An introduction to symplectic geometry

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Abstract

The intent of this series of lectures is two-fold: in the first week, we will provide a quick overview of equivariant symplectic geometry, starting at the very beginning (i.e. with the definition of a symplectic structure). In the second week, we will give a series of loosely connected expository overviews of some themes that consistently arise in current research in this field. The purpose is to familiarize the audience with the basic tools and language of the field.

Topics will include (time permitting): In the first week, we will discuss the definition of a symplectic structure, examples of symplectic manifolds, local normal forms, group actions, Hamiltonian actions, moment maps, symplectic quotients, and Delzant's construction of symplectic toric manifolds. In the second week, we will discuss equivariant cohomology and equivariant Morse theory, localization, moment graphs and GKM theory, Duistermaat-Heckman measure, Kirwan surjectivity. I hope to also discuss related quotient theories (e.g. Kähler and hyperKähler quotients), time permitting.

Preface

These are informal, rough lecture notes from an intensive two-week summer school on symplectic geometry given at Osaka City University in July 2007. Many thanks are due to Professor Mikiya Masuda for taking the initiative to organize the summer school, arranging for notes from my blackboard lectures to be typed up into LaTeX form, and helping to provide an excellent friendly atmosphere throughout the two weeks. I am grateful to Mr. Kaname Hashimoto for his patient LaTeXing as well as his many beautiful figures. Needless to say, similar thanks are also due to all the other participants, whose many questions and good energy made my weeks in Osaka very pleasurable.

Megumi Harada

Lectures on Symplectic Geometry

Prerequisite : language of differential geometry, de Rham theory, Lie groups, Lie group actions.
References :

- Ana Cannas de Silva, *Lectures on symplectic geometry*, Springer LNM.
- R. Bryant, *An Introduction to Lie Groups and Symplectic Geometry*, in Geometry and quantum theory, IRS/Park City Math, vol.I.
- V. Guillemin, V. Ginzburg, Y. Karshon, *Moment Maps, Cobordisms, and Hamiltonian Group Actions*, AMS, MSM vol.98.

Further reading :

- McDuff, D. Salamon, *Introduction to Symplectic Topology.*
- Guillemin, *Moment maps and combinatonal invariants of Hamiltonian T-spaces.*
- V. Guillemin, E. Lerman, S. Sternberg, *Symplectic Fibrations and Multiplicity Diagrams*, Cambridge University Press, 1996.

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Contents

1 Symplectic Linear Algebra

Let V be an m-dimensional vector space over $\mathbb{R}$. Let $\Omega : V \times V \longrightarrow \mathbb{R}$ be a bilinear pairing. The map Ω is **skew-symmetric** if $\Omega(u,v) = -\Omega(v,u)$ for all $u, v \in V$.

Theorem 1.1. (Standard form for skew-symmetric pairings)
Let Ω be a skew-symmetric bilinear map on V. Then there is a basis $u_1, \ldots, u_k, e_1, \ldots, e_n, f_1, \ldots, f_n$ on V such that

1. $\Omega(u_i, v) = 0, \qquad \big({}^\forall v \in V, \quad 1 \le i \le k\big).$
2. $\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0 \qquad (1 \le i, j \le n).$
3. $\Omega(e_i, f_j) = \delta_{i,j} \qquad (1 \le i, j \le n).$

NOTES .

- In matrix notation with respect to such a basis,

$$\omega(u,v) = \begin{pmatrix} & u & \end{pmatrix} \begin{array}{c c} & \begin{array}{ccc} k & n & n \end{array} \\ \begin{array}{c} k \\ n \\ n \end{array} & \begin{pmatrix} 0 & & \\ & 0 & Id_n \\ & -Id_n & \end{pmatrix} \end{array} \begin{pmatrix} \\ v \\ \\ \end{pmatrix}.$$

  Proof. By induction using a skew-symmetric version of the Gram-Schmidt. [Exercise] □

- The dimension of the subspace

$$\mathfrak{U} := \big\{ u \in V \mid \Omega(u,v) = 0, \quad {}^\forall v \in V \big\}$$

  is independent of the choice of basis.

- Similarly n is independent of the choice of basis. This n is called the **rank** of Ω.

Given any bilinear pairing $\Omega : V \times V \longrightarrow \mathbb{R}$, we may define a map

$$\begin{aligned} \widetilde{\Omega} : \quad & V \longrightarrow V^* \\ & v \longmapsto [\widetilde{\Omega}(v)(u) := \omega(u,v)] \end{aligned}$$

Clearly, $\mathrm{Ker}(\widetilde{\Omega}) = \mathfrak{U}$.

Definition 1.1. A skew symmetric bilinear map $\Omega : V \times V \longrightarrow \mathbb{R}$ is **symplectic** (or non-degenerate) if $\widetilde{\Omega}$ is bijective (equivalently $\mathfrak{U} = \{0\}$). We call Ω a **linear symplectic structure** and (V, Ω) a **symplectic vector space**.

NOTES .

- By the standard form theorem, $\mathrm{dim} V = 2n$ is even.
- **A symplectic basis** of (V, Ω) satisfies

$$\Omega(e_i, f_j) = \delta_{i,j}, \qquad \Omega(e_i, e_j) = \Omega(f_i, f_j) = 0 \qquad ({}^\forall i, j\ ).$$

- Special subspaces of symplectic vector spaces
  - $W \subset V$ symplectic if $\Omega|_W$ is non-degenerate.
  - $W \subset V$ isotropic if $\Omega|_W \equiv 0 \quad \big(\text{i.e. } W \subseteq W^\Omega\big).$

[Exercise] Prove if W is isotropic then $\dim W \leq \frac{1}{2}\dim V$.

If W has dimension $= \frac{1}{2}\dim V$, then W is called **Lagrangian**. $W \subset V$ is coisotropic if $W^{\Omega} \subseteq W$, where

$$W^{\Omega} := \left\{ \, v \in V | \; \Omega(v,w) = 0, \quad {}^{\forall} w \in W \right\}.$$

- A symplectomorphism between (V_1, Ω_1) and (V_2, Ω_2) symplectic vector spaces is a linear map $\varphi : V_1 \longrightarrow V_2$, which is a linear isomorphism and $\varphi^* \Omega_2 = \Omega_1$.

2 Symplectic manifolds : definitions and examples

Now, we wish to go from local description to a global description on a manifold.
A "manifold" in this course is C^∞(smooth), Hausdorff, and second-countable. Let M be a manifold and $\omega \in \Omega^2(M)$ a de Rham 2-form, i.e. ${}^\forall p \in M, \omega_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ skew bilinear.

Definition 2.1. The 2-form ω is **symplectic** if

- ω_p is symplectic for all $p \in M$ (i.e. non-degenerate).
- $d\omega = 0$.

("ω is closed and non-degenerate")

Then **a symplectic manifold** is a pair (M, ω) where M is a manifold, ω is a symplectic form on M.

Example 2.1.

(1) Let $M = \mathbb{R}^{2n}$, linear coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$. Then

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

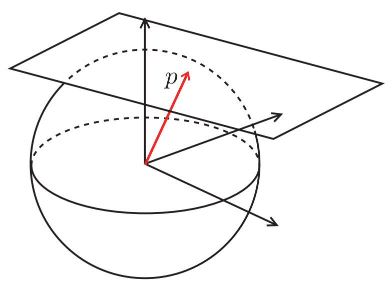
(2) Let $M = \mathbb{C}^n$, complex coordinates $(z_1, \ldots, z_n)$. Then

$$\omega_0 = \frac{\sqrt{-1}}{2} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k.$$

In fact, we can identify (1) and (2) by using the identification

$$\mathbb{C}^n \ni z_k = x_k + \sqrt{-1} y_k \overset{\cong}{\longmapsto} (x_k, y_k) \in \mathbb{R}^n.$$

(3) Let $M = S^2 \subset \mathbb{R}^3$ be the unit sphere. Identify the tangent space to p with vectors in $\mathbb{R}^3$ orthogonal (w.r.t. the standard inner product) to p.



Then $\omega_p(v, w) := \langle p, v \times w \rangle$ defines a closed non-degenerate differential 2-form.

[Exercise] Check assertions in (1), (2) and (3).

(4) The canonical symplectic structure on a cotangent bundle to a manifold.

Let X be an n-dimensional manifold. Then its cotangent bundle $M = T^*X$ is $2n$-dimensional and has a canonical symplectic structure. To describe this, use local coordinates. Suppose

$(\mathfrak{U}, x_1, \ldots, x_n)$ is a local coordinate chart on X. Then $(T^*\mathfrak{U}, \underbrace{x_1, \ldots, x_n}_{\text{base}}, \underbrace{\xi_1, \ldots\ldots, \xi_n}_{\substack{\text{cotangent fiber} \\ \text{coordinates}}})$ is an associated local chart for $M = T^*X$. In these coordinates, define

$$\alpha = \sum_{i=1}^{n} \xi_i dx_i.$$

The 1-form α is "canonical 1-form".

NOTE . It can be checked explicitly that α is independent of choice of coordinates [so in particular defines a global 1-form on M]

(Better way) There is also a coordinate-free description of α: let $p = (x, \xi) \in T^*X$ $(x \in X, \xi \in T_x^*X)$. Consider the vector bundle projection.

$$\begin{array}{ccc} M = T^*X & & T(T^*X) \\ \pi \downarrow & \rightsquigarrow_{\text{take } d} & \downarrow d\pi \\ X & & TX \end{array}$$

NOTE . $\pi : (x, \xi) \longmapsto x$

$$\begin{array}{ccc} T_p(T^*X) & \cong & T_xX \bigoplus T_x^*X \\ & & \downarrow d\pi \\ & & T_xX \end{array}$$

i.e. $d\pi$ "is" projection to the first factor. Then

$$\alpha_p = (d\pi_p)^*(\xi) = \xi \circ d\pi.$$

Then the canonical symplectic structure on T^*X is

$$\omega := -d\alpha.$$

NOTES .

- In coordinates, $\omega := \displaystyle\sum_{i=1}^{n} dx_i \wedge d\xi_i$ (looks like $\mathbb{R}^{2n}$ example).
- ω is exact, hence closed.

