Ulrich ideals in 2-almost Gorenstein rings

based on the work jointly with

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Special Session on Commutative Algebra and its Environs

March 24, 2019

Introduction

What is an Ulrich ideal?

- In 1987, Brennan, Herzog, and Ulrich introduced **M**aximally **G**enerated **M**aximal **C**ohen-**M**acaulay modules.
- In 2014, Goto, Ozeki, Takahashi, Watanabe, and Yoshida generalized the notion of MGMCM module, which they call **Ulrich module and Ulrich ideal**.

Preceding results

- (Goto-Ozeki-Takahashi-Watanabe-Yoshida)

  Determined all the Ulrich ideals of Gorenstein local rings of finite CM-representation type and of dimension at most 2.

- (Goto-Isobe-Kumashiro)

  Studied the relation between Ulrich ideals and birational finite extensions of R, where R is a CM local ring with $\dim R = 1$.

- (Goto-Takahashi-T)

  Studied $\mathbf{R}\mathrm{Hom}_R(R/I, R)$ for Ulrich ideals I in a CM local ring R.

Theorem 1.1 (Goto-Takahashi-T)

Let $(R, \mathfrak{m})$ *be a non-Gorenstein almost Gorenstein ring with* $\dim R = 1$*. Then*

$$\mathcal{X}_R \subseteq \{\mathfrak{m}\}$$

where $\mathcal{X}_R$ *denotes the set of Ulrich ideals in* R*.*

What is an almost Gorenstein ring?

- In 1997, Barucci and Fröberg defined the notion of almost Gorenstein ring for one-dimensional analytically unramified local rings.
- In 2013, Goto, Matsuoka, and Phuong generalized the notion to arbitrary one-dimensional CM local rings.
- In 2015, Goto, Takahashi, and Taniguchi gave the notion of almost Gorenstein local/graded rings of higher dimension.
- In 2019, Chau, Goto, Kumashiro, and Matsuoka defined the notion of 2-**almost Gorenstein rings**.

Question 1.2

How many Ulrich ideals are contained in a given 2-almost Gorenstein ring?

Survey on 2-AGL rings

Setting 2.1

- $(R, \mathfrak{m})$ a CM local ring with $\dim R = 1$
- $\exists\ I \subsetneq R$ an ideal of R s.t. $I \cong \mathrm{K}_R$

Hence, $\exists\ \mathrm{e}_0(I) > 0,\ \mathrm{e}_1(I) \in \mathbb{Z}$ s.t.

$$\ell_R(R/I^{n+1}) = \mathrm{e}_0(I)\binom{n+1}{1} - \mathrm{e}_1(I) \quad \text{for} \quad \forall n \gg 0.$$

Definition 2.2 (Goto-Matsuoka-Phuong)

We say that R is an *almost Gorenstein local ring* (abbr. AGL ring), if $\mathrm{e}_1(I) \leq \mathrm{r}(R)$.

Suppose I contains a reduction $Q = (a)$, i.e. $I^{\ell+1} = QI^{\ell}$ for $\exists \ell \geq 0$.

Let

- $\mathcal{T} = \mathcal{R}(Q) = R[Qt] \subseteq \mathcal{R} = \mathcal{R}(I) = R[It] \subseteq R[t]$
- $\mathcal{S}_Q(I) = I\mathcal{R}/I\mathcal{T}$, $\mathfrak{p} = \mathfrak{m}\mathcal{T}$

and set

$$\operatorname{rank} \mathcal{S}_Q(I) := \ell_{\mathcal{T}_{\mathfrak{p}}}([\mathcal{S}_Q(I)]_{\mathfrak{p}}) = \mathrm{e}_1(I) - [\mathrm{e}_0(I) - \ell_R(R/I)].$$

Then

- R is a Gorenstein ring $\iff$ $\operatorname{rank} \mathcal{S}_Q(I) = 0$.
- R is a non-Gorenstein AGL ring $\iff$ $\operatorname{rank} \mathcal{S}_Q(I) = 1$.

