Universality in Periodicity Manifestations in Turbulent Non- Locally Coupled Map Lattices

Tokuzo SHIMADA and Shou TSUKADA∗)

Department of Physics, School of Science and Technology, Meiji University,
Kanagawa 214-8571, Japan

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Recently, it has been found in a globally coupled map lattice that, even in a region of very weak coupling, maps form periodic cluster attractors through synchronization when a certain condition between the map nonlinearity and the coupling is satisfied. We here investigate how this intriguing cluster formation depends on the non-locality of the couplings. We examine this issue for three types of coupled map lattices, in each case with spatial dimensions \( D = 1, 2, 3 \). We find that for all types of coupling and for all \( D \), the “periodicity manifestations” (cluster attractors and their remnants) occur at the same strength when the fluctuation of the local mean field around the system mean field is the same.

§1. Introduction

The coupled map lattice (CML) has been a vital testing ground for the dynamics of random coupled elements. Local CMLs (nearest-neighbor CMLs) exhibit spatio-temporal chaos and pattern formation.\(^1\),\(^2\) Contrastingly, globally coupled map lattices (GCMLs) eliminate the concept of distance and exhibit rich phases through the subtle balance between the map nonlinearity and the coherence introduced by the averaging interaction.\(^3\) Globally coupled map lattices have been used to model basic features of systems including neural networks, Josephson junction arrays, coupled multi-mode lasers, and fluid vortices.

Recently, it has been found in a GCML that even in the ‘turbulent regime’—the region of very weak coupling and the strong map nonlinearity—maps systematically form either periodic cluster attractors or their remnant states through synchronization when a certain tuning condition between the map nonlinearity and the coupling is satisfied.\(^4\) -\(^7\) Following our previous work\(^5\) we refer to such attractors and their remnant states generically as periodicity manifestations (PMs) that emerge from the turbulence. Periodicity manifestations represent intriguing synchronization phenomena realized with very weak coupling. The element map has many periodic windows in between regions of chaos. Periodicity manifestations are induced when the high-dimensional GCML dynamics reduce to those of only several clusters via the synchronization of maps, and a certain periodic window governs the motion of these clusters.

A turbulent GCML with a large number of maps exhibits weak coherence (so called ‘hidden-coherence’), which leads to a violation of the law of large numbers

∗) After April 1, Hitachi Software Engineering, Co., Ltd., Ogami-cho 6-81, Naka-ku, Yokohama, Kanagawa 231-0015, Japan.

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(LLN) in the statistics of the mean field in time.\textsuperscript{8}) (This does not imply the violation of LLN in the ensemble average, however.\textsuperscript{9}) With the recent discovery of PMs, a turbulent GCML may now be regarded as a system that sensitively mirrors the periodic windows of the element maps in addition to the weak coherence in the background.

The amazing fact that, even with very weak coupling, maps form periodic cluster attractors may have important implications for complex systems of coupled chaotic elements, in particular for the activity of the brain. For instance, an efficient switch between periodic states via chaos\textsuperscript{10}) may be realized even in a system in a turbulent phase, provided that periodic cluster formation is not unique to a GCML. In this note we address the question of how this phenomenon depends on the global coupling features of the coupled map lattice. We examine three types of coupled map lattices, in each case with spatial dimensions $D = 1, 2, 3$. All models interpolate between a GCML and a local CML, but along different paths. In the first two models, $\text{POW}_\alpha$ and $\text{EXP}_{\rho_0}$, the coupling decreases with distance $\rho$ according to a power law, $1/\rho^\alpha$, and an exponential law, $\exp(-\rho/\rho_0)$, respectively. In the third model, $\text{CML}_\kappa$, the interaction is uniform within a certain range ($\rho \leq \kappa$) and vanishes outside.

Related works in the literature can be summarized as follows. For $D = 1$ CML\textsubscript{\kappa}, it was shown that the background coherence becomes visible when the coupling range exceeds a certain threshold.\textsuperscript{11}) The phase diagram in the space of the coupling strength versus the coupling range was examined, and the region of collective chaos was determined.\textsuperscript{12}) The thermodynamic limit of the model was analytically investigated.\textsuperscript{13}) For $D = 1$ power-law coupling CML, it was recently shown that the maximal Lyapunov exponent increases monotonically as the coupling varies from global to local.\textsuperscript{14}) For a $D = 2$ inverse power-law CML, it was shown that the phase diagram is the same as that for a GCML.\textsuperscript{4}) For $D = 1$ coupled Ginzburg-Landau oscillators and coupled Brusselators, it has been found that anomalous spatio-temporal chaos occurs, leading to power-law spatial correlations,\textsuperscript{15}) and this has also been confirmed for $D = 2$ Ginzburg-Landau oscillators.\textsuperscript{16}) All of these studies represent progress, but for the most part they have been limited to a particular model in a particular dimension.

In this note, for the first time we investigate periodic cluster formation and the manner in which it varies in passing from the global limit to the local limit and extensively compare three models for $D = 1, 2, 3$. We find a salient universality such that the PMs occur at the same strength when the fluctuation of the local mean field around the system mean field is the same. Some preliminary results of this paper were reported in Ref. \textsuperscript{17}).