[Exercise] A diffeomorpism $\varphi : X \longrightarrow X$ "lifts" to a symplectomorphism $(d\varphi)^* : T^*X \longrightarrow T^*X$. In other words, $(d\varphi)^*$ is a diffeomorphism and preserves the symplectic structure, i.e. $\varphi^*\omega = \omega$, where ω is the canonical symplectic structure on T^*M.

(5) (WARNING; a bit more advanced) "Coadjoint orbits of Lie groups"

Let G be a compact Lie group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. Denote by $\langle\ ,\ \rangle : \mathfrak{g}^* \times \mathfrak{g} \longrightarrow \mathbb{R}$ the natural pairing.

- Let $\mathrm{Ad} : G \times \mathfrak{g} \longrightarrow \mathfrak{g}$ be the adjoint representation of G on $\mathfrak{g} = T_eG$ induced by action of conjugation (G action on itself).
- Let $\mathrm{Ad}^* : G \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ be the coadjoint representation induced by Ad, defined by $\langle \mathrm{Ad}^*(g, \xi), X\rangle \underset{\text{def}}{=} \langle \xi, \mathrm{Ad}(g^{-1}, X)\rangle$, for all $\xi \in \mathfrak{g}^*, X \in \mathfrak{g}$.

[Exercise]

(a) Let $X^\sharp$ denote the vector field generated by $\{\exp tX\} \subseteq G$ on $\mathfrak{g}^*$, and $X^\sharp_\xi$ is the value of $X^\sharp$ at $\xi \in \mathfrak{g}^*$ in $T_\xi\mathfrak{g}^* \cong \mathfrak{g}^*$. Then $\langle X^\sharp_\xi, Y\rangle = \langle \xi, [X, Y]\rangle$ for all $Y \in \mathfrak{g}$.

(b) For any $\xi \in \mathfrak{g}^*$, define a skew-symmetric bilinear form on $\mathfrak{g}$ by

$$\omega_\xi (X, Y) := \langle \xi, [X, Y] \rangle.$$

Then the kernel of ω_ξ is exactly the Lie algebra of this stabilizer of ξ for the coadjoint representation.

(c) ω_ξ defines a non-degenerate, closed 2-form on the **orbit** of G through ξ in $\mathfrak{g}^*$.
(orbit of $\xi = G \cdot \xi \cong G/G_\xi$)

HINT: Closedness is a consequence of the Jacobi identity in $\mathfrak{g}$. Non-degeneracy follows from (b).

This is called **the canonical symplectic structure** or **Kostant-Kirillov-Souriau symplectic structure** on coajoint orbits.

[Exercise] Work out specifically what (a)-(c) mean for $G = U(n, \mathbb{C})$, $\mathfrak{g} = \sqrt{-1}\mathfrak{H}$ is the set of skew-hermitian $n \times n$ complex matrices. The coadjoint orbits in this case are of the form

$$\left\{ A \in \sqrt{-1}\mathfrak{H} \mid \text{spectrum is } \sqrt{-1}(\lambda_1, \lambda_2, \dots, \lambda_n), \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n, \lambda_i \in \mathbb{R} \right\}.$$

These are **partial flag manifolds**.

Recall : The full flag manifold in $\mathbb{C}^n$ is the manifold of

$$\{0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n \mid \dim_{\mathbb{C}} V_i = i\}.$$

(The correspondence is given by looking at the eigenspaces of A)

3 A (VERY) brief discussion of uniqueness and existence issues

For more details, see Bryant's lecture on the topic, "Symplectic Manifolds II".

$\boxed{\text{Question}}$ Suppose M is a manifold, and ω_0, ω_1 are both symplectic structures on M. Does there exist a diffeomorphism $\psi : M \longrightarrow M$ such that $\psi^*(\omega_1) = \omega_0$?

Partial answer : "Moser trick'

(There are many version. We will write a special case.)

Theorem 3.1. Let M be a compact symplectic manifold, with symplectic structures ω_0 and ω_1. Suppose that

- $\omega_t = (1-t)\omega_0 + \omega_1$ is symplectic for all $t \in [0,1]$
- $[\omega_0] = [\omega_1] \in H^2_{dR}(M)$.

Then there is a diffeomrphism $\psi : M \longrightarrow M$ such that $\psi^*(\omega_1) = \omega_0$.

Proof. (sketch of proof)

- There is $\beta \in \Omega^1(M)$ a 1-form such that $\omega_0 - \omega_1 = d\beta$.
- By non-degeneracy of ω_t for all t, there is a unique vector field $X_t \in \mathrm{Vect}(M)$ such that $\beta = i_{X_t}\omega_t$.
- Let ρ be the isotorpy (i.e. 1-parameter flow) along X_t. Set $\psi = \rho_1$.

Claim : $\psi^*(\omega_1) = \omega_0$.

THE IDEA Notice $\rho_0 = id$, $\rho_0^*\omega_0 = \omega_0$.

Claim : $\left.\frac{d}{dt}\right| \rho_t^*\omega_t = 0$.
Compute

$$\frac{d}{dt}\left(\rho_t^*\omega_t\right) = \underbrace{\mathcal{L}_{X_t}\omega_t}_{di_{X_t}w_t} + \underbrace{\frac{d\omega_t}{dt}}_{\omega_1-\omega_0}.$$

Everything is chosen such that this is zero.

∎

The following theorem follows from Moser's trick. Roughly :

"All symplectic manifolds locally look the same"

Theorem (Darboux)/ Corollary(Moser)
Let (M,ω) be a symplectic manifold and let $p \in M$. Then there exists a coordinate chart $(\mathfrak{U}, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at p such that on $\mathfrak{U}$,

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Remark : In other words, all symplectic manifolds locally look like $(\mathbb{R}^n, \omega)$ as in Example (1) yesterday. In particular, there are no local invariants for symplectic manifolds (compared to e.g.

curvature in Riemannian geometry).

$\boxed{\text{Question}}$ When does a manifold M admit a symplectic structure?

Partial answers :

- The manifold must be even-dimensional, from symplectic linear algebra.
- Here is a simple cohomological condition for M compact. Suppose M^{2n} is symplectic, with form ω. Then ω^n is a $\underbrace{\text{volume form}}_{\text{top-dim'l non-vanishing}}$ on M. [This follows from non-degeneracy and the standard form.] Hence $[\omega^n] \neq 0 \in H^{2n}_{dR}(M) \Longrightarrow [\omega] \neq 0 \in H^{2n}_{dR}(M)$.

Example 3.1. S^{2n} for $n > 1$ **can not** be symplectic.

4 Symplectic group actions

Recall/ Notation :

- A **(left) action** of **Lie group** G **on** M is a group homorphism
$$\psi : G \ni g \longmapsto \psi_g \in \mathrm{Diff}(M).$$
- We often use the evaluation map instead
$$G \times M \ni (g, m) \longmapsto \psi_g(m) =: g \cdot m \in M.$$
The action is **smooth** if this map is smooth.
- In particular, for $G = \mathbb{R}$, we call $\{\psi_t\}$ a 1-parameter group of diffeomorphisms and there is a 1-1 correspondence as follows :
$$\begin{array}{ccc} \{\text{complete vector fields on } M\ \} & \underset{\text{1-1}}{\longleftrightarrow} & \{\text{smooth actions of } \mathbb{R} \text{ on } M\ \} \\ X_p := \left.\dfrac{d}{dt}\right|_{t=0} \psi_t(p) & \longleftarrow & \psi \\ X & \longrightarrow & \exp tX \end{array}$$

For us, we want something more.

Definition 4.1. An action $\psi : G \longrightarrow \mathrm{Diff}(M)$ is a **symplectic action** if
$$\psi : G \longrightarrow \mathrm{Symp}(M) \subseteq \mathrm{Diff}(M),$$
where $\mathrm{Symp}(M)$ is the subgroup of diffeomorphisms preserving ω.

Example 4.1.

(1) $M = \mathbb{R}^{2n},\ \omega_0 = \displaystyle\sum_{i=1}^{n} dx_i \wedge dy_i$. Consider the translation action
$$\begin{array}{ccc} \mathbb{R} \times M & \longrightarrow & M \\ (t, (x_1, y_1, x_2, y_2, \cdots, x_n, y_n)) & \longmapsto & (x_1, y_1 + t, x_2, y_2, \cdots x_n, y_n) \end{array}$$
$$\text{(shift the } y_1 \text{ only)}$$
then it is easy to check for any $t \in \mathbb{R},\ \psi_t^* \omega_0 = \omega_0$

(2) $M = \mathbb{C}$ with standard $\omega_0 = \dfrac{\sqrt{-1}}{2} dz \wedge d\bar{z}$. Let $G = S^1 = \{\ \lambda \in \mathbb{C} \ \mid\ |\lambda| = 1\}$, the circle act by complex multiplication
$$\begin{array}{ccc} S^1 \times \mathbb{C} & \longrightarrow & \mathbb{C} \\ (\lambda, z) & \longmapsto & (\lambda \cdot z = \underbrace{\lambda z}_{\mathbb{C}\text{ multiplication}}) \end{array}$$

Check : $\psi_\lambda^* \omega_0 = \omega_0$.

4.1 Hamiltonian functions

$\boxed{\text{BUT}}$ How do you construct symplectic group actions?

"What do symplectic geometers do with a function?"

By the non-degeneracy of ω, there is a way to move between

$$\begin{array}{ccc} \{\text{vector fields on } (M,\omega)\} & \longleftrightarrow & \{\text{1-forms on } (M,\omega)\} \\ \| & & \| \\ \mathit{\Gamma}(TM) & \underset{\widetilde{\omega}:TM\cong T^*M}{\longleftrightarrow} & \mathit{\Gamma}(T^*M) \end{array}.$$

Hence we can construct 1-parameter groups of diffeomorphisms as follows:

Step 1 :
Given $H \in C^\infty(M)$, consider the 1-form $dH \in \Gamma(T^*M) = \Omega^1(M)$.