Definition 2.3 (Chau-Goto-Kumashiro-Matsuoka)

R is called a *2-almost Gorenstein local ring* (abbr. 2-AGL ring)

$\overset{def}{\Longleftrightarrow}$ $\operatorname{rank} \mathcal{S}_Q(I) = 2$.

Example 2.4

(1) $k[[t^3, t^7, t^8]]$

(2) $k[[t^3, t^7, t^8]] \times_k k[[t]]$

(3) $k[[t^3, t^7, t^8]] \ltimes k[[t]]$

Set $K = a^{-1}I = \left\{\frac{x}{a} \mid x \in I\right\} \subseteq \mathrm{Q}(R)$. Then

K is a fractional ideal of R s.t. $R \subseteq K \subseteq \overline{R}$ and $K \cong \mathrm{K}_R$.

Let $\mathfrak{c} = R : R[K]$. Then

- R is a Gorenstein ring $\iff$ $\mathfrak{c} = R$.
- R is a non-Gorenstein AGL ring $\iff$ $\mathfrak{c} = \mathfrak{m}$

Theorem 2.5 (Chau-Goto-Kumashiro-Matsuoka)

TFAE.

(1) R *is a* 2-AGL *ring.*

(2) $\ell_R(R/\mathfrak{c}) = 2$.

(3) $K^2 = K^3$ *and* $\ell_R(K^2/K) = 2$.

Ulrich ideals

- $(R, \mathfrak{m})$ be a CM local ring with $d = \dim R$.
- $\sqrt{I} = \mathfrak{m}$, I contains a parameter ideal Q of R as a reduction.

Definition 3.1 (Goto-Ozeki-Takahashi-Watanabe-Yoshida)

We say that I is an *Ulrich ideal of R*, if

(1) $I \supsetneq Q$, $I^2 = QI$, and

(2) I/I^2 is R/I-free.

Note that

- (1) $\iff$ $\mathrm{gr}_I(R)$ is a CM ring with $\mathrm{a}(\mathrm{gr}_I(R)) = 1 - d$.
- If $I = \mathfrak{m}$, then (1) $\iff$ R has minimal multiplicity $\mathrm{e}(R) > 1$.

Let I be an Ulrich ideal in R. Then $\mu_R(I) \geq d+1$.

Theorem 3.2 (Goto-Takahashi-T)

$$\mathrm{Ext}^i_R(R/I, R) \text{ is } R/I\text{-free for } \forall i \in \mathbb{Z}.$$

Hence

- R Gorenstein $\iff$ $\mu_R(I) = d+1$, R/I is Gorenstein
- $\mu_R(I) = d+1$ $\iff$ $\mathrm{Gdim}_R(R/I) < \infty$
- R G-regular $\implies$ $\mu_R(I) \geq d+2$

Corollary 3.3

Suppose that $\exists$ K_R and that $\exists$ an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0.$$

If $\mu_R(I) \geq d+2$, then $\mathrm{Ann}_R\, C \subseteq I$.

Main Results

Setting 4.1

- $(R, \mathfrak{m})$ a CM local ring with $\dim R = 1$
- $R \subseteq \exists K \subseteq \overline{R}$ an R-submodule of $\overline{R}$ s.t. $K \cong \mathrm{K}_R$
- $S = R[K]$, $\mathfrak{c} = R : S$
- $\mathcal{X}_R$ the set of Ulrich ideals in R

Recall that

$$R \text{ is a 2-AGL ring} \iff K^2 = K^3 \text{ and } \ell_R(K^2/K) = 2$$
$$\iff \ell_R(R/\mathfrak{c}) = 2$$

Suppose that R **is a** 2-AGL **ring**. Then

- $\mathfrak{c} = R : S = R : K$.
- $\exists$ a minimal system $x_1, x_2, \dots, x_n$ of generators of $\mathfrak{m}$ s.t.

$$\mathfrak{c} = (x_1^2) + (x_2, x_3, \dots, x_n).$$

- $K/R \cong (R/\mathfrak{c})^{\oplus \ell} \oplus (R/\mathfrak{m})^{\oplus m}$ for $\exists\ \ell > 0$, $\exists\ m \geq 0$ s.t. $\ell + m = \mathrm{r}(R) - 1$.