§2. The periodicity manifestations in GCML

Let us briefly survey periodicity manifestations recently found in turbulent GCML.\textsuperscript{4, 5}) For further details, we refer the reader to Ref. \textsuperscript{5}) and references therein.

A homogeneous GCML consisting of $N$ maps on a lattice $\Lambda$ is defined by an
evolution equation
\[ x_P(t + 1) = (1 - \varepsilon)f(x_P(t)) + \varepsilon h(t), \quad P \in \Lambda, \] (2.1)
where the mean field \( h(t) \) is given by
\[ h(t) \equiv \frac{1}{N} \sum_{Q \in \Lambda} f(x_Q(t)), \] (2.2)
and we use the logistic map \( f(x) = 1 - ax^2 \). This GCML evolution consists of the iteration of a two-step process: the parallel mapping of all elements by a common function \( f \) followed by an interaction between elements via their mean field \( h(t) \). We note the important relation
\[ \frac{1}{N} \sum_{P \in \Lambda} x_P(t + 1) = \frac{1}{N} \sum_{P \in \Lambda} f(x_P(t)); \] (2.3)
that is, the mean field is invariant under the interaction. All the non-local models we construct below respect this invariance. The GCML has only two parameters, the map nonlinearity \( a \) and the overall coupling \( \varepsilon \). Yet, through the subtle balance between the randomness among the elements and the coherence introduced by the averaging interaction, it exhibits rich phases in the \((a, \varepsilon)\) plane. In the ‘turbulent regime’ — the regime of very small coupling \( \varepsilon < \sim 0.1 \) and strong nonlinearity \( a \gtrsim 1.80 \) — the maps do not in general form visible clusters; for most of the coupling values, the maps seem to evolve randomly, as revealed by direct observation. However, under a certain condition between \( a \) and \( \varepsilon \), the maps systematically form periodic cluster attractors through synchronization. In Fig. 1, we show examples of them and discuss below their systematics.

The cluster attractors consist of two types. For one type, the maps divide themselves into almost equally populated clusters, which evolve through mutual periodic motion, and the periodicity \( p \) and the number of the clusters \( c \) are equal. We refer to these \((p = c)\)-type attractors as “maximally symmetric cluster attractors” (MSCAs). In Fig. 1(a) and (c) we exhibit the \( p3c3 \) MSCA and the \( p5c5 \) MSCA, respectively. The evolution of a period \( p \) MSCA can be described as
\[ \cdots \rightarrow \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{pmatrix} \rightarrow \begin{pmatrix} X_2 \\ X_3 \\ \vdots \\ X_1 \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} X_p \\ X_1 \\ \vdots \\ X_{p-1} \end{pmatrix} \rightarrow \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{pmatrix} \rightarrow \cdots, \]
where variables \( X_I \) (\( I = 1, \cdots, c = p \)) denote the cluster coordinates. For MSCAs, the mean field is almost time independent, due to the population symmetry among the clusters. Hence, the mean squared deviation (MSD) of the time series of the mean field \( h(t) \) almost vanishes \((10^{-6} - 10^{-3})\). The \( h(t) \) distribution consists of \( p \) sharp peaks, almost degenerate in position. The vanishing fluctuation of the mean field helps to stabilize the clusters. It can be proved that MSCAs are linearly stable by algebraically solving the eigenvalue problem of the linear stability matrix.\(^5\)
T. Shimada and S. Tsukada

Fig. 1. Each row represents an iteration of $N = 10^4$ GCML for $1.1 \times 10^5$ steps from a random initial configuration. Left: Orbits of maps for the last ten steps. Right: The $h(t)$ distribution calculated from $10^5$ steps discarding the first $10^4$ steps. (Normalized as $\int \rho(h) dh = 1$.)

(a) $p6c6$ MSCA formed at $(a, \varepsilon) = (1.90, 0.0352)$, $[(h, r) = (1.773, 0.933)]$. $h(t)$ distribution consists of three sharp peaks, almost degenerate in position due to the population symmetry. The MSD is negligible ($\delta h^2 = 3.52 \times 10^{-4}$ for this event).

(b) $p3c2$ cluster attractor $[(a, \varepsilon; b, r) = (1.90, 0.046; 1.748, 0.920)]$. One cluster is absent, leading to $h(t)$ distribution consisting of three peaks largely separated each other ($\delta h^2 = 0.127$).

(c) and (d) respectively exhibit $p5c5$ MSCA $[(1.663, 0.0118; 1.630, 0.980)]$ and its remnant state $[(1.734, 0.0357; 1.630, 0.940)]$. In (c), at low reduction ($r > r_{th}$), $h(t)$ distribution consists of almost degenerate 5 peaks ($\delta h^2 = 7.1 \times 10^{-4}$). It becomes in (d), at high reduction, a Gaussian distribution with an enhanced MSD ($\delta h^2 = 3.1 \times 10^{-3}$) — so called hidden-coherence. This is confronted with the distribution at no correlation, calculated with $\varepsilon \equiv 0$ using same values for $a$ and $N$: This LLN prediction yields $\delta h^2 = 3.3 \times 10^{-5}$.