Step 2 :
By the non-degeneracy, there is a unique vector field $X_H \in \Gamma(TM) = \mathrm{Vect}(M)$ such that $dH = i_{X_H}\omega = \widetilde{\omega}(X_H)$. X_H is called "the Hamiltonian vector field associated to H".

Step 3 :
(Technical point: assume M compact or that X_H complete) Integrate X_H to a 1-parameter family $\{\rho_t\}$ of diffeomorphisms $\rho_t : M \longrightarrow M$ such that $\rho_0 = id_M$ and $\left.\dfrac{d}{dt}\right|_{t=0} \rho_t(p) = X_H(p)$.

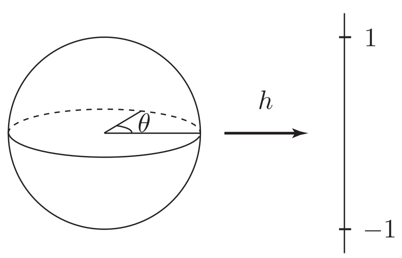
Claim : $\{\rho_t\}$ is a symplectic $\mathbb{R}$-action.

Proof. We wish to show $\rho_t^*\omega = \omega$ for all t.

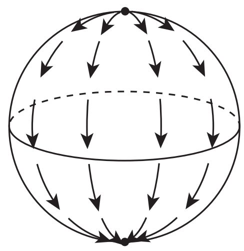
$$\begin{aligned} \frac{d}{dt}(\rho_t^*\omega) &= \rho_t^*\mathcal{L}_{X_H}\omega \\ &\underset{\text{Cartan}}{=} \rho_t^*\left(\underbrace{di_{X_H}\omega}_{d^2H=0} + \cancel{i_{X_H}d\omega}^{\,0}\right) \\ &= 0 \end{aligned}$$

∎

Example 4.2. The difference between Riemannian and symplectic geometry (at least, in terms of generating vector fields!).
$M = S^2, \omega =$ area from away from poles $= d\theta \wedge dh$
$h =$ height, $\theta =$ angle

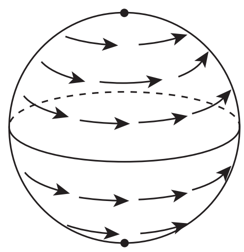


Consider the Hamiltonian function $H = h$, the height function on the sphere S^2. Then the north and south poles are the critical points (so the gradient vector field $= 0$ there). But what happens elsewhere? If we deal with "usual" Riemannian geometry, think of h as a Morse function, use usual metric on S^2, then



But in symplectic geometry, we ask for vector field such that

$$dh = i_{X_h}(d\theta \wedge dh) \Longrightarrow X_h = \frac{\partial}{\partial \theta}.$$



(This is the difference between using symmetric versus skew-symmetric pairing.)

NOTES .

- The Hamiltonian vector field $X_h = \dfrac{\partial}{\partial \theta}$ generates an $\mathbb{R}$-action that is **periodic**, i.e. an S^1-action on S^2 by rotation.
- The level sets of the Hamiltonian function are **preserved** by this S^1-action. Indeed, in general
$$dh(X_h) = i_{X_h}\omega(X_{X_h}) = \omega(X_h, X_h) = 0$$
- The fixed points $\{N, S\}$ of the S^1-action are exactly the critical points of h. i.e.
$$\mathrm{Crit(h)} = (S^2)^{S^1}.$$

All of these observations (and generalizations) will be useful to us later.

4.2 Symplectic versus Hamiltonian vector fields

In §4.1 we generated a symplectic $\mathbb{R}$-action $\{\rho_t\}$ using a Hamiltonian function. But they don't all have to arise this way.

Definition 4.2. A vector field X on M preserving ω. i.e. $\mathcal{L}_X\omega = 0$ is called **a symplectic vector field**. So certainly

$$\text{Hamiltonian vector field} \Longrightarrow \text{symplectic vector field.}$$

Looking again at same computation :

$$\begin{aligned}\mathcal{L}_X\omega &= di_X\omega + \cancelto{0}{i_X d\omega} \\ &= di_X\omega,\end{aligned}$$

we immediately see

$$\begin{aligned} X \text{ is symplectic} &\iff i_X\omega \text{ closed} \\ X \text{ is Hamiltonian} &\iff i_X\omega \text{ exact (i.e.} i_x\omega = dH)\end{aligned}$$

So :

- Any symplectic vector field is **locally** Hamiltonian.
- $H^1_{dR}(M)$ is the obstruction for symplectic vector field to be Hamiltonian.

Example 4.3. $M = (S^1 \times S^1, (\theta_1, \theta_2))$, $\omega = d\theta_1 \wedge d\theta_2$, the S^1-actions twisting coordinates separately. $X_1 = \dfrac{\partial}{\partial\theta_1}, X_2 = \dfrac{\partial}{\partial\theta_2}$ are symplectic but NOT Hamiltonian.

So

$$\mathrm{Vect}^{\mathrm{Ham}}(M) \subseteq \mathrm{Vect}^{\mathrm{Symp}}(M) \subseteq \mathrm{Vect}(M) \tag{1}$$

Next theorem : (motivated by our interest in G-action on $M \rightsquigarrow \mathfrak{g} \to \mathrm{Vect}(M)$)
Analyze the interaction of these subspaces with the natural Lie bracket on $\mathrm{Vect}(M)$.

Recall : Vector fields are differential operators on $C^\infty(M)$ defined by :

$$X \cdot f := \mathcal{L}_X f = df(X).$$

Moreover, given two vector fields X, Y, we define the Lie bracket $[X,Y]$ as follows :

$$\mathcal{L}_{[X,Y]} = \mathcal{L}_X(\mathcal{L}_Y f) - \mathcal{L}_Y(\mathcal{L}_X f) = [\mathcal{L}_X, \mathcal{L}_Y]f \qquad \big({}^{\forall} f \in C^\infty(M)\big)$$

FACT : For any form α on M,

$$\begin{aligned} i_{[X,Y]}\alpha &= \mathcal{L}_X i_Y\alpha - i_Y\mathcal{L}_X\alpha \\ &= [\mathcal{L}_X, i_Y]\alpha\end{aligned}$$

$\boxed{\text{Question}}$ How does this bracket behave with respect to these inclusions (1) ?

Theorem 4.1. Let (M, ω) be symplectic, $X, Y \in \mathrm{Vect}^{\mathrm{Symp}}(M)$. Then $[X,Y] \in \mathrm{Vect}^{\mathrm{Ham}}(M)$.

Proof.

$$\begin{aligned} i_{[X,Y]}\omega &= \mathcal{L}_X i_Y\omega - i_Y\mathcal{L}_X\omega \\ &= di_X i_Y\omega - i_X \cancelto{0}{di_Y d\omega} + i_Y \cancelto{0}{i_X d\omega} - i_Y di_X \cancelto{0}{\omega} \\ &= d(\omega(Y,X)).\end{aligned}$$

Here $\omega(Y,X)$ is a Hamiltonian function for $[X,Y]$. $\square$

Corollary 4.1. $\mathrm{Vect}^{\mathrm{Ham}}(M) \subseteq \mathrm{Vect}^{\mathrm{Symp}}(M) \subseteq \mathrm{Vect}(M)$ inclusions are Lie algebra homomorphisms. Expressions of the form $\omega(X,Y)$ have another name for $X, Y \in \mathrm{Vect}^{\mathrm{Ham}}(M)$. We have already discussed the map

$$\begin{array}{ccc} C^\infty(M) & \longrightarrow & \mathrm{Vect}(M) \\ f & \longmapsto & X_f \end{array} \tag{2}$$

Definition 4.3. The Poisson bracket of $f, g \in C^\infty(M)$ is (on a sympelctic (M, ω)).

$$\{f, g\} := \omega(X_f, X_g) \in C^\infty(M).$$

[Exercise] $\{\ ,\ \}$ makes $C^\infty(M)$ a Lie algebra. i.e. satisfies Jacobi.

Corollary 4.2. (of above Theorem) (2) is a Lie algebra **anti**-homomorphism.

Proof. From above computation

$$X_{\{f,g\}} = X_{\omega(X_f, X_g)} = [X_g, X_f] = -[X_f, X_g].$$

$\square$

NOTE . $\varphi([X, Y]) = -[\varphi(X), \varphi(Y)]$.

5 Moment maps and Hamiltonian group action

5.1 Definitions

Recall : We know what it means for an $\mathbb{R}$-action to be Hamiltonian.

$\boxed{\text{Question}}$ What "should" we mean for an ordinary compact, connected Lie group to "act Hamiltonianly"?

Idea : To say the condition in terms of the 1-parameter subgroups, but then place an extra condition to ensure compatibility with the Lie **group** structure on G.

Suppose $G \curvearrowright (M, \omega)$ symplectic, i.e. group homomorphism $G \longrightarrow \mathrm{Symp}(M, \omega)$. Then for any $X \in \mathfrak{g}$ consider the corresponding $\{\exp tX\} \subseteq G$. Let $X^\sharp$ denote the vector space on M, generated by $\{\exp tX\}$. From before, the $\mathbb{R}$-action on M is Hamiltonian exactly when there is a lift

$$\begin{tikzcd} C^\infty(M) \ni \Phi(X) \arrow[r] & X^\sharp \in \mathrm{Vect}(M) \\ & X \in \mathfrak{g} \arrow[u] \arrow[ul, dotted] \end{tikzcd}$$

such that $d(\Phi(X)) = i_{X^\sharp}\omega$. (i.e. $\Phi(X)$ is the Hamiltonian function generating $X^\sharp$.)

Definition 5.1. Let G be a compact, connected Lie group, acting symplectically on (M, ω). The action is **Hamiltonian** if there exists a **comoment map** such that

$$\begin{tikzcd} C^\infty(M) \arrow[r] & \mathrm{Vect}(M) \\ & \mathfrak{g} \arrow[u] \arrow[ul, "\Phi^*"] \end{tikzcd}$$

commutes, and

1. For all $X \in \mathfrak{g}, \Phi^*(X) := \Phi^X$ is a Hamiltonian function for $X^\sharp$.
2. Φ^* is a Lie algebra homomorphism (standard bracket on $\mathfrak{g}$, Poisson bracket on $C^\infty(M)$). In other words,
$$\Phi^*([X,Y]) = \left\{\Phi^X, \Phi^Y\right\}.$$

NOTES .

