AGL rings arising as fiber products

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1 Introduction

The fiber product

$$A = R \times_T S = \{(a, b) \in R \times S \mid f(a) = g(b)\}$$

is the subring of $R \times S$, where

$$R \xrightarrow{f} T \text{ and } T \xleftarrow{g} S$$

are homomorphisms of rings. Hence we have the exact sequence

$$0 \longrightarrow A \xrightarrow{\iota} R \times S \xrightarrow{\begin{bmatrix} f \\ -g \end{bmatrix}} T$$

of A-modules.

Question 1.1

When is $R \times_T S$ an AGL ring?

Preceding results

- Ogoma ([7])
  the Gorensteinness of fiber product $A = R \times_T S$, where R is a CM local ring, S is a equi-dimensional Noetherian local ring with (S_1)
- D'Anna, Shapiro, Ananthnarayan-Avramov-Moore ([3, 8, 1])
  the Gorensteinness of fiber product $A = R \times_{R/I} R$, where R is a Noetherian local ring
- Nasseh-Sather-Wagstaff-Takahashi-VandeBogert ([4])
  the CM fiber products of finite CM type

Example 1.2

Let $R = k[[X, Y]]/(X^a - Y^b)$, $S = k[[Z, W]]/(Z^c - W^d)$ with $a, b, c, d \geq 2$.
Then

$$A = R \times_k S \cong k[[X, Y, Z, W]]/\left[(X, Y)\cdot(Z, W) + (X^a - Y^b, Z^c - W^d)\right]$$

is a CM local ring with $\underline{\mathrm{r}(A) = 3}$.

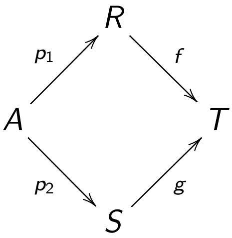
How about the AGL property?

2 Basic facts

For homomorphisms $f : R \to T$, $g : S \to T$, we consider

$$A = R \times_T S = \{(a, b) \in R \times S \mid f(a) = g(b)\} \subseteq B = R \times S.$$

Then



where $p_1 : A \to R, (x, y) \mapsto x$, $p_2 : A \to S, (x, y) \mapsto y$.

Lemma 2.1

Suppose f and g are surjective.

(1) *A is a Noetherian ring* $\iff$ *R, S are Noetherian rings*

(2) *(A, J) is a local ring* $\iff$ *$(R, \mathfrak{m}), (S, \mathfrak{n})$ are local rings*

When this is the case, $J = (\mathfrak{m} \times \mathfrak{n}) \cap A$.

(3) *$(R, \mathfrak{m}), (S, \mathfrak{n})$ are CM,* $\dim R = \dim S = d > 0$, depth $T \geq d - 1$ $\implies$ *(A, J) is CM and* $\dim A = d$.

Proof.

Consider

$$0 \longrightarrow A \xrightarrow{\iota} B = R \times S \xrightarrow{\varphi} T \longrightarrow 0$$

where $\varphi = \left[\begin{smallmatrix} f \\ -g \end{smallmatrix}\right]$. □

Let $(R, \mathfrak{m})$, $(S, \mathfrak{n})$ be Noetherian local rings, $k = R/\mathfrak{m} = S/\mathfrak{n}$, and $f : R \to k$, $g : S \to k$ the canonical maps.

Proposition 2.2

(1) $v(A) = v(R) + v(S)$.

(2) $\dim R = \dim S > 0 \implies \mathrm{e}(A) = \mathrm{e}(R) + \mathrm{e}(S)$.

(3) *If* R, S *are CM and* $\dim R = \dim S = 1$,

$$A = R \times_k S \text{ is Gorenstein } \iff R \text{ and } S \text{ are DVRs.}$$

Proof.

$J^{\ell+1} = \mathfrak{m}^{\ell+1} \times \mathfrak{n}^{\ell+1}$ $(\forall \ell \geq 0)$, since $J = \mathfrak{m} \times \mathfrak{n}$.

