On the almost Gorenstein property of determinantal rings

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Introduction

- $2 \leq t \leq m \leq n$ integers
- $X = [X_{ij}]$ an $m \times n$ matrix of indeterminates over an infinite field k
- $S = k[X] = k[X_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n]$ the polynomial ring
- $\mathrm{I}_t(X)$ the ideal of S generated by the $t \times t$ minors of the matrix X
- $R = S/\mathrm{I}_t(X)$

Fact 1 ([2, 3])

- *R is a Cohen-Macaulay normal domain*
- $\dim R = mn - (m - (t-1))(n - (t-1))$
- $\mathrm{K}_R = Q^{n-m}(-(t-1)m)$

where $Q = \mathrm{I}_{t-1}(Y)R$ and $Y = [X_{ij}]$ is an $m \times (t-1)$ matrix obtained from X by choosing the first $t-1$ columns.

Therefore R is level, $\mathrm{a}(R) = -(t-1)n$, and

$$R \text{ is Gorenstein} \quad \Longleftrightarrow \quad m = n.$$

Question 1.1

When do the determinantal rings satisfy almost Gorenstein property?

Therefore R is level, $\mathrm{a}(R) = -(t-1)n$, and

$$R \text{ is Gorenstein} \iff m = n.$$

Question 1.1

When do the determinantal rings satisfy almost Gorenstein property?

Theorem 1.2 (Goto-Takahashi-T, 2015)

Let $R = k[R_1]$ be a Cohen-Macaulay homogeneous ring with $d = \dim R > 0$. Suppose that R is not a Gorenstein ring and $|k| = \infty$. Then TFAE.

(1) *R is an almost Gorenstein graded ring and level.*

(2) *$\mathrm{Q}(R)$ is a Gorenstein ring and $\mathrm{a}(R) = 1 - d$.*

Corollary 1.3

$R = S/\mathrm{I}_t(X)$ is an almost Gorenstein graded ring $\iff$ $m = n$, or $m \neq n$ and $m = t = 2$.

Set $M = R_+$. Then

$$\begin{aligned} R = k[X]/\mathrm{I}_t(X) : AGG \quad &\Longrightarrow \quad R_M = (k[X]/\mathrm{I}_t(X))_M : AGL \\ &\Longleftrightarrow \quad k[[X]]/\mathrm{I}_t(X) : AGL \end{aligned}$$

Question 1.4

Does the implication

$$R_M = (k[X]/\mathrm{I}_t(X))_M : AGL \quad \Longrightarrow \quad R = k[X]/\mathrm{I}_t(X) : AGG$$

hold true?

Theorem 1.5

Suppose that k is a field of characteristic 0. Then TFAE.

(1) *R is an almost Gorenstein graded ring.*

(2) *R_M is an almost Gorenstein local ring.*

(3) *Either $m = n$, or $m \neq n$ and $m = t = 2$.*

Preliminaries

Setting 2.1

- $(R, \mathfrak{m})$ a Cohen-Macaulay local ring with $d = \dim R$
- $|R/\mathfrak{m}| = \infty$
- $\exists$ K_R the canonical module of R

Definition 2.2

We say that R is *an almost Gorenstein local ring*, if $\exists$ an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$.

Look at an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules. If $C \neq (0)$, then C is Cohen-Macaulay and $\dim_R C = d - 1$.

Set $\overline{R} = R/[(0) :_R C]$.

Then $\exists\ f_1, f_2, \ldots, f_{d-1} \in \mathfrak{m}$ s.t. $(f_1, f_2, \ldots, f_{d-1})\overline{R}$ forms a minimal reduction of $\overline{\mathfrak{m}} = \mathfrak{m}\overline{R}$. Therefore

$$\mathrm{e}^0_{\mathfrak{m}}(C) = \mathrm{e}^0_{\overline{\mathfrak{m}}}(C) = \ell_R(C/(f_1, f_2, \ldots, f_{d-1})C) \geq \ell_R(C/\mathfrak{m}C) = \mu_R(C).$$

Thus

$$\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C) \Longleftrightarrow \mathfrak{m}C = (f_1, f_2, \ldots, f_{d-1})C.$$

Hence C is a *maximally generated maximal Cohen-Macaulay $\overline{R}$-module* in the sense of B. Ulrich, which is called *an Ulrich R-module*.