- Condition 1 says that each $\{\exp tX\}$ is Hamiltonian.
- Condition 2 says that the map is compatible with the Lie group structure on G.
- This is the "dual" version of the usual definition of a moment map.

Preliminaries/ Recall : Any Lie group G acts on itself by conjugation :

$$G \times G \ni (g, a) \longmapsto gag^{-1} \in G$$

The derivate at $1 \in G$ of $\psi_g : a \longmapsto gag^{-1}$ is a linear map

$$\mathrm{Ad_g} : \mathfrak{g} = T_1 G \longrightarrow \mathfrak{g}, \quad \text{so } \mathrm{Ad_g} \in \mathrm{GL}(\mathfrak{g}).$$

Letting $g \in G$ vary, we get the **adjoint representation**

$$\begin{array}{rcl} \mathrm{Ad} : G & \longrightarrow & \mathrm{GL}(\mathfrak{g}) \\ g & \longmapsto & \mathrm{Ad_g} \end{array}$$

Moreover,

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp tX}(Y) = [X, Y], \qquad {}^\forall X, Y \in \mathfrak{g}.$$

Now consider the natural pairing

$$\begin{array}{ccc} \mathfrak{g}^* \times \mathfrak{g} & \longrightarrow & \mathbb{R} \\ (\xi, X) & \longmapsto & \xi(X) \end{array}$$

We define the coadjoint representation $\mathrm{Ad}^* : G \longrightarrow \mathrm{GL}(\mathfrak{g}^*)$ by, given $\xi \in \mathfrak{g}^*$

$$\langle \mathrm{Ad}^* \xi, X \rangle = \langle \xi, \mathrm{Ad}_{g^{-1}} X \rangle \qquad {}^\forall X \in \mathfrak{g}.$$

Now note that we can repackage the comoment map information $\Phi^* : \mathfrak{g} \longrightarrow C^\infty(M)$ by

$$\Phi : M \longrightarrow \mathfrak{g}^*$$

such that for all $X \in \mathfrak{g}$,

$$\begin{aligned} \langle \Phi(m), X \rangle &\stackrel{\text{def}}{=} (\Phi^*(X))(m) \\ &= \Phi^X(m). \end{aligned}$$

Definition 5.2. Let G be a compact, connected Lie group, acting symplectically on (M, ω). The action is **Hamiltonian** if there exists a moment map

$$\Phi : M \longrightarrow \mathfrak{g}^*,$$

such that

1. For all $X \in \mathfrak{g}$, $\langle \Phi, X \rangle$ is a Hamiltonian function for $X^\sharp$.
2. Φ is G-equivariant with respect to the given G-action on M and the coadjoint action on $\mathfrak{g}^*$, i.e.
$$\Phi(g \cdot m) = \mathrm{Ad}^*_g \Phi(m), \qquad {}^\forall m \in M, {}^\forall g \in G.$$

Remark : In the diagram for comoment maps,

$$\begin{tikzcd} C^\infty(M) \arrow[r, "\text{Lie alge anti-homo}^m"] & \mathrm{Vect}(M) \\ & \mathfrak{g} \arrow[ul, "\text{Lie alge homo}"] \arrow[u, "\text{Lie alge anti-homo}^{\mathrm{m}}"'] \end{tikzcd}$$

5.2 Examples of Hamiltonian actions and moment maps

I. Classical (physics) examples

The name “moment map” comes from being a generalization of the linear and angular momentum from classical mechanics.

- Let $G = SO(3) =$ “rotations in $\mathbb{R}^3$” $= \{A \in GL(3, \mathbb{R}) \mid A^t A = Id,\ \det A = 1\}$. Then $\mathfrak{so}(3) = \mathrm{Lie}(SO(3)) = \{A \in \mathfrak{gl}(3, \mathbb{R}) \mid A + A^t = 0\}$.

  Identify $\mathfrak{so}(3) \cong \mathbb{R}^3$ via

$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \longmapsto \vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3.$$

Then the Lie algebra bracket on $\mathfrak{so}(3)$ (the usual matrix commutator) can be identified with the usual cross product in $\mathbb{R}^3$:

$$[A, B] = AB - BA \longmapsto \vec{a} \times \vec{b},$$

and under the identification $\mathfrak{so}(3) \cong \mathfrak{so}(3)^* \cong \mathbb{R}^3$, the adjoint and coajoint actions correspond to the usual $SO(3)$ action on $\mathbb{R}^3$.

[Exercise] Consider $SO(3) \curvearrowright T^*\mathbb{R}^3$, with the standard symplectic structure. Then this action is Hamiltonian with moment map

$$\varPhi(\vec{q}, \vec{p}) = \vec{q} \times \vec{p} \qquad \text{“angular momentum”}$$

- Consider the $\mathbb{R}^3$-action on $(T^*\mathbb{R}^3, \omega_0)$ by translation :

$$\varPsi_{\vec{a}}(\vec{q}, \vec{p}) = (\vec{q} + \vec{a}, \vec{p}).$$

  This action is Hamiltonian with moment map

$$\varPhi(\vec{q}, \vec{p}) = \vec{p}. \qquad \text{“linear momentum”}$$

II. “Linear algebra” example

- Let $G = T^n = (S^1)^n = \left\{(e^{\sqrt{-1}t_1}, e^{\sqrt{-1}t_2}, \dots, e^{\sqrt{-1}t_n}) \mid t_k \in \mathbb{R}\right\}$. Consider the diagonal orbit of T^n on $\mathbb{C}$:

  $(e^{\sqrt{-1}t_1}, e^{\sqrt{-1}t_2}, \dots, e^{\sqrt{-1}t_n}) \cdot (z_1, z_2, \dots, z_n) = (e^{\sqrt{-1}t_1} z_1, e^{\sqrt{-1}t_2} z_2, \dots, e^{\sqrt{-1}t_n} z_n) \in \mathbb{R}^n \cong \mathrm{Lie}(T^n)$

[Exercise] This is Hamiltonian, with moment map

$$\varPhi(z_1, \dots, z_n) = -\frac{1}{2}(||z_1||^2, \dots, ||z_n||^2) \in \mathbb{R}^n \cong \mathrm{Lie}(T^n)$$

General fact : Suppose a compact connected Lie group $G \curvearrowright (M, \omega)$ Hamiltonianly with moment map $\varPhi_G$. Suppose $H \subseteq G$ closed Lie subgroup. Then the restriction of the action to $H \curvearrowright (M, \omega)$ is also Hamiltonian, and the moment map $\varPhi_H$ is given by

$$M \xrightarrow{\varPhi_G} \mathfrak{g}^* \xrightarrow{\pi} \mathfrak{h}^*$$

[If $H \subseteq G$, then $\mathfrak{h} \hookrightarrow \mathfrak{g}$ induces dual projection $\pi : \mathfrak{g}^* \longrightarrow \mathfrak{h}^*$]

- In particular, consider the S^1 subgroup in T^n given by inclusion

$$e^{\sqrt{-1}\theta} \longmapsto (e^{\sqrt{-1}m_1\theta}, \dots, e^{\sqrt{-1}m_n\theta}), \qquad \text{for } m_k \in \mathbb{Z}.$$

  (Think of the m_i as the different “speeds” of the rotation on different coordinates.)

  the corresponding moment map is

$$\varPhi(z_1, \dots, z_n) = -\frac{1}{2}\sum_{k=1}^{n} m_k ||z_k||^2$$

- Consider the natural action of $T^n \subseteq U(n, \mathbb{C}) \curvearrowright (\mathbb{C}, \omega_0)$. This is Hamiltonian with moment map

$$\varPhi(z) = \frac{\sqrt{-1}}{2} zz^* \in \mathfrak{u}(n, \mathbb{C})$$

General fact : If $G \curvearrowright (M_1, \omega_1) \xrightarrow{\Phi_1} \mathfrak{g}^*$ and $G \curvearrowright (M_2, \omega_2) \xrightarrow{\Phi_2} \mathfrak{g}^*$, then $G \curvearrowright M_1 \times M_2$ is also Hamiltonian with moment map the sum

$$M_1 \times M_2 \underset{\Phi_1 + \Phi_2}{\longrightarrow} \mathfrak{g}^*$$

- Using the above general fact and the previous example, then we see that the action of $U(n, \mathbb{C})$ on $M(n \times n, \mathbb{C}) \cong \mathbb{C}^{n^2}$ by conjugation $(A, X) \longmapsto AXA^*$ is Hamiltonian with moment map

$$\Phi(X) = \frac{\sqrt{-1}}{2}[X, X^*].$$

- Consider the action of G on a coadjoint orbit

$$G \cdot \xi \subseteq \mathfrak{g}^* \qquad \text{for } \xi \in \mathfrak{g}^*.$$

  This is Hamiltonian, with moment map the inclusion

$$G \cdot \xi \hookrightarrow \mathfrak{g}^*.$$

FACT : Moment maps may not be unique. (Ref. for more details ; Ana Cannas de Silva lecture on existence and uniqueness) Indeed, the Hamiltonian equation only involves the derivative of $\Phi^X, X \in \mathfrak{g}$. So in particular one could add global constant and still satisfy $d\Phi^X = i_{X^\sharp}\omega$.
$\Longrightarrow$ e.g. for T^n-action, moment map are determined up to a constant, if Φ is a moment map, then

$$\Phi' = \Phi + c,$$

for some $c \in (\mathbb{R}^n)^*$ constant is also a moment map.

[in general, can add any element in $[\mathfrak{g}, \mathfrak{g}]^0 \subseteq \mathfrak{g}^*$, which is annihilator in $\mathfrak{g}^*$ of $[\mathfrak{g}, \mathfrak{g}]$.]