(1) $\ell_A(J/J^2) = \ell_k([\mathfrak{m}/\mathfrak{m}^2] \oplus [\mathfrak{n}/\mathfrak{n}^2]) = \ell_R(\mathfrak{m}/\mathfrak{m}^2) + \ell_S(\mathfrak{n}/\mathfrak{n}^2)$.

(2)
$$\begin{aligned}\ell_A(A/J^{\ell+1}) &= \ell_A(A/J) + \ell_A(J/J^{\ell+1}) \\ &= 1 + \left[\ell_R(\mathfrak{m}/\mathfrak{m}^{\ell+1}) + \ell_S(\mathfrak{n}/\mathfrak{n}^{\ell+1})\right] \\ &= 1 + \left\{[\ell_R(R/\mathfrak{m}^{\ell+1}) - 1] + [\ell_S(S/\mathfrak{n}^{\ell+1}) - 1]\right\} \\ &= \left[\ell_R(R/\mathfrak{m}^{\ell+1}) + \ell_S(S/\mathfrak{n}^{\ell+1})\right] - 1\end{aligned}$$

(3) $(\Rightarrow)$ By $0 \to A \xrightarrow{\iota} B \xrightarrow{\varphi} k \to 0$,

$$0 \to A : B \xrightarrow{\iota} A \to \mathrm{Ext}_A^1(A/J, A) \to 0.$$

Hence, $J = A : B$. Thus, because A is Gorenstein and $A : J = J : J$,

$$R \times S = B = A : (A : B) = A : J = J : J = (\mathfrak{m} : \mathfrak{m}) \times (\mathfrak{n} : \mathfrak{n}).$$

Therefore, $R = \mathfrak{m} : \mathfrak{m}$ and $S = \mathfrak{n} : \mathfrak{n}$, whence R, S are DVRs. □

3. AGL rings

Suppose $(R, \mathfrak{m})$ a CM local ring, $d = \dim R$, $\sharp(R/\mathfrak{m}) = \infty$, $\exists \, \mathrm{K}_R$.

Definition 3.1 (Goto-Takahashi-T)

We say that R is *an almost Gorenstein local ring*, if $\exists$ an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$.

We have

- R is a Gorenstein ring $\Rightarrow$ R is an AGL ring.
- $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C) \Leftrightarrow \mathfrak{m}C = (f_1, f_2, \ldots, f_{d-1})C$, for some $f_1, f_2, \ldots, f_{d-1} \in \mathfrak{m}$

Suppose $\dim R = 1$ and $R \subseteq \exists K \subseteq \overline{R}$ s.t. $K \cong \mathrm{K}_R$. Then

Remark 3.2 (Goto-Matsuoka-Phuong, Goto-Takahashi-T, Kobayashi)

$$R \text{ is an AGL ring} \Leftrightarrow \mathfrak{m}K \subseteq R \Leftrightarrow \mathfrak{m}K = \mathfrak{m} \Leftrightarrow \mathfrak{m}K \cong \mathfrak{m}.$$

Example 3.3

(1) $k[[t^e, t^{e+1}, \ldots, t^{2e-3}, t^{2e-1}]]$ $(e \geq 4)$

(2) $k[[X, Y, Z]]/(X, Y) \cap (Y, Z) \cap (Z, X)$

(3) $k[[t^4, t^5, t^6]] \ltimes (t^4, t^5, t^6)$

(4) 1-dimensional CM rings of finite CM-representation type

(5) 2-dimensional rational singularity

(6) $k[[X_1, X_2, \ldots, X_n, Y_1, Y_2, , \ldots, Y_n]]/\mathrm{I}_2\begin{pmatrix} X_1 & X_2 & \cdots & X_n \\ Y_1 & Y_2 & \cdots & Y_n \end{pmatrix}$ $(n \geq 2)$

4 Results in dimension one

Setting 4.1

- $(R, \mathfrak{m})$, $(S, \mathfrak{n})$ CM local rings, $\dim R = \dim S = 1$
- $k = R/\mathfrak{m} = S/\mathfrak{n}$, $f : R \to k$, $g : S \to k$ canonical maps
- $A = R \times_k S \subseteq B = R \times S$, $J = \mathfrak{m} \times \mathfrak{n}$ (the maximal ideal of A)

Then

- $\mathrm{Q}(A) = \mathrm{Q}(B) = \mathrm{Q}(R) \times \mathrm{Q}(S)$
- $\overline{A} = \overline{B} = \overline{R} \times \overline{S}$

We assume that $\mathrm{Q}(A) = \mathrm{Q}(R) \times \mathrm{Q}(S)$ is a Gorenstein ring, $\exists \mathrm{K}_A$, and $\sharp k = \infty$. Hence, all the rings A, R, and S possess fractional canonical ideals.