(e) and (f) respectively exhibit $p5c3$ $[(1.663, 0.0120; 1.630, 0.980)]$ and its remnant state $[(1.734, 0.036; 1.630, 0.940)]$. The $h(t)$ distribution in (e) turns into an overlapping Gaussian distribution in (f), but still keep tracks of previous cluster orbits. Note that in the evolution plot for remnant states, orbits of only one percent of maps are shown.
As the strength of the coupling increases, the number of clusters $c$ decreases one by one. Thus, associated with each MSCA, there are $(p > c)$-type cluster attractors in the sequence of $c = p - 1, p - 2, \cdots$. See Fig. 1(b) and (e) for the $p3c2$ and $p5c3$ cluster attractor, respectively. In contrast to the case of MSCA, the mean field of the $(p > c)$-type attractor oscillates periodically due to the unbalance caused by the missing clusters. The $h(t)$ distribution now consists of $p$ peaks, largely separated each other, and the MSD is extremely large $[O(1)]$. The MSD is solely determined by the cluster orbits and the population ratios between the clusters, and hence is independent of $N$.

The evolution of maps in the cluster attractors can be directly observed in a numerical iteration of (2.1), if one knows where to find their formation in the parameter space. Let us work out the location of the formation of a period $p$ MSCA in the $(a, \varepsilon)$ plane. For the MSCA, the mean field is a time-independent constant $[h(t) = h^*]$. Therefore the simple time-independent scaling
\[ y_P(t) = (1 - \varepsilon + \varepsilon h^*)^{-1} x_P(t) \] (2.4)
transforms (2.1) into a standard logistic map,
\[ y_P(t + 1) = 1 - b y_P^2(t). \] (2.5)
Note that $b$ is ‘reduced’ from $a$ at the rate
\[ r \equiv \frac{b}{a} = (1 - \varepsilon) (1 - \varepsilon(1 - h^*)) < 1. \] (2.6)
A higher reduction (smaller $r$) is achieved as the coupling $\varepsilon$ is increased. It is useful to note that $r \approx 1 - 2\varepsilon$ for small $\varepsilon$. The scaled maps $y_P$ evolve with period $p$, following the period $p$ MSCA; hence $b$ must be in the interval of a period $p$ window of the logistic map. There is a further constraint from (2.4),
\[ \overline{y}_b = \langle y_P \rangle_A = (1 - \varepsilon + \varepsilon h^*)^{-1} h^*. \] (2.7)
The first equality here means that the average of the period $p$ orbit points of a logistic map (with $b$ dependence made explicit) is equal to the snapshot average of the maps $y_P$. This equality holds since the $y_P$ are scaled copies of the maps in a MSCA, where the system configuration is time independent, excluding cyclic permutations of the clusters. Eliminating $h^*$ from (2.6) and (2.7) we find that the model parameters $a$ and $\varepsilon$ must be on a foliation curve $^5$ defined by
\[ (a, \varepsilon)^b(r) = \left( \frac{b}{r}, 1 - \frac{r \overline{y}_b}{2} - \sqrt{r(1 - \overline{y}_b) + \left( \frac{r \overline{y}_b}{2} \right)^2} \right) \] (2.8)
over the $(a, \varepsilon)$ plane. This curve emanates from the window point $(b, 0)$ at $r = 1$ and flows in the parameter space as $r$ decreases from 1 (as the reduction becomes higher). The GCML that produces a period $p$ MSCA must have $a$ and $\varepsilon$ set along the foliation curves from a period $p$ window.
The most prominent PMs are those induced by the period three window which resides at \( b = 1.75 - 1.7903 \). They are induced at any \( r \) along the respective foliation curves from the period three window. For \( a = 1.90 \), the period three window induces a \( p3c3 \) MSCA and its bifurcation, a \( p6c6 \) MSCA for \( \varepsilon = 0.037 - 0.041 \) and \( 0.032 - 0.037 \), respectively, and a \( p3c2 \) cluster attractor at the adjacent higher range of coupling, \( \varepsilon = 0.037 - 0.050 \).

For the foliation of all other windows, there is a threshold \( r_{th} \) with respect to the cluster formation. \( r_{th} \) is around 0.95 for most of the dominant windows, but for some narrow windows, \( r_{th} \geq 0.99 \). We recapitulate our findings below.\(^5\)

(i) For \( r \geq r_{th} \) ("upstream the foliation curve"), maps form MSCA (negligible MSD) and \((p > c)\)-type clusters \([\text{MSD} \approx O(1)]\), respectively, along the curves of the period \( p \) window and in the adjacent region of higher \( \varepsilon \). This is just the same with the case of period three window.

(ii) For \( r \leq r_{th} \) ("downstream the foliation curve"), clusters are no longer formed. Nevertheless, there occurs the same structure in the MSD of the \( h(t) \) distribution — MSD is suppressed along the foliation curves of a window, while it turns out very large in the adjacent region of higher \( \varepsilon \). Because the MSD maintains approximately the same value all along the foliation curve, the sequence of periodic windows of the element map induces a successive valley-peak structure in the MSD curve, measured as a function of \( \varepsilon \) for any \( a \).\(^5\) For a local CML, contrastingly, the MSD curve does not exhibit any structure, and the MSD simply follows the LLN as the number of maps \( N \) increases.

