Ulrich ideals of dimension 1

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Introduction

1. In 1987 B. Ulrich and the other authors [BHU] introduced **M**aximally **G**enerated **M**aximal **C**ohen–**M**acaulay modules.
2. In 2012 S. Goto and the others [GOTWY] generalized the notion of MGMCM module, which they call **Ulrich module/ideal**.

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Contents

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1. $(A, \mathfrak{m})$ a **Cohen–Macaulay** local ring, $\dim A = 1$
2. I an **$\mathfrak{m}$–primary ideal** in A, $n = \mu_A(I)$
3. I contains a parameter ideal $Q = (a)$ of A as a reduction.

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We say that I is an **Ulrich ideal** of A, if

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2. I/I^2 is A/I–free.

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Let $\mathcal{X}_A$ be the set of **Ulrich ideals** in A.

Theorem 3 ([GOTWY])

Suppose that A is of **finite C–M representation type**. *Then $\mathcal{X}_A$ is a finite set.*

Let

$$A = k[[t^{a_1}, t^{a_2}, \ldots, t^{a_\ell}]] \subseteq k[[t]] = \overline{A}$$

be the numerical semigroup ring over a field k, where $0 < a_1, a_2, \ldots, a_\ell \in \mathbb{Z}$ such that $\mathrm{GCD}(a_1, a_2, \ldots, a_\ell) = 1$.
Let $\mathcal{X}_A^g = \{$Ulrich ideals in A generated by **monomials** in $t\}$.

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The set $\mathcal{X}_A^g$ is finite.

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We continue the research [GOTWY], providing a practical method for counting Ulrich ideals in dimension 1.

A brief survey

Lemma 5

Suppose that $I^2 = QI$. Then TFAE.

1. *I is an* **Ulrich ideal** *of A.*
2. *I/Q is a* **free** *A/I–module.*

Proof.

The equivalence of conditions (1) and (2) follows from the **splitting** of the sequence

$$0 \to Q/QI \to I/I^2 \to I/Q \to 0.$$

When this is the case, $I/Q \cong (A/I)^{n-1}$, since $Q = (a)$ is generated by a part of a minimal basis of I. □

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Let I be an Ulrich ideal and look at the isomorphism

$$I/Q \cong (A/I)^{n-1}.$$

Then we get the following.

Corollary 6

1. $Q : I = I$.
2. $0 < (n-1) \cdot \mathrm{r}(A/I) = \mathrm{r}_A(I/Q) \leq \mathrm{r}(A/Q) = \mathrm{r}(A)$,

where $\mathrm{r}(A) = \ell_A(\mathrm{Ext}_A^1(A/\mathfrak{m}, A))$. *Hence* $n \leq \mathrm{r}(A) + 1$.

Therefore, if A is a **Gorsenstein** ring, A/I is a Gorenstein ring, $n = 2$, and I is a **good ideal** in the sense of [GIW].

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$$\mathbb{F}_\bullet : \quad \cdots \to F_i \overset{\partial_i}{\to} F_{i-1} \to \cdots \to F_1 \overset{\partial_1}{\to} F_0 \to A/I \to 0$$

be a **minimal** free resolution of A/I and $\beta_i = \mathrm{rank}_A F_i \quad (i \geq 0)$.

Theorem 7 ([GOTWY])

$$\beta_i = \begin{cases} (n-1)^{i-1} \cdot n & (i \geq 1), \\ 1 & (i = 0) \end{cases}$$

for $i \geq 0$. Hence $\beta_i = \binom{1}{i} + (n-1)\beta_{i-1}$ for $\forall i \geq 1$.

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Look at the exact sequence

$$0 \to Q \to I \to (A/I)^{\oplus(n-1)} \to 0.$$

Corollary 8 ([GOTWY])

A minimal free resolution of I is obtained by those of Q and $(A/I)^{\oplus(n-1)}$.

Corollary 9 ([GOTWY])

$\mathrm{Syz}_A^{i+1}(A/I) \cong [\mathrm{Syz}_A^i(A/I)]^{\oplus(n-1)}$ *for all $i \geq 1$. Hence*

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for all $i \geq 1$, if A is a Gorenstein local ring.

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Theorem 10 ([GOTWY])

Let I and J be Ulrich ideals of A. Then $I = J$ if and only if

$$\mathrm{Syz}_A^i(A/I) \cong \mathrm{Syz}_A^i(A/J)$$

for **some** $i \geq 0$.

Example 11

Suppose that A is a **Gorenstein** local ring of dimension 1 and I an **Ulrich** ideal of A. Then $\mu_A(I) = 2$. We write $I = (a, x)$ $(x \in A)$ where $Q = (a)$ is a reduction of I. Then $x^2 = ay$ for some $y \in I$, since $I^2 = aI$, and a minimal free resolution of A/I is given by

$$\mathbb{F}_\bullet : \quad \cdots \to A^2 \xrightarrow{\begin{pmatrix} -x & -y \\ a & x \end{pmatrix}} A^2 \xrightarrow{\begin{pmatrix} -x & -y \\ a & x \end{pmatrix}} A^2 \xrightarrow{\begin{pmatrix} a & x \end{pmatrix}} A \xrightarrow{\varepsilon} A/I \to 0.$$

Hence $I \cong I^*$.

