an almost Gorenstein ring?

- In 1997, Barucci and Fröberg defined the notion of almost Gorenstein rings for one-dimensional analytically unramified local rings.
- In 2013, Goto, Matsuoka, and Phuong generalized the notion to arbitrary one-dimensional CM local rings.
- In 2015, Goto, Takahashi, and Taniguchi gave the notion of almost Gorenstein local/graded rings of arbitrary dimension.

# Survey on AG rings

- $(R, \mathfrak{m})$ a CM local ring with $d = \dim R$, $|R/\mathfrak{m}| = \infty$
- $\exists$ $\mathrm{K}_R$ the canonical module of $R$.

## Definition 2.1

We say that $R$ is an *almost Gorenstein local ring* (abbr. AGL ring), if $\exists$ an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of $R$-modules s.t. $\mu_R(C) = \mathrm{e}(C)$

where

$$\mathrm{e}(C) = \lim_{n \to \infty} (d-1)! \cdot \frac{\ell_R(C/\mathfrak{m}^{n+1}C)}{n^{d-1}}.$$

If $C \neq (0)$, then $C$ is CM and $\dim_R C = d - 1$. Besides

$$\mu_R(C) = \mathrm{e}(C) \iff \mathfrak{m}C = (f_2, f_3, \dots, f_d)C$$

for $\exists\ f_2, f_3, \dots, f_d \in \mathfrak{m}$. Hence $C$ is *an Ulrich R-module*.

**Example 2.2**

- $k[[t^3, t^4, t^5]]$.
- $k[[X, Y, Z]]/(X, Y) \cap (Y, Z) \cap (Z, X)$.
- $k[[t^3, t^4, t^5]] \ltimes (t^3, t^4, t^5)$.
- $k[[t^3, t^4, t^5]] \times_k k[[t^3, t^4, t^5]]$.
- 1-dimensional finite CM–representation type.
- 2-dimensional rational singularity.

- $R = \bigoplus_{n \geq 0} R_n$ a CM graded ring, $d = \dim R$, $\exists$ $\mathrm{K}_R$
- $(R_0, \mathfrak{m})$ a local ring, $|R_0/\mathfrak{m}| = \infty$

**Definition 2.3**

We say that $R$ is an *almost Gorenstein graded ring* (abbr. AGG ring), if

$$\exists \quad 0 \to R \to \mathrm{K}_R(-a) \to C \to 0$$

of graded $R$-modules s.t. $\mu_R(C) = \mathrm{e}(C)$

where $a = \mathrm{a}(R)$, $M = \mathfrak{m}R + R_+$.

- $R$ is an AGG ring $\implies$ $R_M$ is an AGL ring.
- The converse is not true in general.

## Example 2.4

Let $S = k[X_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\ ]$ $(2 \leq m \leq n)$ and set

$$R = S/\mathrm{I}_t(X)$$

where $2 \leq t \leq m$, $X = [X_{ij}]$. Then

$$R \text{ is an AGG ring} \iff m = n, \text{ or } m \neq n \text{ and } t = m = 2.$$

## Example 2.5

Let $R = k[X_1, X_2, \ldots, X_d]$ $(d \geq 1)$ and $1 \leq n \in \mathbb{Z}$. Then

- If $d \leq 2$, then $R^{(n)} = k[R_n]$ is an AGG ring.
- If $d \geq 3$, then

$$R^{(n)} \text{ is an AGG ring} \iff n \mid d, \text{ or } d = 3 \text{ and } n = 2.$$

# Main results (parameter ideals)

Let

- $(R, \mathfrak{m})$ a CM local ring with $d = \dim R \geq 3$
- $a_1, a_2, \ldots, a_r \in \mathfrak{m}$ a subsystem of parameters in $R$ $(r \geq 3)$
- $Q = (a_1, a_2, \ldots, a_r)$
- $\mathcal{R} = \mathcal{R}(Q) = R[Qt] \subseteq R[t]$, $M = \mathfrak{m}\mathcal{R} + \mathcal{R}_+$

Then

- $\mathcal{R} \cong R[X_1, X_2, \ldots, X_r]/\mathrm{I}_2\begin{pmatrix} X_1 & X_2 & \cdots & X_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}$ is a CM ring.
- $\dim \mathcal{R} = d + 1$ and $\mathrm{a}(\mathcal{R}) = -1$.

## Theorem 3.1

- *$\mathcal{R}$ is an AGG ring $\iff$ $R$ is a RLR and $a_1, a_2, \ldots, a_r$ is a regular system of parameters in $R$*
- *$\mathcal{R}_M$ is an AGL ring $\iff$ $R$ is a RLR*

## Key for the proof

- The Eagon-Northcott complex
- Proposition 3.2

## Proposition 3.2

*Let $(B, \mathfrak{n})$ be a Gorenstein local ring, $I$ an ideal of $B$. Suppose that $A = B/I$ is a non-Gorenstein AGL ring. If $\operatorname{pd}_B A < \infty$, then $B$ is a RLR.*

Proof. May assume $|B/\mathfrak{n}| = \infty$. Choose an exact sequence

$$0 \to A \to \mathrm{K}_A \to C \to 0$$

s.t. $C$ is an Ulrich $A$-module. Then $\operatorname{pd}_B C < \infty$. Take an $A$-regular sequence $f_1, f_2, \ldots, f_{d-1} \in \mathfrak{n}$ s.t.

$$\mathfrak{n}C = (f_1, f_2, \ldots, f_{d-1})C$$

where $d = \dim A$. Set $\mathfrak{q} = (f_1, f_2, \ldots, f_{d-1})$. Since $f_1, f_2, \ldots, f_{d-1}$ is a regular sequence on $C$, $\operatorname{pd}_B C/\mathfrak{q}C < \infty$. Hence $B$ is a RLR, because $C/\mathfrak{q}C = C/\mathfrak{n}C$ is a vector space over $B/\mathfrak{n}$. $\square$

# Main results (integrally closed ideals)

Let

- $(R, \mathfrak{m})$ be a Gorenstein local ring with $\dim R = 2$
- $I$ an $\mathfrak{m}$-primary ideal in $R$
- $I$ contains a parameter ideal $Q$ s.t. $I^2 = QI$
- $J = Q : I$

## Proposition 3.3

*Suppose that $\exists\ f \in \mathfrak{m}$, $g \in I$, and $h \in J$ s.t.*

$$IJ = gJ + Ih \quad \text{and} \quad \mathfrak{m}J = fJ + \mathfrak{m}h.$$

*Then $\mathcal{R}(I)$ is an AGG ring.*

## Theorem 3.4

*Let* $(R, \mathfrak{m})$ *be a two-dimensional RLR with* $|R/\mathfrak{m}| = \infty$, *and* $I = \overline{I}$. *Then* $\mathcal{R}(I)$ *is an AGG ring.*

## Corollary 3.5

*Let* $(R, \mathfrak{m})$ *be a two-dimensional RLR with* $|R/\mathfrak{m}| = \infty$. *Then* $\mathcal{R}(\mathfrak{m}^\ell)$ *is an AGG ring for* $\forall \ell > 0$.

# Thank you for your attention.