(iii) For \( r \leq 0.95 \), the \( h(t) \) distribution is always Gaussian with an enhanced MSD along the foliation curves of a window, while it is always an overlapping Gaussian distribution in the adjacent region of higher \( \varepsilon \). The former case represents so-called hidden-coherence. Let us examine this in Fig. 1. As we discussed above, the \( h(t) \) distribution [shown in (c)] for the \( p5c5 \) MSCA (\( r \equiv 0.98 \) for this example) consists of almost degenerate five sharp peaks. Now, at \( r \equiv 0.94 \), clusters disintegrate and upstream sharp peaks disappears. The resulting downstream \( h(t) \) distribution [(d)] is a simple Gaussian distribution, but with sizably enhanced MSD compared with the prediction from the LLN (the hidden coherence).\(^8\) Contrastingly, the upstream distribution with apparently separated peaks [(e)] for \( p5c3 \) cluster attractor changes into an overlapping Gaussian distribution [(f)] downstream. In the case of \((p > c)\) cluster attractors, the downstream distribution clearly keep tracks of the upstream cluster attractor orbits.

(iv) The hidden-coherence is thus restricted to downstream regions of MSCAs. As noted above, MSD for MSCAs vanishes due to the population symmetry and MSD for \((p > c)\) clusters is determined by the population ratios between the clusters. In both cases, coherence governs the system, map dynamics reduce to the cluster dynamics, and MSD becomes independent of \( N \). The MSD at hidden-coherence is also independent of \( N \). The entire turbulent regime is governed by the coherence. After one knows PMs, the question why the hidden coherence exists may be reversed to why it is hidden. It may be because hidden-coherence is remnant of MSCA: Upstream, population symmetry erases the orbit information entirely, and downstream the coherence is "hidden" in the simple Gaussian distribution.\(^5\).
In this note we generically refer to the two types of cluster attractors plus their tracks observed in the MSD peak as “periodicity manifestations”. We use the MSD curve as a succinct representation of them in the analysis below.

§3. Non-locally coupled map models and periodicity manifestations

3.1. A power law model: \( \text{POW}_\alpha \)

As an extension of a GCML, let us consider the model

\[
x'_P = (1 - \varepsilon)f(x_P) + \varepsilon h_P, \quad P \in \Lambda,
\]

\[
h_P = \sum_{Q \in \Lambda} W_{PQ} f(x_Q)
\]

\[
\equiv c^{(\alpha)} f(x_P) + d^{(\alpha)} \sum_{\rho = 1}^{\rho_{\text{max}}} \frac{1}{\rho^\alpha} \sum_{P \in \Lambda_{\rho}(P)} f(x_Q).
\] (3.1)

Here the \( t \) dependence of the variables \( x_P, x_Q \) and \( h_P \) is suppressed for brevity, and \( x'_P \) represents \( x_P(t + 1) \). In a GCML, the interaction between the maps occurs via the overall system mean field \( h(t) \). In this extension, a map at a site \( P \) couples to other maps via a local mean field \( h_P(t) \). \( \Lambda_{\rho}(P) \) is a sub-lattice of \( \Lambda \) consisting of maps at an equal distance \( \rho \) from \( P \). For the simple analytic estimates given below, we approximate it by a set of points on the boundary of a \((2\rho + 1)^D\) square (cube) for \( D = 2 \) (3). The number of maps in \( \Lambda_{\rho} \) is then given by \( n_{\rho} = 2, 8\rho \) and \( 24\rho^2 + 2 \) for \( D = 1, 2 \) and 3, respectively. We impose periodic boundary conditions, and the maximum ‘radius’ of \( \Lambda_{\rho} \) is given by \( \rho_{\text{max}} = (N^{1/D} - 1)/2 \). We normalize the weights as \( \sum_{Q \in \Lambda} W_{PQ} = 1 \). This gives the constraint on the coefficients

\[
c^{(\alpha)} + d^{(\alpha)} S^{(\alpha)} = 1 \text{ with } S^{(\alpha)} = \sum_{\rho = 1}^{\rho_{\text{max}}} \frac{n_{\rho}}{\rho^\alpha}.
\] (3.2)

Normalization and the reciprocity condition \( W_{PQ} = W_{QP} \) guarantee the invariance relation (2.3). They also give the relation

\[
\frac{1}{N} \sum_{P \in \Lambda} h_P(t) = h(t); \quad (3.3)
\]

that is, the average of the local mean fields is equal to the mean field of the whole system at any time \( t \).

Let us make (3.1) an interpolation model between the GCML and the local CML. In order to coincide with the GCML at \( \alpha = 0 \), the coefficients must be \( c^{(0)} = d^{(0)} = 1/N \). The local CML is given by

\[
x'_P = f(x_P) + \frac{\varepsilon}{n_1 + 1} \left( \sum_{Q \in \Lambda_1(P)} f(x_Q) - n_1 f(x_P) \right).
\] (3.4)

In order to coincide with this for \( \alpha \to \infty \), the coefficients must be \( c^{(\infty)} = d^{(\infty)} = 1/(n_1 + 1) \). In both limits, \( c = d \). Therefore, we set \( c^{(\alpha)} = d^{(\alpha)} \) for all \( \alpha \) as the simplest
interpolation. Normalizing the couplings by (3.2), we now obtain a one-parameter extension of a GCML, called POW$\alpha$. It has the local mean field

$$h_P = \frac{1}{1 + S^{(\alpha)}} \left( f(x_P) + \sum_{\rho=1}^{\rho_{\text{max}}} \frac{1}{\rho^\alpha} \sum_{Q \in A_\rho(P)} f(x_Q) \right). \tag{3.5}$$

In order to investigate the PMs, the MSD of the system mean field $h(t)$ is computed for a 100($\varepsilon$) x 50($\alpha$) grid. At each grid point, the POW$\alpha$ is iterated from a random initial configuration, and $h(t)$ is sampled over the interval $t = 10^3 - 2 \times 10^3$. The analysis is then repeated for $D = 1, 2, 3$. The same amount of data are also generated for EXP$\rho_0$ and CML$\kappa$, carrying out the computation in the $(\rho_0, \varepsilon)$ and $(\kappa, \varepsilon)$ planes, respectively.