So e.g. for $T^n \curvearrowright (\mathbb{C}^n, \omega_0)$ standard action

$$\Phi(z_1, z_2, \ldots, z_n) = -\frac{1}{2}(||z_1||^2, \ldots, ||z_n||^2) + (\lambda_1, \lambda_2, \ldots, \lambda_n), \qquad \text{for any } \lambda_i \in \mathbb{R}$$

is also a T^n-moment map.

6 Symplectic quotients

6.1 Preliminaries : orbit space and slices of group actions

Suppose $G \curvearrowright M$ is any action.

Recall : We say that the G-action is **free** if all stabilizer groups (isotropy groups) are trivial. The **orbit space** $M/G = M/\sim$ where the equivalence relation is $p \sim q$, if there exist a $g \in G$ such that $g \cdot p = q$ has a natural topology, i.e. the weakest topology with respect to which the projection

$$M \longrightarrow M/G$$

is continuous, i.e. the quotient topology.

Example 6.1. In general, the quotient topology can be quite "bad", e.g. non-Hausdorff.
Let $G = \mathbb{C}^\times = \mathbb{C} \setminus \{0\}, M = \mathbb{C}^n$, and consider the G-action given by usual scalar multiplication. For $z \in \mathbb{C}^n, z \neq 0$, the G-orbit is the punctured complex line spanned by z. But $\mathbb{C}^\times \cdot \{0\} = \{0\}$. i.e. the stabilizer of $\{0\}$ is $\mathbb{C}^\times$.

$$M/G = \mathbb{C}^n/\mathbb{C}^\times = \mathbb{C}P^{n-1} \sqcup \{0\}$$

The only open set containing $\{0\}$ in the quotient topology is all of M/G

$$\Longrightarrow \text{NOT Hausdorff !}$$

However, to obtain a Hausdorff quotient, only need to remove $\{0\}$. Then $\mathbb{C}P^{n-1} = (\mathbb{C} \setminus \{0\})/\mathbb{C}^\times$.

IMPORTANT NOTE . We can also build $\mathbb{C}P^{n-1}$ by taking

$$S^{2n-1}/S^1 = \mathbb{C}P^{n-1}.$$

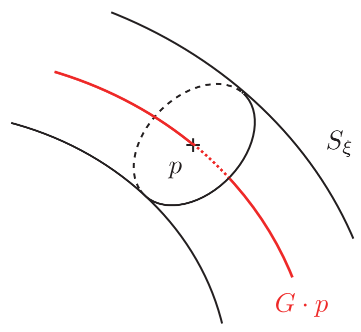
Note that the S^{2n-1} is exactly a level set of a S^1-moment map for the action of $S^1 \curvearrowright (\mathbb{C}^n, \omega_0)$.

Idea : Taking a level set of a moment map is analogous to "removing unstable points" in the sense of geometric invariant theory in algebraic geometry.

One more ingredient :

Theorem 6.1. *If a compact Lie group G acts freely on a manifold M, then M/G is a manifold and $\pi : M \longrightarrow M/G$ is a principal G-bundle.*

This is a consequence of "**the slice theorem**" (one version) :
Let G compact Lie group $\curvearrowright M$. Suppose G acts freely at $p \in M$. For sufficiently small slice $S_\xi, \eta : G \times S_\xi \longrightarrow M$ maps $G \times S_\xi$ diffeomorphically onto G-invariant neighborhood $\mathcal{U}$ of the G-orbit of p.



POINT : Quotients by free actions of compact Lie groups are manifolds.

6.2 Marden-Weinstein-Meyer symplectic quotients

Our goal

Theorem 6.2. (Marden-Weinstein; Meyer)
Let (M, ω, G, Φ) be a Hamiltonian G-space, with G compact connected Lie group.
Let $i : \Phi^{-1}(0) \hookrightarrow M$ be the inclusion map. Assume that G acts freely on $\Phi^{-1}(0)$. Then

- $M_{\text{red}} := \Phi^{-1}(0)/G$ the orbit space is a manifold.
- $\pi : \Phi^{-1}(0) \longrightarrow M_{\text{red}}$ is a principal G-bundle.
- There is a symplectic form ω_{red} on M_{red}, called **the reduced symplectic form**, such that $i^*\omega = \pi^*\omega_{\text{red}}$ on $\Phi^{-1}(0)$

$$\begin{tikzcd} \Phi^{-1}(0) \arrow[r, hook, "i"] \arrow[d, "\pi"] & (M, \omega) \\ (M_{\text{red}} := \Phi^{-1}(0)/G,\ \omega_{\text{red}}) & \end{tikzcd}$$

Proof. First two assertions follow immediately from the slice theorem and its consequences.

Claim : Let (G, M, ω, Φ) Hamiltonian. Let $p \in M$ and $\mathfrak{g}_p = \text{Lie}(G_p)$ the Lie algebra of the stabilizer of p. Then $d\Phi_p : T_pM \longrightarrow T_{\Phi(p)}\mathfrak{g}^* \cong \mathfrak{g}^*$ has

- $\operatorname{Ker} d\Phi_p = (T_p(G \cdot p))^{\omega_p}$.
- Im $d\Phi_p = \mathfrak{g}_p^0 :=$ annihilator of $\mathfrak{g}_p$ in $\mathfrak{g}^*$.

Proof. (claim) Stare at the equation $\langle d\Phi_p(v), X\rangle = \omega_p(X_p^\sharp, v)$ for all $X \in \mathfrak{g}$ and $v \in T_pM$. □

Corollary of Claim : If G acts freely on $\Phi^{-1}(0)$, then for $p \in \Phi^{-1}(0)$

$\Longrightarrow$ Im $d\Phi_p = \mathfrak{g}^*$ since $\mathfrak{g}_p = 0$.

$\Longrightarrow$ $\mathrm{d}\Phi_p$ is surjective for all $p \in \Phi^{-1}(0)$.

$\Longrightarrow$ 0 is a regular value of Φ.

$\Longrightarrow$ $\Phi^{-1}(0)$ is a closed submanifold of M, $\dim \Phi^{-1}(0) = \dim M - \dim G$.

$\Longrightarrow$ $\operatorname{Ker} d\Phi_p = T_p\Phi^{-1}(0)$.

Note also that since G acts on $\Phi^{-1}(0)$,

$$\begin{aligned} T_p(G \cdot p) \subseteq \operatorname{Ker} d\Phi_p &= T_p\Phi^{-1}(0) \\ &= T_p(G \cdot p)^{\omega_p}. \end{aligned}$$

So the G-orbit through p is **isotropic**, i.e. $\omega_p|_{T_p(G \cdot p)} \equiv 0$.

Claim : (Symplectic linear algebra)
Let (V, ω) be a symplectic vector space. Suppose $U \subseteq V$ isotropic subspace. Then ω induces a canonical symplectic structure Ω on U^ω/U.

Proof. (claim)[Exercise]
[Define $\Omega([u_1], [u_2]) = \omega(u_1, u_2)$ for $u_1, u_2 \in U^\omega/U$] □

Corollary 6.1. Since a point $[p] \in \Phi^{-1}(0)/G$ has tangent space

$$\operatorname{Ker} d\Phi_p / T_p(G \cdot p) = T_p(G \cdot p)^{\omega} / T_p(G \cdot p),$$

ω induces a canonical symplectic structure ω_{red} on $T_{[p]}(\Phi^{-1}(0)/G)$. This is well-defined because ω is G-invariant.

Now, by construction, $\pi^*\omega_{\text{red}} = i^*\omega$. Moreover, ω_{red} is closed because

$$\pi^* d\omega_{\text{red}} = d\pi^*\omega_{\text{red}} = di^*\omega = i^* d\omega = i^* 0 = 0$$

and π is injective (since π is the projection of a fiber bundle). So $(M_{\text{red}}, \omega_{\text{red}})$ is a symplectic manifold of dimension $\dim M - 2\dim G$. □

NOTE . (More common in literature) $M_{\text{red}} = M \underset{0}{/\!\!/} G$ where 0 signifies "the value at which you reduce"

Remarks :

- There is an "equivariant version" with the same assumptions as above. Suppose

  $$G \times H \curvearrowright (M, \omega) \underset{\Phi_G \oplus \Phi_H}{\longrightarrow} \mathfrak{g}^* \oplus \mathfrak{h}^*,$$

  so G-action and H-action commute. Suppose Φ_H is G-invariant. Then the H-action descends to an action on $M \underset{0}{/\!\!/} G$ and is Hamiltonian, with moment map

  $$(\Phi_H)_{\text{red}} : M \underset{0}{/\!\!/} G \longrightarrow \mathfrak{h}^*$$

  satisfying

  $$(\Phi_H)_{\text{red}} \circ \pi = \Phi_H \circ i.$$

  [This is the situation for the construction of symplectic toric manifolds.]

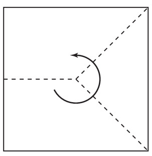
- There is nothing special about $0 \in \mathfrak{g}^*$. For any regular value $\xi \in \mathfrak{g}^*$, can define

  $$M \underset{\xi}{/\!\!/} G := \Phi^{-1}(\xi)/G_\xi$$

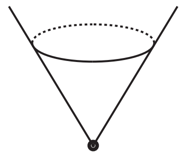
  also symplectic by similar computation if G_ξ acts freely on $\Phi^{-1}(\xi)$.
  (The dimension computation will be different.)

Digression : Some comments about orbifolds arising as symplectic quotients.

An orbifold is, **VERY ROUGHLY**, is a topological space which "look like $\mathbb{R}^n/\Gamma$ for Γ a finite group".
N.B. We won't get into technical details today!
Locally, the picture is :



$\mathbb{Z}/3\mathbb{Z}$ acting by twists of angle $\dfrac{2\pi}{3}$



singular at the origin
(smooth elsewhere).