The Gorenstein case

Definition 12 ([GIW])

We say that I is a **good ideal** of A, if

1. $I^2 = QI$ and
2. $Q : I = I$.

Let $\mathcal{V}_A$ be the set of intermediate rings $A \subsetneq B \subseteq \mathrm{Q}(A)$ such that B is a finitely generated A–module and put

$$\mathcal{Y}_A = \{I \mid I \text{ is a good ideal of } A\},$$

$$\mathcal{Z}_A = \{B \in \mathcal{V}_A \mid B \text{ is a Gorenstein ring}\}.$$

Hence $\mathcal{X}_A \subseteq \mathcal{Y}_A$ and $\mathcal{Z}_A \subseteq \mathcal{V}_A$.

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Lemma 13 (Main Lemma)

We have a well–defined bijective map

$$\varphi : \mathcal{Z}_A \to \mathcal{Y}_A, \quad B \mapsto A : B,$$

where for each $B \in \mathcal{Z}_A$, $A : B \in \mathcal{X}_A \Leftrightarrow \mu_A(B) = 2$.

Proof.

Let $B \in \mathcal{Z}_A$ and put $J = A : B$. Then $J = bB$ for some $b \in J$, since B is a Gorenstein ring and $J \cong \mathrm{K}_B$. Let $\mathfrak{q} = bA$. Then $J^2 = \mathfrak{q}J$ and $\mathfrak{q} : J = A : B = J$, so that J is a **good** ideal of A. If $J \in \mathcal{X}_A$, then $\mu_A(B) = \mu_A(J) = 2$. Suppose that $\mu_A(B) = 2$. Then $J/\mathfrak{q}$ is cyclic, since $\mathfrak{q}$ is a minimal reduction of J. Hence $J/\mathfrak{q} \cong A/J$, because $\mathfrak{q} : J = J$. Thus $J \in \mathcal{X}_A$. □

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Theorem 14

Let $A = k[[t^n, t^{n+1}, \ldots, t^{2n-2}]]$ $(n \geq 3)$. Then

$$\mathcal{X}_A = \begin{cases} \{(t^4, t^6)\} & (n = 3), \\ \{(t^4 - \lambda t^5, t^6) \mid \lambda \in k\} & (n = 4), \\ \quad \emptyset & (n \geq 5). \end{cases}$$

Proof of the case: $n = 2q + 1$ $(q \geq 2)$. Let $I \in \mathcal{X}_A$ and $S = \frac{I}{a}$. Then

$$t^n V \subseteq k[[t^n, t^{n+1}, \ldots, t^{2n-1}]] \subseteq S,$$

since t^{2n-1} is the generator of the socle of $\mathrm{Q}(A)/A$. Let

$$\mathcal{C} := S : V = t^c V \ (c \geq 0).$$

Then $c \leq n = 2q + 1$. We put $\ell = \ell_S(V/S)$. Hence

$$2\ell = c,$$

since S is a **Gorenstein** ring. Thus $\ell \leq q$.
Look at

$$\overline{S} := S/\mathfrak{m}S \supsetneq J := \mathfrak{m}_{\overline{S}} \supsetneq J^2 = (0).$$

Take $\xi \in \mathfrak{m}_S$ so that $J = (\overline{\xi})$. Then $\overline{\xi} \neq 0$ and $\overline{\xi}^2 = 0$ in $\overline{S}$.

Proof of the case: $n = 2q+1$ $(q \geq 2)$ (continued).

Hence

$$\xi^2 \in \mathfrak{m}S \subseteq t^n V \ \text{ and } \ S = A + A\xi,$$

because $S/\mathfrak{m}S = k + k\overline{\xi}$. Therefore $2{\cdot}\mathrm{o}(\xi) \geq n = 2q+1$, so that $\mathrm{o}(\xi) \geq q+1$. Thus

$$S = A + A\xi \subseteq T := k[[t^{q+1}, t^{q+2}, \ldots, t^{2q+1}]].$$

Hence $S = T$, because

$$\ell_A(V/T) = q \ \text{ and } \ \ell_S(V/S) \leq q.$$

This is impossible. Thus $\mathcal{X}_A = \emptyset$. □

For some special class of one–dimensional Cohen–Macaulay local rings possessing **finite C–M representation type**, we have the following, where $k[[X, Y]]$ and $k[[t]]$ are the formal power series rings over a field k, and x, y denote the images of X, Y in the corresponding ring.

Theorem 15

The following assertions hold true.