Fig. 2. (a) The MSD surface for POW$\alpha$ over the ($\alpha, \varepsilon$) plane for $D = 1, 2, 3$. The system size $N$ is, respectively, 51$^2$, 51$^2$ and 13$^3$, and $a = 1.90$ for all. (b) The MSD curves at the marked points (a-h) as sampled for $D = 1$. The corresponding curves on each MSD surface are depicted by bold lines. The number associated with each dominant peak specifies the periodicity of the responsible element map window. For period-three window effects, the responsible cluster attractors ($p3c3$ and $p3c2$) are indicated.

In Fig. 2(a), the resulting MSD surfaces over the ($\alpha, \varepsilon$) plane are displayed for $D = 1, 2, 3$. The leftmost sections of the three surfaces (by definition, all are the
same MSD curve in the GCML limit) exhibit the effects of periodic windows at full strength.\textsuperscript{5) } Passing from the global to the local limit, the effects diminish. This happens most quickly in the case $D = 1$ and more gradually in higher dimensions. Apart from this, the three surfaces are remarkably similar; if one picks a certain MSD curve (a section of the MSD surface) at a given value of $\alpha$ for $D = 1$, one can then find almost the same curve for $D = 2, 3$ at some other $\alpha'$ and $\alpha''$. We will see how to determine such $\alpha'$ and $\alpha''$ from $\alpha$ in Sec. 4.

We find that the $\alpha$ interval can be naturally divided into three typical regions (I, II, III), determined by the strength of window effects. (See below for details.) Furthermore, to investigate the correspondence of $\alpha$ between dimensions, it is useful to pick ‘marked points’ in the interval of $\alpha$ values, dimension by dimension, at which characteristic MSD curves are realized. Eight such marked points (a-h) are chosen, and the MSD curves (for $D = 1$) are shown in Fig. 2(b).

Region I (the GCML limit to d): Here, all kinds of PMs that occur in a GCML can be observed. In particular, both $p3c3$ MSCA and $p3c2$ cluster attractors are formed. (The formation time increases from approximately 100 steps to 1000 steps as $\alpha$ is increased.) From the GCML limit up to a, PMs emerge at full strength — they are indistinguishable from those in the GCML. From a to d, the peak-valley structure, except for that due to $p3$ clusters, gradually diminishes. The peak due to the $p5$ window starts diminishing at a and it becomes half of the original height at b. At c, all the sub-dominant peak-valley structures vanish, and even the $p5$ peak vanishes at d.

Region II (d-f): This is the region of $p3$ PMs only. Here, $p3$ clusters are not formed, but their remnants are seen as overlapping $h(t)$ distributions. This region starts from d, and the $p3c2$ peak disappears at e. At f, only a broad MSD peak is seen in the MSD curve.

Region III (f-h): This is the region of background coherence only. A broad MSD peak can be seen around the foliation zone of the $p3$ window. At f and in the $\varepsilon$ region near the top of the peak, the temporal correlator\textsuperscript{*}) decays with a $p3$ motion characterized by an exponential envelope. At g, the correlator fails to detect the periodicity everywhere, but there remains a broad MSD enhancement. At h, the MSD enhancement disappears completely.

The transition points $T_1$, $T_2$ and $T_3$ between the regions are d, f and h, respectively. We note that at the first transition point $T_1$ (d), $\alpha \approx 0.9, 1.9, 2.9$, approximately in the ratio $1 : 2 : 3$, for $D = 1, 2, 3$, respectively. We have verified using the marked points that the rule $\alpha : \alpha' : \alpha'' \approx 1 : 2 : 3$ holds to a good approximation over the entire region up to the first transition point ($\alpha \lesssim 1.0$ with $D = 1$).

\textsuperscript{*}) The correlator is defined by $C(\tau) \equiv \langle \hat{x}(t + \tau) \cdot \hat{x}(t) / |\hat{x}(t + \tau)| |\hat{x}(t)| \rangle$, with the relative vector $\hat{x}(t) \equiv (x_1(t) - h(t), \cdots, x_N(t) - h(t))$. Here, $\langle \cdots \rangle$ represents the average over the last $10^3$ steps of the iteration.\textsuperscript{5)
3.2. A coupled map lattice with exponentially decaying couplings: \( \text{EXP}_{\rho_0} \)

This model has the local mean field

\[
h_P = \frac{1}{1 + S(\rho_0)} \left( f(x_P) + \sum_{\rho=1}^{\rho_{\text{max}}} w_{\rho,\rho_0} \sum_{Q \in A_\rho} f(x_Q) \right),
\]

where \( w_{\rho,\rho_0} \equiv \exp\left(-\frac{(\rho - 1)}{\rho_0}\right) \) is the exponentially decaying coupling and \( S(\rho_0) \equiv \sum_{\rho=1}^{\rho_{\text{max}}} n_{\rho} w_{\rho,\rho_0} \). This reduces to a GCML as \( \rho_0 \to \infty \), and the local CML as \( \rho_0 \to 0 \).