Lemma 2.3

Let R be an almost Gorenstein local ring and choose an exact sequence

$$0 \to R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0$$

of R-modules s.t. $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$. If $\varphi(1) \in \mathfrak{m}\,\mathrm{K}_R$, then R is a RLR.

Therefore

$$\mu_R(C) = \mathrm{r}(R) - 1$$

provided R is not a RLR.

Corollary 2.4

Let R be an almost Gorenstein local ring but not Gorenstein. Choose an exact sequence

$$0 \to R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0$$

of R-modules s.t. C is an Ulrich R-module.

Then

$$0 \to \mathfrak{m}\varphi(1) \to \mathfrak{m}\,\mathrm{K}_R \to \mathfrak{m}C \to 0$$

is an exact sequence of R-modules.

Hence

$$\mu_R(\mathfrak{m}\,\mathrm{K}_R) \leq \mu_R(\mathfrak{m}) + \mu_R(\mathfrak{m}C).$$

Survey on the resolution of determinantal rings

Setting 3.1

- $t \geq 1$, $m \geq n \geq 1$ integers
- $(S, \mathfrak{n})$ a Noetherian local ring s.t. $\mathbb{Q} \subseteq S$
- F, G free S-modules with $\mathrm{rank}_S F = m + t - 1$, $\mathrm{rank}_S G = n + t - 1$
- $\phi = (r_{ij}) : F \to G$ a S-linear map s.t. $r_{ij} \in \mathfrak{n}$

Let $\lambda(m, n)$ be the Young tableau consisting of rectangle of n rows of m squares, where the i-th row contains the numbers $(i-1)m+1, (i-1)m+2, \dots, im$ in increasing order.

$$
\lambda(m,n) \quad = \quad
\begin{array}{ccccc}
\boxed{1} & \boxed{2} & \cdots & \boxed{m-1} & \boxed{m} \\
\boxed{m+1} & \boxed{m+2} & \cdots & \boxed{2m-1} & \boxed{2m} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\boxed{} & \boxed{} & \cdots & \boxed{} & \boxed{mn}
\end{array}
$$

- k an integer s.t. $0 \leq k \leq mn$
- $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ a partition of k s.t. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, $\sum_{i=1}^{n} \lambda_i = k$, and $\lambda_i \leq m$ for $1 \leq \forall i \leq n$

Definition 3.2

We define the tableaux λ_F, λ_G as follows.

- The i-th column of λ_F consists of λ_i squares which contain the numbers of the $(n-i+1)$-th row of $\lambda(m,n)$ in reverse order.
- λ_G is the tableau derived from $\lambda(m,n)$ by removing the numbers of λ_F.

Example 3.3

Consider the case where $m=4$, $n=3$, $k=5$, and $\lambda=(3,2,0)$. Then

$\lambda(m,n) =$

1	2	3	4
5	6	7	8
9	10	11	12

, $\lambda_F =$

12	8
11	7
10	

, and $\lambda_G =$

1	2	3	4
5	6		
9			

To the square in the (i,j) position of $\lambda(m,n)$ we associate:

- The square in the (i,j) position of $\lambda(m, n+t-1)$ if $j-i > m-n$.
- The string of t squares from the (i,j) position to the $(i+t-1, j)$ position if $j-i = m-n$.
- The square in the $(i+t-1, j)$ position if $j-i < m-n$.

Example 3.4

Consider the case where $m=4$, $n=3$, $k=5$, $\lambda=(3,2,0)$, and $t=3$. Then

$\lambda(m, n+t-1) =$

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16
17	18	19	20

, $\lambda(m,n) =$

1	2	3	4
5	6	7	8
9	10	11	12

.

Definition 3.5

We define the tableaux $\lambda_F(t)$, $\lambda_G(t)$ as follows.