$\rightsquigarrow$ the quotient will be a "cone"

THE POINT, for us : It is very easy to produce examples of orbifolds by taking symplectic quotients. For simplicity assume $G = T$ abelian, so $G_\xi = G$ always. From our computations above,

$$\begin{aligned} \xi \text{ is a regular value} \quad &\Longrightarrow \quad d\Phi_p \text{ surjective at all } p \in \Phi^{-1}(\xi) \\ &\Longrightarrow \quad \mathfrak{g}_p = 0 \qquad \forall p \in \Phi^{-1}(\xi) \\ &\Longrightarrow \quad G_p \text{ is discrete } \subseteq G \\ &\Longrightarrow \quad G_p \text{ is finite } (G: \text{ compact}) \\ &\underset{\text{by slice Thm}}{\Longrightarrow} \quad \Phi^{-1}(\xi)/G \text{ is an orbifold} \end{aligned}$$

since the small transverse slice S_ξ/G_p serves as local model for $[p]$ in quotient.

By Sard's Theorem **most** values are regular, so **most** symplectic quotients (at different values) are orbifolds.

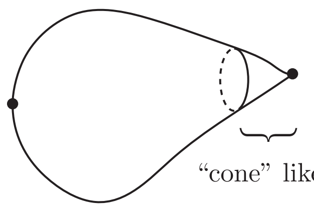
Example 6.2. **(1)** $S^1 \curvearrowright \mathbb{C}^2$ by $e^{\sqrt{-1}\theta}(z_1, z_2) = \left(e^{\sqrt{-1}\theta} z_1, e^{\sqrt{-1}\theta} z_2\right)$ for $m \geq 2$.

$$\text{Moment map} = -\frac{1}{2}(\|z_1\|^2 + m\|z_2\|^2).$$

Any $\xi < 0$ is a regular value, with level set on "ellipsoid".

NOTE . The action of S^1 on $\Phi^{-1}(\xi)$ is NOT free ! If $z_1 \neq 0$, then the action is free.

BUT if $z_1 = 0$, then stabilizer $\cong \mathbb{Z}/m\mathbb{Z}$. The reduced space is called a **teardrop orbifold** with a single singularity "of type $\mathbb{Z}/m\mathbb{Z}$".



"cone" like above, cone angle $\dfrac{2\pi}{m}$

(2) More generally, **weighted projective space** are obtained by $S^1 \curvearrowright \mathbb{C}^n$ by

$$e^{\sqrt{-1}\theta}(z_1, z_2, \ldots, z_n) = \left(e^{\sqrt{-1}m_1\theta} z_1, e^{\sqrt{-1}m_2\theta} z_2, \ldots, e^{\sqrt{-1}m_n\theta} z_n\right) \qquad \text{for } m_i \in \mathbb{Z}_{\geq 1}$$

has symplectic quotient $\mathbb{C}P^{n-1}_{(m_1, m_2, \ldots, m_n)}$.

6.3 Symplectic quotients revisited

More "advanced" example

I. Gauge theory and moduli space of flat connections : a sketch

- Historical origins : Atiyah-Bott 1982, "The Yang-Mills equation over Riemann surface, Phylosophical Transactions of the Royal Society of London, series A, 308 (1982) 523-615".
- This is an infinite-dimensional symplectic quotient.

Let G be a compact Lie group, B a compact manifold.

Recall :

Definition 6.1. A principal G-bundle over B is a manifold P with a smooth map $\pi : P \longrightarrow B$ such that

- G acts on the left freely on P.
- B is the orbit space for $G \curvearrowright P$, and it is the orbit projection.
- There is an open cover of B such that for all $\mathcal{U}$ open in the cover, there corresponds a map
$$\varphi_{\mathcal{U}} : P \supseteq \pi^{-1}(\mathcal{U}) \longrightarrow \mathcal{U} \times G$$
such that for all $p \in \pi^{-1}(\mathcal{U}), \varphi_{\mathcal{U}}(p) = (\pi(p), s_{\mathcal{U}}(p))$, and $s_{\mathcal{U}}(p)$ is G-equivariant.

Also often represented by diagram

$$\begin{array}{ccc} \text{"structure group"} = G & \hookrightarrow & P = \text{"total space"} \\ & & \downarrow \pi \\ & & B = \text{"base space"} \end{array}$$

Example 6.3.

$$\begin{array}{ccc} S^1 & \hookrightarrow & S^3 \\ & & \downarrow \pi \\ & & \mathbb{C}P^1 = S^2 \end{array}$$

Choose a basis $X_1, \ldots, X_k$ of $\mathfrak{g}$. Since G-action on P is free, $X_1^\sharp, \ldots, X_k^\sharp \in \mathrm{Vect}(P)$ linearly independent at all $p \in P$.

Definition 6.2. Vertical bundle V of $P :=$ rank k subbundle of TP spanned by $X_1^\sharp, \ldots, X_k^\sharp \in \mathrm{Vect}(P)$.

[Exercise] Prove V is independent of the choice of basis. **HINT :** $V = \mathrm{Ker}(d\pi)$.

V is canonically defined on a principal bundle; what makes things interesting is the choice of a complementary subspace in TP.

Definition 6.3. A (Ehresmann) **connection on a principal bundle** P is a choice of a G-invariant subbundle H of TP complementary to V,
$$TP \cong V \oplus H.$$
H is called a **horizontal bundle**.

Another way to describe a connection is by constructing a 1-form on P which "projects to V".

Definition 6.4. A connection form on a principal bundle P is a Lie algebra-value 1-form

$$A = \sum_{i=1}^{k} a_i \otimes X_i \in \Omega^1(P) \otimes \mathfrak{g}$$

such that

- A is vertical, i.e. $i_{X^\sharp} A = X$.
- A is G-invariant with respect to the diagonal action of G on $\Omega^1(P) \otimes \mathfrak{g}$.

[Exercise] Show that a connection form determines a connection and vice versa by the formula

$$H = \operatorname{Ker} A = \{v \in TP \mid A(v) = 0\}$$

The splitting $TP = V \oplus H$ induces splitting

- $T^*P = V^* \oplus H^*$
- $\bigwedge^2 T^*P = \bigwedge^2 V^* \oplus (V^* \wedge H^*) \oplus \bigwedge^2 H^*$
- $\Omega^2(P) = \Omega^2_{\text{vert}} \oplus \Omega^2_{\text{mix}} \oplus \Omega^2_{\text{horiz}}$.

By definition, $A \in \Omega^1_{\text{vert}}(P) \otimes \mathfrak{g}$. The exterior derivative may have mixed components

$$dA = (dA)_{\text{vert}} + (dA)_{\text{mix}} + (dA)_{\text{horiz}}.$$

FACTS :

- $(dA)_{\text{vect}}(X, Y) = [X, Y]$
- $(dA)_{\text{mix}} = 0$

Hence the "interesting" part is $(dA)_{\text{horiz}}$.

Definition 6.5. The curvature form of a connection is

$$\text{curve } A := (dA)_{\text{horiz}} \in \Omega^2_{\text{horiz}} \otimes \mathfrak{g}.$$

A connection A is **flat** if curve $A = 0$.

Symplectic structure on the space of connections. Fix a principal bundle $P \xrightarrow{\pi} B$. Let $\mathcal{A}$ be the space of connections on P.

FACTS :

- If A_0 is a connection form $\in \mathcal{A}$, and if $a \in (\Omega^1_{\text{horiz}} \oplus \mathfrak{g})^G$, then $A_0 + a$ is another connection form.
- If A_0, B are connection forms, $A_0 - B \in (\Omega^1_{\text{horiz}} \oplus \mathfrak{g})^G$
$\Longrightarrow \mathcal{A}$ is an affine space modeled on $(\Omega^1_{\text{horiz}} \oplus \mathfrak{g})^G$. To describe a symplectic structure on $\mathcal{A}$, it suffices to describe one on $(\Omega^1_{\text{horiz}} \oplus \mathfrak{g})^G$.

[Exercise] Observe that the projection $\pi : P \longrightarrow B$ induces a pullback map, for all k

$$\pi^* : \Omega^k(B) \longrightarrow \Omega^k(P).$$

Prove π^* is an isomorophism onto its image $(\Omega^k_{\text{horiz}}(P))^G$.

Special case : Assume B is a Riemann surface (compact, 2-dimensional, oriented no boundary with Riemannian structure). Assume G is equipped with a G-invariant metric on $\mathfrak{g}$.

Let be $a, b \in (\Omega^1_{\text{horiz}} \oplus \mathfrak{g})^G$ with respect to the basis $X_1, \ldots, X_k$ of $\mathfrak{g}$

$$a = \sum_{i=1}^{k} a_i \otimes X_i, \qquad b = \sum_{i=1}^{k} b_i \otimes X_i.$$

The symplectic form is given by
"first pair the $\mathfrak{g}$ using the metric on $\mathfrak{g}$, then integrate the $a_i \wedge b_i$ over B".
Namely

$$\begin{array}{cccc} \omega_{\mathcal{A}} : (a, b) \longmapsto & \displaystyle\sum_{i,j=1}^{k} a_i \wedge b_j \langle X_i, Y_j \rangle & \xmapsto[\text{integrate over } B]{} & \displaystyle\int_B \sum_{i,j=1} a_i \wedge b_j \langle X_i, Y_j \rangle \\ & \pitchfork & & \pitchfork \\ & \Omega^2_{\text{horiz}}(P) & & \mathbb{R} \\ & \cong \Omega^2(B) & & \end{array}$$

FACT : $\omega_{\mathcal{A}}$ is non-degenerate $\rightsquigarrow$ $\mathcal{A}$ is **symplectic**.

Gauge group action and the moment map

For $P \longrightarrow B$ a principal G-bundle, a diffeomorphism $P \longrightarrow P$ commuting with the G-action determines a diffeomorphism $f_{\text{base}} : B \longrightarrow B$.

Definition 6.6. A diffeomorphism $f : B \longrightarrow B$ such that $f_{\text{base}} = id_B$ is a **gauge transformation**.

The gauge group $\mathcal{G} :=$ subgroup of gauge transformations. The derivative $df : TP \longrightarrow TP$ for $f \in \mathcal{G}$ takes a connection to another connection. So $\mathcal{G} \curvearrowright \mathcal{A}$.