As shown in Fig. 3, the PMs diminish again in passing from the global to the local limit. Furthermore, the MSD curves that appear in the process are remarkably similar to those that we saw in \( \text{POW}_\alpha \).

We note that, in an analysis of the predictability for \( D = 1 \), there is a statement in the literature that the case of an exponential law coupling is the same as the case of local coupling.\(^{18}\) However, we have verified that as \( \rho_0 \) approaches the system size, the predictability time reaches a plateau value roughly equal to that in the case of the GCML.

We will elucidate how to compare \( \rho_0 \) with \( \alpha \) at the marked points in Sec. 4.

3.3. A coupled map lattice with an interaction range \( \kappa \): \( \text{CML}_\kappa \)

The models considered above maintain the all-to-all coupling feature of a GCML. Let us now consider \( \text{CML}_\kappa \), given by

\[
h_P = \frac{1}{K} \left( f(x_P) + \sum_{\rho=1}^{\kappa} \sum_{Q \in A_\rho(P)} f(x_Q) \right),
\]

where \( K = (2\kappa + 1)^D \) is the number of maps within the range \( \kappa \). The PMs diminish again via the same pattern of MSD curves (the figure is omitted) as \( \kappa \) decreases.

In \( \text{CML}_\kappa \), we find a remarkably simple rule that relates the PMs in various dimensions. That is, \( \text{the same PMs occur for all dimensions if the neighborhood encloses the same number of maps} \). For instance, the value of range \( \kappa \) at \( T_1 \) is \( 77 - 92, 5 - 6, 2 - 3 \) for \( D = 1, 2, 3 \), respectively. These values differ significantly.

Fig. 3. The MSD surfaces for \( \text{EXP}_{\rho_0} \) over the \((\rho_0,\varepsilon)\) plane. Here, \( a = 1.90 \) and \( N = 51^2, 51^2, 13^3 \) for \( D = 1, 2, 3 \) respectively. The MSD curves at the marked points (a-h) are almost the same as those in Fig. 2(b).
However, these values of $\kappa$ correspond to the values $155 - 185, 121 - 169, 125 - 343$ for $K$, which are independent of $D$ within the errors. The large error for $D = 3$ results from the cubic dependence of $K$ on $\kappa$ in this case. To avoid this particular difficulty in the discrete range model, we have also computed the values of $K$ using a refined neighborhood, a set of lattice points $Q$ around $P$ satisfying $\sum_{i=1}^{D} (\Delta \rho_i)^2 \leq \kappa^2$. With this neighborhood, we find much better agreement: For $T_1$ we find $K = 155 - 185, 149 - 185, 147 - 179$ with $D = 1, 2, 3$, respectively. Finally, we note that $\kappa \approx 2 \rho_0$ for all marked points.

In summary, we have found the following results.
(R-I) In $POW_\alpha$, the values of $\alpha$ roughly exhibit a $1 : 2 : 3$ ratio for $D = 1, 2, 3$ at the marked points a-d ($\alpha \approx 1$ with $D = 1$).
(R-II) In $CML_\kappa$, the quantity that essentially determines the PMs is the number of maps $K$ within the interaction range $\kappa$.
(R-III) In $EXP_\rho_0$, we can interpolate between the global and local limits by varying the coupling range $\rho_0$. The value $\rho_0$ acts like the range $\kappa$ in $CML_\kappa$ ($\kappa \approx 2 \rho_0$).

§4. Universality in the periodicity manifestations

The numerical results given above can be understood in terms of a simple hypothesis.

4.1. A working hypothesis

The difference between a GCML and other non-local models is only in regard to the interaction step. In a GCML, all the maps contract uniformly to $h(t)$ by a factor $1 - \varepsilon$ at each time step, while in other non-local models, a map $f(x_P(t))$ is contracted to the local mean field $h_P(t)$, which is distributed around the overall system mean field $h(t)$ [see Eq. (3.3)]. Therefore, when the variance of $\xi_P(t) = h_P(t) - h(t)$ over the lattice is large, some distortion of the map configuration is unavoidably introduced. Contrastingly, when the variance is small at each $t$, such a distortion is avoided, and the non-local system may evolve in the same way as a GCML. Thus, we naturally expect that the deviation from the global limit is controlled by the variance of $\xi_P(t)$.