1. $\mathcal{X}_{k[[t^3,t^4]]} = \{(t^4, t^6)\}$.
2. $\mathcal{X}_{k[[t^3,t^5]]} = \emptyset$.
3. $\mathcal{X}_{k[[X,Y]]/(Y(X^2-Y^{2\ell+1}))} = \{(x, y^{2\ell+1}), (x^2, y)\}$, where $\ell \geq 1$.
4. $\mathcal{X}_{k[[X,Y]]/(Y(Y^2-X^3))} = \{(x^3, y)\}$.
5. $\mathcal{X}_{k[[X,Y]]/(X^2-Y^{2\ell})} = \{(x^2, y), (x - y^\ell, y(x + y^\ell)), (x + y^\ell, y(x - y^\ell))\}$, where $\ell \geq 1$ and $\operatorname{ch} k \neq 2$.

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The non–Gorenstein case

Theorem 16

Let $(V, \mathfrak{n})$ be a Cohen–Macaulay local ring with $\dim V = 1$. Let $A = V[Y]/(Y^n)$ $(n \geq 2)$. Then $\sharp\mathcal{X}_A = \infty$.

Proof.

Suppose $n = 2q + 1$ $(q \geq 1)$ and let a be a parameter for V. For each $\ell > 0$, let $I = I_\ell := (a^{2\ell} - y, a^\ell y^q)$, where y is the image of Y in A. Then $I^2 = (a^{2\ell} - y)I$, while $A/(a^{2\ell} - y) \cong V/(a^{2\ell n})$ and $A/I \cong V/(a^{\ell n})$. Hence $\ell_V(I/(a^{2\ell} - y)) = \ell_V(A/I)$. Therefore $I/(a^{2\ell} - y) \cong A/I$ as A–modules, so that $I_\ell = I \in \mathcal{X}_A$. Hence $\sharp\mathcal{X}_A = \infty$.

For the case $n = 2q$ $(q \geq 1)$, consider $I = I_\ell := (a^\ell, y^q)$. □

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Theorem 17

Suppose that $A = \widehat{A}$ and A is a reduced ring. Let $\overline{A}$ be the integral closure of A in the total quotient ring. Then

$$\mathcal{X}_A = \{\mathfrak{m}\}, \quad \text{if} \ \ \mathfrak{m}\overline{A} \subseteq A \ \ \text{and} \ \ A \neq \text{a RLR}.$$

Proof.

The ring $\overline{A}$ is a finitely generated A–module and $\mathfrak{m}\overline{A} = \mathfrak{m}$. Take $a \in \mathfrak{m}$ so that $\mathfrak{m} = a\overline{A}$. Then $\mathfrak{m}^2 = a\mathfrak{m}$ and $\mu_A(\mathfrak{m}) > 1$. Thus $\mathfrak{m} \in \mathcal{X}_A$. Conversely, let $I \in \mathcal{X}_A$ and choose a reduction $Q = (a)$ of I. Then $\mathfrak{m}\frac{I}{a} \subseteq A$, since $\frac{I}{a} \subseteq \overline{A}$. Hence $\mathfrak{m}I \subseteq Q$. Therefore $I = \mathfrak{m}$, since I/Q is A/I–free. Thus $\mathcal{X}_A = \{\mathfrak{m}\}$. □

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Corollary 18

Let $n \geq 2$ and $A = k[[t^n, t^{n+1}, \ldots, t^{2n-1}]]$. Then $\mathcal{X}_A = \{\mathfrak{m}\}$.

Corollary 19

Let $(S, \mathfrak{n})$ be a RLR with $\dim S = n \geq 2$. Let $\mathfrak{n} = (X_1, X_2, \ldots, X_n)$ and put $A = S/\bigcap_{i=1}^{n}(X_j \mid j \neq i)$. Then $\mathcal{X}_A = \{\mathfrak{m}\}$.

Corollary 20

Let K/k $(K \neq k)$ be a finite extension of fields. Assume that there are no proper intermediate fields between K and k. We put

$$V = K[[t]] \quad \text{and} \quad A = k[[tK]].$$

Then $\mathcal{X}_A = \{tV\}$.

Using value semigroups

Let $V = k[[t]]$.

Example 21

1. Let $f, g \in V$ such that $\mathrm{o}(f) = 3, \mathrm{o}(g) = 5$. We put $A = k[[f, g]]$. Then $\mathcal{X}_A = \emptyset$.
2. Let $f, g \in V$ such that $\mathrm{o}(f) = 3, \mathrm{o}(g) = 4$. We put $A = k[[f, g]]$. Then $\mathcal{X}_A = \{(g, f^2)\}$.
3. Let $A = k[[f_5, f_6, f_7, f_8]]$, where $f_i \in V$ such that $\mathrm{o}(f_i) = i$ for $5 \leq \forall i \leq 8$. Then $\mathcal{X}_A = \emptyset$.

Thank you very much for your attention!

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