AMAZING FACT : (insight of Atiyah and Bott '82)
$\mathcal{G} \curvearrowright \mathcal{A}$ is Hamiltonian, with moment map the curvature.

$$\Phi : A \longmapsto \text{curv} A \in (\Omega^2_{\text{horiz}}(P) \otimes \mathfrak{g})^G \cong \text{Lie}(\mathfrak{g}) = \text{Lie}(\text{Map}(B, G))$$

$\Longrightarrow$ the symplectic quotient is $\Phi^{-1}(0)/\mathcal{G} =$ "moduli space of flat connections modulo gauge equivalence.

II. Quiver varieties :

Reference : a good first reference : Nakajima, "Varieties associated with quivers, in Representation theory of algebras and related topics, CMS conference proceedings 19, AMS (1996) 139–157."

Let $Q = (\mathcal{I}, E)$ be a finite oriented graph

$\mathcal{I} =$ vertices $\cong \{1, 2, \ldots, n\}$

$E =$ oriented edges.

(For an arrow $a \in E, t(a) =$ (tail of a) $\in \mathcal{I}, h(a) =$ head of a)

$$t(a) \xrightarrow{a} h(a)$$

[Assume Q has no cycle]

Such a graph is called a **quiver**.

Definition 6.7. A (finite-dimensional) representation of quiver is a collection of (finite-dimensional) vector space over $\mathbb{C}, V = (V_k)_{k\in\mathcal{I}}$ and linear maps.

$$B_a : V_{t(a)} \longrightarrow V_{h(a)}, \qquad {}^{\forall}a \in E$$

The **dimension vector** of representation is $d_V := (d_k := \dim V_k)_{k\in\mathcal{I}} \in \mathbb{Z}^n_{\geq 0}$.

There are natural notions of morphisms between representations, direct sum, etc. Fix a particular dimension vector $d = (d_k)_{\in\mathcal{I}}$. Then the set of isomorphism classes of representations of $Q = (\mathcal{I}, E)$ of dimension vector d_V is the set of $PG(V)$-orbits of

$$E_Q(V, V) := \bigoplus_{a\in E} \mathrm{Hom}(V_{t(a)}, V_{h(a)}),$$

where $\dim V_k = d_k$ and $G(V) := \prod_k GL(V_k)$ acts by conjugation (i.e. change-of-basis) on each factor :

$$(B_a)_{a\in E} \longmapsto \left(g_{h(a)} B_a g^{-1}_{t(a)}\right)_{a\in E}.$$

However, the naive quotient is BAD $\longrightarrow$ so we need to "rip out unstable points" in the sense of GIT, or symplectic quotient. Instead of $GL(V)$, consider the subgroup

$$U(V) := \prod_{k\in\mathcal{I}} U(V_k, \mathbb{C})$$

and take a symplectic quotient at a value $\alpha \in \mathbb{Z}(\mathfrak{u}(V))$
$\Longrightarrow$ the result is called the moduli space of (α-semistable) representations of the quiver Q.

MOTIVATIONS

- There are variations on this construction
  - "framed" quiver varieties, attach extra W_k ($\mathbb{C}$-vectors space) at each $k \in \mathcal{I}$.
  - "double the arrows" [a hyperkähler analogue].
- This (and related) construction gives a different construction and perspective on previously known objects :
  - flag varieties.
  - [Kronheimer] singularities $\mathbb{C}/\Gamma$ and their resolutions.
  - [Kronheimer-Nakajima] moduli spaces of ASD-connections on ALE spaces.
- There is a similarity between the theory of quiver varieties with moduli space of flat connection over a Riemann surface.
- [Nakajima] Geometric constructions of representations of Kac-Moody algebras.

Reference : (for recent developments on the subject) AIM website on recent workshop "Arithmetic harmonic analysis on character and quiver varieties" and problems/ notes therein.

(back to basics)

[Exercises]

(i) Recall : natural action of $U(k,\mathbb{C}) \curvearrowright \mathbb{C}^{k\times n}$ has a moment map

$$\Phi(A) = \frac{\sqrt{-1}}{2} AA^* + \frac{Id}{2\sqrt{-1}}.$$

Show that $\mathbb{C}^{k\times n} \mathbin{/\!\!/}_0 U(k,\mathbb{C}) \cong \underbrace{Gr(k,\mathbb{C}^n)}_{\substack{\text{Grassmannian of} \\ k-\text{planes in } \mathbb{C}^n}}$

(ii) Recall : the S^1-action on $(\mathbb{C}^{n+1}, \omega_0)$ given by scalar multiplication has

$$\Phi(z_1, \ldots, z_{n+1}) = -\frac{1}{2} \sum_{k=1}^{n+1} \|z_k\|^2 + \frac{1}{2}.$$

Then show that $\mathbb{C}^{n+1} \mathbin{/\!\!/}_0 S^1 \cong \mathbb{C}P^n$. Check that ω_{red} is the usual Fubini-Study form on $\mathbb{C}P^n$, and write down in explicit coordinates a formula for a moment map of the residual T^n-action.

7 Convexity

Before heading to symplectic toric manifold, some results about images of T-moment maps.

WARNING : Until further notice, we focus on the case $G = T$ compact torus.

Observations about T-moment maps :

- Since T is **abelian**, the conjugation, adjoint, and coadjoint actions are all **trivial**. In particular, T-moments are **T-invariant**. So $\Phi(T \cdot p)$ is a single point.
- Similarly, since $[X, Y] = 0$ for all $X, Y \in \mathfrak{t}$, then $\left\{ \varPhi^X, \varPhi^Y \right\} = 0 = \varPhi^{[X,Y]}$. In fact this is just the beginning !!

Definition 7.1. (Atiyah ; Guillemin-Sternberg)
Let $(M, \omega, T^m, \varPhi)$ be a compact connected Hamiltonian T^m-space. Then

- the level sets of $\varPhi$ are connected
- the image of $\varPhi$ is **convex**
- the image of $\varPhi$ is the convex hull of the images of the fixed points M^T

HISTORICAL REMARK

This convexity result tied equivariant symplectic geometry with combinatorics via polytopes!
$\cdots$ started a "whole new era" in modern symplectic geometry. In particular, a lot of equivariant topological information is encoded in the combinatorics!

We will not prove this, but some comments:

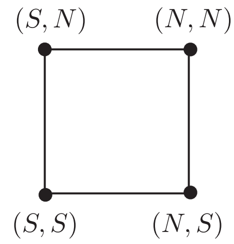
- "Main example": $T^m \curvearrowright (\mathbb{C}^m, \omega_0) \xrightarrow[\varPhi]{} -\dfrac{1}{2}(\|z_1\|^2, \ldots, \|z_m\|^2) + \lambda$
  has image (the translate of) an **orthant** as image - certainly convex. Note that the unique T^m-fixed point goes to the vertex.
- Next main example: $T \hookrightarrow T^m \curvearrowright (\mathbb{C}^m, \omega_0)$ by $t \longmapsto ((\exp \alpha_1)(t), \cdots, (\exp \alpha_m)(t))$ for same weights $\alpha_i \in \mathrm{Lie}(T)^*_{\mathbb{Z}}$. Then the corresponding moment map for the T-action is

  $$\varPhi(z_1, \cdots, z_m) = -\frac{1}{2} \sum_{i=1}^{m} \alpha_i \|z_i\|^2 + \lambda'.$$

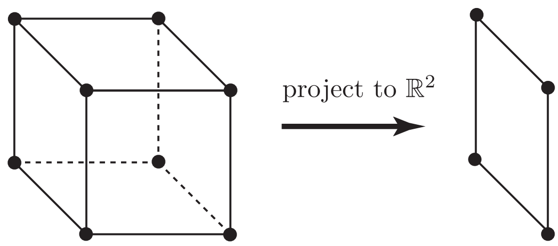
  So the image consists of non-positive linear combinations of same weights (up to translation). Also, assuming all $\alpha \neq 0$, again the only T fixed point goes to vertex.
- Locally near T-fixed points, an "equivariant Dorboux theorem" says that T-action near a T-fixed point looks like the example, i.e. a linear T-action on $\mathbb{C}^m \rightsquigarrow$ so **locally**, near a point in M^T, the image is convex.
- **Global** convexity statement comes from **connectedness** of $\varPhi^{-1}(\xi)$.

Example 7.1.

(i) $S^1 \times S^1 \curvearrowright S^2 \times S^2$ each S^1 acting (separately) on corresponding S^2. The fixed points: $\{(N, N), (N, S), (S, N), (S, S)\}$ where N, S denote north and south poles. Moreover, the moment map is $\varPhi((x_1, y_1, z_1), (x_2, y_2, z_2)) = (z_1, z_2)$. So the image is $[-1, 1] \times [-1, 1]$, and is the convex hull of the image of fixed points



(ii) (M^T need not be isolated)
For $S^1 \times S^1 \times S^1 \curvearrowright S^2 \times S^2 \times S^2$, the same computations shows that the moment map image is cube.



Consider the subgroup $T^2 = S^1 \times S^1 \times \{1\} \subseteq S^1 \times S^1 \times S^1 \curvearrowright S^2 \times S^2 \times S^2$ and get moment map image a square, but the fixed points are not isolated.

Now: symplectic toric manifolds. We will see that when the torus is "big enough" or really "as large as possible", then the moment map image completely determines the Hamiltonian T-space.

Definition 7.2. A T-action on M is **effective** if $T \longrightarrow \mathrm{Diff}(M)$ is **injective**.

Henceforth we always assume all T-actions are effective (otherwise there is no point in discussing $\dim T$ compared to $\dim M$).

Claim : If (M, ω, T, Φ) is a Hamiltonian T-space with an effective action, then

$$\dim T \leq \frac{1}{2} \dim M.$$

Proof. If T acts effectively, then there is a T-orbit $T \circ p$ of dimension $m := \dim T$. On the other hand, we have already seen

$$T_p(T \circ p) \subseteq \operatorname{Ker} d\Phi_p = T_p(T \circ p)^{\omega_p}$$

so $T_p(T \circ p)$ is isotropic. For $(T \circ p)$ is m-dimensional, we conclude from symplectic linear algebra

$$\dim T = m \leq \frac{1}{2} \dim M.$$

□

When the dimension is as large as possible, we give them a name.