By definition, $\xi_P(t)$ is a weighted sum of maps given by

$$\xi_P = \sum_{Q \in \Lambda} \left( W_{PQ} - \frac{1}{N} \right) f(x_Q),$$  \hspace{1cm} (4.1)

where $W_{PQ}$ is the weight factor involved in taking the average of the maps $f(x_Q)$ to obtain the local mean field $h_P$ [see Eq. (3.1)]. Let us suppose that the spatial correlation between the maps is negligible at each time $t$. Then, the variance of $\xi_P$ over the lattice $\Lambda$ may be simply evaluated using $\sum_{Q \in \Lambda} W_{PQ} = 1$ as

$$\langle \xi_P \rangle_\Lambda \equiv \left( h_P - h \right)^2 \right\rangle_\Lambda \approx \mathcal{F} \left( (f(x_P) - h)^2 \right\rangle_\Lambda,$$  \hspace{1cm} (4.2)
with
\[ F \equiv \sum_Q (W_{PQ})^2 - \frac{1}{N}. \quad (4.3) \]

The factor \( F \) represents the suppression of the variance of \( \xi_P \) that occurs when taking the weighted average of independent maps characterized by the same snapshot distribution at time \( t \). In the global limit, we have \( W_{PQ} \to 1/N \) and \( F \to 0 \) (strictly no variance). For intermediate couplings and large \( N \) the factor \( 1/N \) may be ignored, and \( F \) is solely determined by the couplings of the models.

Under the two assumptions that the quantity governing the PMs is the variance of \( \xi_P(t) \) and that the correlation between the maps at each time \( t \) is negligible, we propose the working hypothesis that PMs emerge universally in all non-local models whenever the models have the same factor \( F \). Let us examine how this works below.

4.2. The factor \( F \) in each model

In CML, the suppression factor \( F \) is simply given by
\[ F = 1/K - 1/N \approx 1/K. \quad (4.4) \]

This succinctly accounts for the result (R-II). [Actually, we were led to the working hypothesis by (R-II).]

In POW, \( F \) is given by
\[ F(\alpha) = \frac{1}{(1 + S_D^{(\alpha)})^2} \left( 1 + \sum_{\rho=1}^{\rho_{\text{max}}} \frac{n_{\rho,D}}{\rho^{2\alpha}} \right) - \frac{1}{N}. \quad (4.5) \]

The leading order in \( N \) approximation for \( F(\alpha) \) are given in Table I. We find in particular \( F \approx \log N/4N \) at \( \alpha = 1/2, 1, 3/2 \) for \( D = 1, 2, 3 \), respectively. This conforms, remarkably, to the 1 : 2 : 3 rule, (R-I).

The quantity \( F^{(\rho_0)}_D \) in EXP is obtained from (4.5) by substituting \( S_D^{(\rho_0)} \) and \( w_{\rho,\rho_0}^2 \) into \( S_D^{(\alpha)} \) and \( 1/\rho^{2\alpha} \), respectively. The discussion below includes a test of the hypothesis regarding \( F^{(\rho_0)}_D \).

Table I. The leading order in \( N \) approximation of the suppression factor \( F \) in POW for values of \( \alpha \) ranging from 0 (global limit) to \( \infty \) (local limit). The top, middle, bottom rows list, respectively, \( F_\alpha \) for each dimension \( D = 1, 2, 3 \). Here, \( \zeta(x) \equiv \zeta(x) \).

<table>
<thead>
<tr>
<th>( D )</th>
<th>( \frac{3}{2} )</th>
<th>( 2 )</th>
<th>( \frac{5}{2} )</th>
<th>( 3 )</th>
<th>( 5 )</th>
<th>( \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \ln N/4N )</td>
<td>( \pi^2/12N )</td>
<td>( \zeta^2/2\zeta )</td>
<td>( \cdots )</td>
<td>( \cdots )</td>
<td>( \frac{1}{3} )</td>
</tr>
<tr>
<td>1</td>
<td>( 1/8N )</td>
<td>( \ln N/4N )</td>
<td>( \pi^2/96\sqrt{N} )</td>
<td>( \zeta^2/2\ln^2 N )</td>
<td>( \zeta^2/8\zeta^2 )</td>
<td>( \cdots )</td>
</tr>
<tr>
<td>2</td>
<td>( 1/24N )</td>
<td>( 1/3N )</td>
<td>( \ln N/4N )</td>
<td>( \pi^2/36N )</td>
<td>( \zeta^2/48N )</td>
<td>( \pi^4/240\ln^2 N )</td>
</tr>
</tbody>
</table>

4.3. Overall test of the working hypothesis

In Fig. 4(a) we first compare POW with CML at \( T_1 \) (the marked point d). We wish to test whether for each \( D \) the values of \( F \) at \( T_1 \) evaluated for POW and
CML\(\kappa\) are equal or not. Let us first consider POW\(\alpha\). The values of \(\alpha\) at \(T_1\) (for \(D = 1, 2, 3\)) are represented by the three vertical bands (with errors resulting from the ambiguity involved in the pattern identification of MSD curves). Then, for each \(D\), the point at which the \(F\) curve and the vertical band cross represents the value of \(F\) in POW\(\alpha\) at \(T_1\). The case of CML\(\kappa\) is simpler, because the transition occurs at approximately the same value of \(K\) for any \(D\). The value of \(F\) for CML\(\kappa\) at \(T_1\) is evaluated using (4.4) for each \(D\) and then averaged over the value of \(D\). This is represented by the single horizontal band. Now, if every curve passes through the respective crossing junction of horizontal and vertical bands, it is implied that, for each \(D\), POW\(\alpha\) and CML\(\kappa\) have the same \(F\).