Theorem 4.2

TFAE.

(1) $A = R \times_k S$ *is an* AGL *ring.*

(2) $A = R \times_k S$ *is a* GGL *ring.*

(3) R *and* S *are* AGL *rings.*

Preliminaries for the proof of Theorem 4.2

We have $R \subseteq K \subseteq \overline{R}$, $K \cong \mathrm{K}_R$, and $S \subseteq L \subseteq \overline{S}$, $L \cong \mathrm{K}_S$.

Firstly, suppose R and S are not DVRs. Then $K : \mathfrak{m} \subseteq \overline{R}$, $L : \mathfrak{n} \subseteq \overline{S}$.

Hence, because $R : \mathfrak{m} \not\subseteq K$ and $S : \mathfrak{n} \not\subseteq L$, we have

$$K : \mathfrak{m} = K + R{\cdot}g_1, \quad L : \mathfrak{n} = L + S{\cdot}g_2$$

for some $g_1 \in (R : \mathfrak{m}) \setminus K$ and $g_2 \in (S : \mathfrak{n}) \setminus L$. We set

$$X = (K \times L) + A{\cdot}g$$

with $g = (g_1, g_2) \in \overline{A}$. Then we have

Lemma 4.3

$$A \subseteq X \subseteq \overline{A} \quad \textit{and} \quad X \cong \mathrm{K}_A.$$

Theorem 4.4

Suppose R and S are not DVRs. TFAE.

(1) $A = R \times_k S$ *is an* AGL *ring.*

(2) R *and* S *are* AGL *rings.*

Proof.

Note A is AGL $\Leftrightarrow JX = J \ (= \mathfrak{m} \times \mathfrak{n})$, while

$$\begin{aligned} JX &= (\mathfrak{m} \times \mathfrak{n}) \cdot [(K \times L) + A{\cdot}g] \\ &= (\mathfrak{m}K + \mathfrak{m}{\cdot}g_1) \times (\mathfrak{n}L + \mathfrak{n}{\cdot}g_2) \\ &= \mathfrak{m}(K + R{\cdot}g_1) \times \mathfrak{n}(L + S{\cdot}g_2) \\ &= \mathfrak{m}{\cdot}(K : \mathfrak{m}) \times \mathfrak{n}{\cdot}(L : \mathfrak{n}) = \mathfrak{m}K \times \mathfrak{n}L. \end{aligned}$$

Therefore

$$A \text{ is AGL } \Leftrightarrow \ \mathfrak{m}K = \mathfrak{m}, \mathfrak{n}L = \mathfrak{n} \ \Leftrightarrow \ R, S \text{ are AGL.}$$

□

Proof of (1) $\Leftrightarrow$ (3) in Theorem 4.2

Assume R **is a DVR** but S is not. Choose X so that $A \subseteq X \subseteq \overline{A}$ and $X \cong \mathrm{K}_A$. Then $\mathrm{K}_B = X : B \cong R \times L$. Therefore

$$X : B = \xi \cdot (R \times L)$$

for some $\xi = (\xi_1, \xi_2) \in \mathrm{Q}(A)$.