- $\lambda_F(t)$ is the tableau constructed by replacing each square of λ_F by the associated square or string of squares of $\lambda(m, n+t-1)$.
- $\lambda_G(t)$ is the tableau obtained from $\lambda(m, n+t-1)$ by removing the squares of $\lambda_F(t)$.

Example 3.6

Consider the case where $m = 4$, $n = 3$, $k = 5$, $\lambda = (3, 2, 0)$, and $t = 3$. Then

$\lambda_F =$

12	8
11	7
10	

, so that $\lambda_F(t) =$

12	8
16	7
20	11
19	15
18	

. Therefore $\lambda_G(t) =$

1	2	3	4
5	6		
9	10		
13	14		
17			

.

Definition 3.7

We put

$$C_k = C_k(t) = \sum_{|\lambda|=k} e(\lambda_F(t))F \otimes_S e(\lambda_G(t))G$$

for every $0 \leq k \leq mn$, where

$$\begin{aligned} e(\lambda_F(t))F &:= e(\lambda_F(t))(F \otimes_S F \otimes_S \cdots \otimes_S F) \\ e(\lambda_G(t))G &:= e(\lambda_G(t))(G \otimes_S G \otimes_S \cdots \otimes_S G). \end{aligned}$$

Therefore

$$0 \to C_{mn} \to C_{mn-1} \to \cdots \to C_1 \to C_0 \to S/\mathrm{I}_t(\phi) \to 0$$

gives a minimal S-free resolution of $S/\mathrm{I}_t(\phi)$.

How to compute the rank of C_k

- Find all partitions λ with $|\lambda| = k$.
- Find the Young diagrams $\lambda_F(t)$, $\lambda_G(t)$.
- Compute the ranks of $e(\lambda_F(t))F$, $e(\lambda_G(t))G$

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition, H a free S-module of rank $r \geq 0$. Let

$$\Delta(x_1, x_2, \ldots, x_r) = \prod_{i<j}(x_i - x_j)$$

where $x_1, x_2, \ldots, x_r \in \mathbb{Z}$.

Put $\ell_i = \lambda_i + r - 1$ for $1 \leq i \leq r$. Then

$$\mathrm{rank}_S e(\lambda)H = \frac{\Delta(\ell_1, \ell_2, \ldots, \ell_r)}{\Delta(r-1, r-2, \ldots, 0)}.$$

Proposition 3.8

There are equalities

$$\mathrm{rank}_S C_{mn} = \frac{\prod_{j=0}^{m-n-1}\left(\prod_{i=0}^{n-1}(t+i+j)\right) 1!\cdot 2!\cdots(n-2)!\cdot(n-1)!}{(m-n)!\cdot(m-n+1)!\cdots(m-2)!\cdot(m-1)!}$$

$$\mathrm{rank}_S C_{mn-1} = \frac{\prod_{j=0}^{m-n-1}\left(\prod_{i=1}^{n-1}(t+i+j)\right)\prod_{i=0}^{m-n-2}(t+i)\,(t+m-1)\,1!\cdot 2!\cdots(n-2)!\cdot n!}{(m-n-1)!\cdot(m-n+1)!\cdot(m-n+2)!\cdots(m-2)!\cdot(m-1)!}$$

provided $m \neq n$.

Proof of Theorem 1.5

Theorem 1.5

Suppose that k is a field of characteristic 0. Then TFAE.

(1) $k[X]/\mathrm{I}_t(X)$ *is an almost Gorenstein graded ring.*

(2) $k[[X]]/\mathrm{I}_t(X)$ *is an almost Gorenstein local ring.*

(3) *Either $m = n$, or $m \neq n$ and $m = t = 2$.*

In what follows, let

- k a field of characteristic 0
- $S = k[[X]]$
- $R = S/\mathrm{I}_t(X)$
- $\mathfrak{m} = (x_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq n)$ the maximal ideal of R