We find the three curves pass through respective junctions quite accurately. The largest deviation occurs for \(D = 1\), where the evaluated values of \(F\) are \((8.5 \pm 1.5) \times 10^{-3}\) and \((5.5 \pm 0.5) \times 10^{-3}\) for POW\(\alpha\) and CML\(\kappa\), respectively.

In Fig. 4(b) we compare CML\(\kappa\) with POW\(\alpha\) in the left panel and CML\(\kappa\) with EXP\(\rho_0\) in the right panel, now with respect to all eight marked points. To facilitate this overall comparison avoiding the overlap of vertical bars from different dimensions, we omit the vertical bands entirely and depict only the crossing junctions between horizontal and vertical bars by crossed error-bars. We find that the curves pass through the junctions quite accurately. We thus conclude that the hypothesis...
accounts for the observed behavior of PMs with respect to all marked points [with \( F \) ranging from \( 10^{-4} \) to \( O(1) \)] and for \( D = 1, 2, 3 \). This in particular explains (R-III).

A few remarks are in order.

(a) The 1 : 2 : 3 rule in \( \text{POW}_\alpha \): The curves in \( \text{POW}_\alpha \) approximately coincide after a scale transformation of \( \alpha \) made in the ratio 1 : 1/2 : 1/3 up to \( F_D \approx 2 \times 10^{-2} \). This explains (R-I).

(b) Missing transition points: In \( \text{CML}_\kappa \), using a refined neighborhood, all three transition points are observed for \( D = 1, 2, 3 \). These are indicated by the horizontal bands. Note that above \( T_3 \), collective behavior is absent. In the other two models, which use a coarse neighborhood, all transition points are still observed for \( D = 1 \) but \( T_3 \) is missing for \( D = 2 \), and both \( T_2 \) and \( T_3 \) are missing for \( D = 3 \). The curves of \( F \) explain the difference succinctly: They are constrained by the limiting values \( 1/3^D \) (see Table I), so they can pass through only the lowest two (one) bands for \( D = 2 \) (3). We have numerically checked that the missing transition points are retrieved in both \( \text{POW}_\alpha \) and \( \text{EXP}_\rho_0 \) using refined neighbors.

(c) Approximation check: Let us determine if our assumption that the spatial correlation is negligible is legitimate. Firstly, we should note that clustering such as \( p_{3c} \) is in the map values and not in the spatial distribution. We have checked that over the entire turbulent regime of the three models, no visible spatial clusters are formed. Furthermore, we have directly checked that the approximation in (4.2) is quite accurate in three models. The case for \( \text{POW}_\alpha \) is presented in Fig. 5. The

![Fig. 5. In \( \text{POW}_\alpha \), the ratio \( \left\langle \xi_P^2 \right\rangle_A \equiv \left\langle (h_P-h)^2 \right\rangle_A \) averaged over 100 steps is compared with \( F^{(\alpha)} \), varying \( \alpha \) from 0.5 to 8.0 with \( \Delta \alpha = 0.5 \) (increasing \( F^{(\alpha)} \)). For \( D = 1 \), \( \alpha = 0.3 \) is added. Points from 10 random initial configurations are overlaid. Boxes (a), (b), (c) and (d) for \( \varepsilon = 0.02, 0.08, 0.0352 \) and 0.045, respectively.](image)
left two boxes there contain the results for \( \varepsilon = 0.02 \) and 0.08, where no visible synchronization occurs, and the right two boxes contain the results for \( \varepsilon = 0.0352 \) and 0.045, where, respectively, the \( p3c3 \) and \( p3c2 \) states are formed (for \( F < 10^{-2} \)). We find that for all dimensions and for all coupling ranges, the approximation is quite accurate for values of \( F \) from \( 10^{-5} \) to \( O(1) \), which fully covers the range of \( F \) in Fig. 4. The spread of the data points observed in the \( p3c2 \) cluster attractor is due to the different ratios of the map populations in the two clusters formed from different initial configurations.

(d) Inhomogeneous map lattices: Even if inhomogeneity is introduced randomly to the nonlinearity of maps \( (\alpha_p \rightarrow \alpha_p \pm \delta \alpha_p) \), none of the features of the PMs discussed in this note are changed for \( \delta \alpha_p < \delta \alpha = 0.01 \). For larger \( \delta \alpha \), the effect is similar to that of a decrease of non-locality. For instance, with \( \delta \alpha = 0.03 \), the sequence of marking points starts from the point d (rather than a). We have also verified that the PMs occur in a non-local CML of circle maps.

§5. Conclusion

In this note we have focused our attention on the recently discovered periodicity manifestations in the turbulent regime of a GCML. We have conducted an extensive statistical analysis of three non-locally coupled map lattices with \( D = 1, 2, 3 \) and examined to what extent they depend on the non-locality of the models. We have noted that the essential deviation of the non-local CML from the GCML results from the variance of the local mean field around the overall mean field. We have analytically evaluated the suppression factor \( F \) of the variance under the approximation that the spatial correlation of maps in the turbulent regime is negligible and checked that this values of \( F \) agrees quite closely with the numerical result. We have found the salient universality feature that for a fixed value of \( F \), the periodicity manifestations exist at the same value of the strength for any type of non-local coupling or for any spatial dimension of the lattice, to a good approximation.

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References

16) H. Nakao, Chaos 9 (1999), 902.
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