Theorem 4.2

Suppose that R is a 2-AGL *ring with minimal multiplicity. Then*

$$\mathcal{X}_R = \begin{cases} \{\mathfrak{c}, \mathfrak{m}\} & \text{if } K/R \text{ is } R/\mathfrak{c}\text{-free}, \\ \{\mathfrak{m}\} & \text{otherwise.} \end{cases}$$

Theorem 4.3

Suppose that R is a 2-AGL ring and K/R is not $R/\mathfrak{c}$-free. Let M be a finitely generated R-module. If

$$\operatorname{Ext}_R^p(M, R) = (0) \quad \textit{for} \quad \forall p \gg 0,$$

then $\operatorname{pd}_R M < \infty$. *Hence, R is G-regular.*

Example 4.4

Let $R = k[[t^6, t^8, t^{10}, t^{11}]] \subseteq V = k[[t]]$ (k is a field).

(1) R is a 2-AGL ring, $\mathfrak{c} = (t^6, t^8, t^{10}) \in \mathcal{X}_R$.

(2) Let $I \in \mathcal{X}_R$. Then, $\mu_R(I) = 2, 3$, and $\mu_R(I) = 3 \Leftrightarrow I = \mathfrak{c}$.

(3) If ch $k \neq 2$, then the set of two-generated Ulrich ideals is

$$\left\{(t^6 + c_1 t^8 + c_2 t^{10}, t^{11}) \mid c_1, c_2 \in k\right\}$$
$$\cup \left\{(t^8 + c_1 t^{10} + c_2 t^{12}, t^{11}) \mid c_1, c_2 \in k\right\}.$$

(4) If ch $k = 2$, then the set of two-generated Ulrich ideals is

$$\left\{(t^6 + c_1 t^8 + c_2 t^{10}, t^{11}) \mid c_1, c_2 \in k\right\}$$
$$\cup \left\{(t^8 + c_1 t^{10} + c_2 t^{12}, t^{11} + dt^{12}) \mid c_1, c_2, d \in k\right\}$$
$$\cup \left\{(t^6 + c_1 t^8 + c_2 t^{11}, t^{10} + dt^{11}) \mid c_1, c_2, d \in k, d \neq 0\right\}.$$

In what follows, let

- $0 < a_1, a_2, \ldots, a_\ell \in \mathbb{Z}$ $(\ell > 0)$ s.t. $\gcd(a_1, a_2, \ldots, a_\ell) = 1$
- $H_1 = \langle a_1, a_2, \ldots, a_\ell \rangle$
- $0 < \alpha \in H_1$ an odd integer s.t. $\alpha \neq a_i$ for $1 \leq \forall i \leq \ell$
- $H = \langle 2a_1, 2a_2, \ldots, 2a_\ell, \alpha \rangle$
- $R_1 = k[[H_1]]$, $R = k[[H]] \subseteq V = k[[t]]$ (k a field)
- $\mathfrak{m}_1$ (resp. $\mathfrak{m}$) the maximal ideal of R_1 (resp. R)

Note that $\mu_R(\mathfrak{m}) = \ell + 1$ and R is a free R_1-module of rank 2.

Theorem 4.5

Suppose that R_1 is a non-Gorenstein AGL ring. Then

(1) *R is a 2-AGL ring, $\mathfrak{c} = \mathfrak{m}_1 R$, and $\mu_R(\mathfrak{c}) = \ell \geq 3$.*

(2) $\mathfrak{c} \in \mathcal{X}_R \iff R_1$ *has minimal multiplicity.*

(3) *R doesn't have minimal multiplicity. Therefore, $\mathfrak{m} \notin \mathcal{X}_R$.*

(4) *Let $I \in \mathcal{X}_R$. Then $\mu_R(I) = 2$ or $I = \mathfrak{c}$.*

(5) *The set of two-generated monomial Ulrich ideals is*

$$\{(t^{2m}, t^{\alpha}) \mid 0 < m \in H_1,\ \alpha - m \in H_1,\ 2(\alpha - 2m) \in H\}.$$

Thank you for your attention.