Let

$$0 \to F \to G \to \cdots \to S \to R \to 0 \qquad (\sharp)$$

be a minimal S-free resolution of R. Then

$$\mathrm{rank}_S F = \frac{\displaystyle\prod_{j=0}^{n-m-1}\left(\prod_{i=0}^{m-t}(t+i+j)\right) 1!\cdot 2!\cdots(m-t-1)!\cdot(m-t)!}{(n-m)!\cdot(n-m+1)!\cdots(n-t-1)!\cdot(n-t)!}.$$

Moreover

$$\mathrm{rank}_S G = \frac{\displaystyle\prod_{j=0}^{n-m-1}\left(\prod_{i=1}^{m-t}(t+i+j)\right)\prod_{i=0}^{n-m-2}(t+i)\cdot n\cdot 1!\cdot 2!\cdots(m-t-1)!\cdot(m-t+1)!}{(n-m-1)!\cdot(n-m+1)!\cdot(n-m+2)!\cdots(n-t-1)!\cdot(n-t)!}$$

provided $m \neq n$.

Take the K_S-dual of $(\sharp)$, we get the presentation

$$G \to F \to \mathrm{K}_R \to 0$$

of R-modules so that

$$\mu_R(\mathfrak{m}\,\mathrm{K}_R) \geq mn \cdot \mathrm{r}(R) - \mathrm{rank}_S G.$$

Let

$$\alpha = \frac{\prod_{j=0}^{n-m-1}\left(\prod_{i=1}^{m-t}(t+i+j)\right)\prod_{i=0}^{n-m-2}(t+i)\cdot 1!\cdot 2!\cdots(m-t-1)!\cdot(m-t)!}{(n-m-1)!\cdot(n-m+1)!\cdot(n-m+2)!\cdots(n-t-1)!\cdot(n-t)!}.$$

Then

$$\mathrm{r}(R) = \frac{t+n-m-1}{n-m}\cdot\alpha, \quad \mathrm{rank}_S G = n\cdot(m-t+1)\cdot\alpha$$

Proof of Theorem 1.5

We may assume that $m \neq n$. Since R is an almost Gorenstein local ring, $\exists$ an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules s.t. $C \neq (0)$ is an Ulrich R-module.

Then

$$0 \to \mathfrak{m} \to \mathfrak{m}\,\mathrm{K}_R \to \mathfrak{m}C \to 0$$

whence

$$\begin{aligned} \mu_R(\mathfrak{m}\,\mathrm{K}_R) &\leq \mu_R(\mathfrak{m}) + \mu_R(\mathfrak{m}C) \\ &\leq mn + (d-1)(\mathrm{r}(R)-1) \end{aligned}$$

because $\mathfrak{m}C = (f_1, f_2, \ldots, f_{d-1})C$ for $\exists\ f_i \in \mathfrak{m}$.

Proof of Theorem 1.5

Therefore

$$mn \cdot \mathrm{r}(R) - \mathrm{rank}_S G \le \mu_R(\mathfrak{m} \mathrm{K}_R) \le mn + (d-1)(\mathrm{r}(R) - 1)$$

which yields that

$$(mn - (d-1))\,(\mathrm{r}(R) - 1) \le \mathrm{rank}_S G.$$

Hence

$$\{(m-(t-1))(n-(t-1))+1\}\left(\frac{t+n-m-1}{n-m} \cdot \alpha - 1\right) \le n(m-(t-1))\alpha.$$

Then a direct computation shows that $t = 2$, whence $m = 2$ as desired. □

Thank you so much for your attention.

References

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Setting 5.1

- $R = \bigoplus_{n \geq 0} R_n$ a Cohen-Macaulay graded ring with $d = \dim R$
- $(R_0, \mathfrak{m})$ a local ring
- $\exists$ the graded canonical module K_R
- $M = \mathfrak{m}R + R_+$

Definition 5.2

We say that R is *an almost Gorenstein graded ring*, if $\exists$ an exact sequence

$$0 \to R \to \mathrm{K}_R(-\mathrm{a}(R)) \to C \to 0$$

of graded R-modules such that $\mu_R(C) = \mathrm{e}^0_M(C)$.

Notice that

- R is an almost Gorenstein graded ring
  $\Longrightarrow$ R_M is an almost Gorenstein local ring.