On the other hand, by $0 \to A \xrightarrow{\iota} B \xrightarrow{\varphi} k = A/J \to 0$, we get

$$0 \longrightarrow X : B \longrightarrow X \longrightarrow A/J \longrightarrow 0.$$

Hence $JX \subseteq X : B \subseteq X$. Thus

Lemma 4.5

$$\begin{aligned} X : B \ \subseteq \ X \ &\subseteq \ (X : B) : J = (\xi_1 R \times \xi_2 L) : J \\ &= \ \xi_1 \cdot (R : \mathfrak{m}) \times \xi_2 \cdot (L : \mathfrak{n}). \end{aligned}$$

Proof of (1) $\Leftrightarrow$ (3) in Theorem 4.2

Corollary 4.6

$$J(X:B) \subseteq JX \subseteq J \cdot [\xi_1(R:\mathfrak{m}) \times \xi_2(L:\mathfrak{n})].$$

(1) $\Rightarrow$ (3) We have $JX = J$. Hence

$$\mathfrak{n}{\cdot}\xi_2 L \subseteq \mathfrak{n} \subseteq \mathfrak{n}{\cdot}\xi_2(L:\mathfrak{n}) = \xi_2{\cdot}\mathfrak{n}L$$

because $\mathfrak{n}(L:\mathfrak{n}) = \mathfrak{n}L$. Thus $\mathfrak{n} = \xi_2{\cdot}\mathfrak{n}L \cong \mathfrak{n}L$, so that S is AGL.

(3) $\Rightarrow$ (1) We have $JX = [JX \cap A \cdot (1,0) \cdot \xi] + J\xi$, and

- $JX \cap A \cdot (1,0) \cdot \xi \subseteq J\xi \;\Rightarrow\; JX = J\xi \cong J$
- $JX \cap A \cdot (1,0) \cdot \xi \not\subseteq J\xi \;\Rightarrow\; JX = \xi(R \times \mathfrak{n}) \cong \xi(\mathfrak{m} \times \mathfrak{n}) = \xi J \cong J$

This will prove that A is AGL. □

Theorem 4.2

$R \times_k S$ *is an* AGL *ring* $\Leftrightarrow$ R *and* S *are* AGL *rings.*

Letting $S = R$, we have

Corollary 4.7

$R \times_{R/\mathfrak{m}} R$ *is* AGL $\Leftrightarrow$ R *is* AGL $\Leftrightarrow$ $R \ltimes \mathfrak{m}$ *is* AGL.

Comment to the case of 2-AGL rings

Let $\mathfrak{c} = R : R[K]$.

- R is a Gorenstein ring $\Leftrightarrow$ $\mathfrak{c} = R$
- R is a non-Gorenstein AGL ring $\Leftrightarrow$ $\mathfrak{c} = \mathfrak{m}$

We also have

Theorem 4.8

$$R \times_{R/\mathfrak{c}} R \textit{ is } 2\text{-AGL} \quad \Leftrightarrow \quad R \textit{ is } 2\text{-AGL} \quad \Leftrightarrow \quad R \ltimes \mathfrak{c} \textit{ is } 2\text{-AGL}$$

Theorem 4.9

$$R \times_k S \textit{ is } \text{2-AGL} \quad \Leftrightarrow \quad \begin{array}{l} R \textit{ is } \text{AGL}, \ S \textit{ is } \text{2-AGL}, \quad \textit{or} \\ R \textit{ is } \text{2-AGL}, \ S \textit{ is } \text{AGL} \end{array}$$

Example 4.10

(1) $k[[t^3, t^7, t^8]] \times_k k[[t]]$

(2) $k[[t^3, t^7, t^8]] \times_k k[[t^3, t^4, t^5]]$

5. Higher dimensional cases

- $(R, \mathfrak{m})$, $(S, \mathfrak{n})$ CM local ring with $d = \dim R = \dim S > 0$
- $(T, \mathfrak{m}_T)$ a RLR with $\dim T = d - 1$, $\sharp(T/\mathfrak{m}_T) = \infty$.
- $f : R \to T$, $g : S \to T$ surjective
- $A = R \times_T S, J = (\mathfrak{m} \times \mathfrak{n}) \cap A$.

Then A is a CM local ring with $\dim A = d$.

Proposition 5.1

$$A = R \times_T S \text{ is Gorenstein } \Leftrightarrow \ R \text{ and } S \text{ are RLRs.}$$

Theorem 5.2

Assume that $\exists\, \mathrm{K}_A$ *and that* $\mathrm{Q}(A)$ *is a Gorenstein ring. Then TFAE.*

(1) $A = R \times_T S$ *is an* AGL *ring.*

(2) R *and* S *are* AGL *rings.*

Thank you for your attention.

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