Definition 7.3. A (symplectic) toric manifold is a compact, connected, symplectic manifold (M, ω) with an effective Hamiltonian action of a torus T with $\dim T = \frac{1}{2} \dim M$ and a choice of moment map $\Phi : M \longrightarrow \mathfrak{t}^*$.

8 The Delzant construction of symplectic toric manifolds

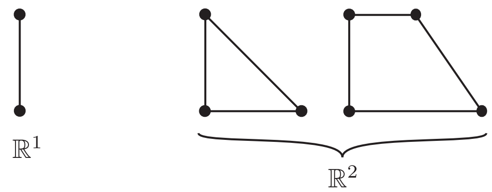
- The beautiful explicit construction is via symplectic quotients.
- Symplectic toric manifolds form a rich class of "first example" on which to test many theories in equivariant symplectic geometry such as Goresky-Kottwitz-MacPherson theory.
- Kirwan surjectivity, etc.

Idea : We have already seen that $\Phi(M)$ is always a convex polytope. For toric manifolds, the image entirely determines M.

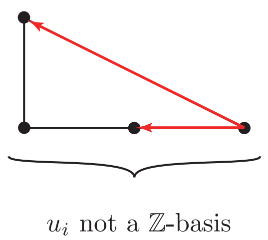
Definition 8.1. **A Delzant polytope** $\triangle \subseteq \mathbb{R}^n$ is a convex polytope such that

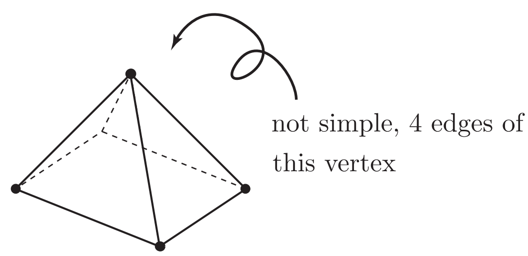
- **simple**
- it is **rational**, that is, each edge emanating from a vertex p is of the form $p + tu, 0 \leq t < \infty$, where $u \in \mathbb{Z}^n$.
- it is **smooth**, that is, for each p vertex, the edges $\{p + tu_i\}_{i=1}^n$ meeting p are such that the $\{u_i\}$ can be chosen to be a **basis of** $\mathbb{Z}^n$.

Example 8.1.



Non-examples .





Theorem 8.1. (Delzant) Symplectic toric manifold are classified by Delzant polytopes. More specifically, there is a 1-1 correspondence

$$\left\{ \begin{matrix} \text{symplectic} \\ \text{toric manifolds} \\ (M^{2n}, \omega, T^n, \Phi) \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Delzant polytopes} \\ \triangle \subseteq \mathbb{R}^n \end{matrix} \right\}$$

The most important part of this proof is the **existence** part, that is, given $\triangle$, we construct a toric manifold with $\Phi(M) = \triangle$. We will focus on this part.

Another description of Delzant polytopes :

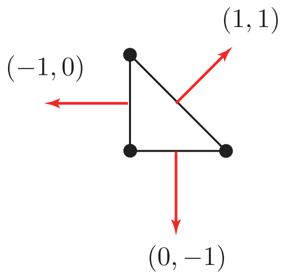
- A facet of a polytope is an $(n-1)$-dimensional face of $\triangle$.
- A lattice vector $v \in \mathbb{Z}^n$ is primitive if it cannot be written $v = ku, k \in \mathbb{Z}, |k| > 1, u \in \mathbb{Z}^n$. e.g. $(1,3)$ is primitive, but $(2,4) = 2(1,2)$ is not.

Write $\triangle$ as an intersection of half-spaces:

$$\triangle = \left\{x \in (\mathbb{R}^n)^* \mid \langle x, v_i \rangle \leq \lambda_i,\ \lambda_i \in \mathbb{R},\ 1 \leq {}^{\forall} i \leq d\right\}$$

where
$d =$ number of facets of $\triangle$
$n =$ dimension of $\triangle$
$v_i =$ primitive outward-pointing normals to facets F_i of $\triangle$.
($(\mathbb{R}^n)^* \cong \mathbb{R}^n$, using standard inner product.)

Example 8.2.



$$\begin{matrix} n = 2 \\ d = 3 \end{matrix} \qquad \Longrightarrow \triangle = \left\{ x \in (\mathbb{R}^2)^* \;\middle|\; \begin{matrix} \langle x, (-1,0)\rangle \leq 0 \\ \langle x, (0,-1)\rangle \leq 0 \\ \langle x, (1,1)\rangle \leq 0 \end{matrix} \right\}$$

NOTE . The $\lambda = (\lambda_1, \ldots, \lambda_d)$ gives the **affine** structure of hyperplane arrangement. Using the normal vectors v_i, define

$$\begin{matrix} \mathbb{R}^d = \mathrm{Lie}(T^d) & \xrightarrow{\pi} & \mathrm{Lie}(T^n) = \mathbb{R}^n \\ \epsilon_i & \longmapsto & v_i \end{matrix}$$

This is surjective, with $k = \mathrm{Ker}(\pi)$:

$$0 \longrightarrow k \underset{i}{\longrightarrow} \mathbb{R}^d \underset{\pi}{\longrightarrow} \mathbb{R}^n \longrightarrow 0$$

which integrates to, at the level of Lie groups,

$$\{1\} \longrightarrow \underbrace{K}_{(d-n)\text{-dim subtorus}} \longrightarrow T^d \longrightarrow T^n \longrightarrow \{1\}$$

Consider the standard action $T^d \curvearrowright \mathbb{C}^d$. Restricting action to $K \curvearrowright \mathbb{C}^d$, we have a moment map for the K-action

$$i^*\varPhi = -\frac{1}{2} i^* \left[\left(\|z_1\|^2, \ldots, \|z_d\|^2 \right) + (\text{constant}) \right].$$

If we choose the constant to be

$$\lambda = \underbrace{(\lambda_1, \ldots\ldots, \lambda_n)}_{\substack{\text{constants determining affine} \\ \text{structure of hyperplanes}}}$$

then $\mathbb{C}^d \mathbin{/\!\!/}_0 K = (i^*\varPhi)^{-1}(0)/K = Z_\triangle / K$, a **symplectic toric manifold** with residual $T^n = T^d / T^{d-n}$-action and **residual moment image equal to** $\triangle$.

Combinatorics in action : the cohomology of toric varieties
As we have just seen, a symplectic toric manifold $M_\triangle$ is completely determined by the polytope. So in particular topological invariants should be computable in terms of $\triangle$.

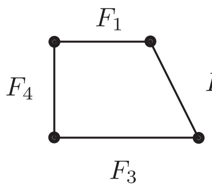
MOTIVATION :
For example, one can compute $H^*(M_\triangle))$ in terms of $\triangle$. There are many proofs of this formula, but one of them is via the Morse theory of the moment map and the "Kirwan method", which we will see later.

Recall :

$$\triangle = \{X \in (\mathbb{R}^n)^* \mid \langle x, v_i \rangle \leq \lambda_i, 1 \leq i \leq d\}$$
$$v_i = \text{primitive outward-pointing normal to facets } F_i.$$

Now: when do certain subsets of facets intersect?

Example 8.3.



$$F_1 \cap F_2 \neq \emptyset$$
but
$$F_1 \cap F_3 = \emptyset$$

$\rightsquigarrow$ encode this combinatorial information by defining

$$\sum := \left\{ \mathcal{I} \subseteq \{1, 2, \ldots, d\} \;\middle|\; \bigcap_{j \in \mathcal{I}} F_j = \emptyset \right\}$$

So, for example, in the above example we have

$$\sum = \{\{1,3\}, \{2,4\}, \{1,2,3\}, \{1,3,4\}, \{1,2,4\}, \{1,2,3,4\}\}$$

We will start with the polynomial ring $\mathbb{C}[u_1, \ldots, u_d]$ (one variable for each facet).
Define the Stanley-Reisner ideal

$$\mathcal{J} := \left\{ \prod_{j_k \in \mathcal{I}} u_{j_k} \;\middle|\; \mathcal{I} \in \sum \right\}$$
$$= \left\{ \prod_{j \in \mathcal{I}} u_j \;\middle|\; \bigcap_{j \in \mathcal{I}} F_j = \emptyset \right\}$$

Now define the ideal

$$\mathcal{K} := \left\{ \underbrace{\sum_i \langle u_i, \xi \rangle u_i}_{\text{linear polynomial}} \;\middle|\; \xi \in (\mathbb{R}^n)^* \right\}$$

Theorem 8.2.

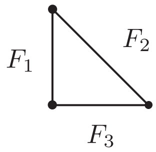
$$H^*(M_\triangle) \cong \mathbb{C}[u_1, \ldots, u_d]/\mathcal{J} + \mathcal{K}$$

Example 8.4.

(1)

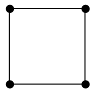
$$\mathbb{C}[u_1, u_2]/\langle u_1 - u_2, u_1 u_2\rangle$$
$$\cong \mathbb{C}[u]/\langle u^2\rangle$$

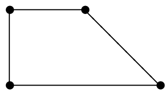
(2)



$$\mathbb{C}[u_1, u_2, u_3]\langle u_1 - u_2, u_2 - u_3, u_1 u_2 u_3\rangle$$
$$\cong \mathbb{C}[u]/\langle u^3\rangle$$

[Exercise]

(1) Try 

(2) Try 

These computations ought to be understood in terms of **equivariant** cohomology, NOT ordinary cohomology.

9 Equivariant cohomology

Let G be a compact Lie group. Suppose $G \curvearrowright M$ smooth.

ROUGHLY, the idea of equivariant cohomology is that it "ought" to be

$$H_G^*(M) = \text{"the ordinary cohomology of the orbit space } M/G\text{"}.$$

Going about this naively does not work well, e.g. if G is not acting freely.

SOLUTION "Force" the G-action to be free, but without changing the topology of M.

FACT : For G compact Lie group, there exists a principal G-bundle

$$\begin{tikzcd} EG \arrow[d] & G \arrow[l, hook'] \\ BG & \end{tikzcd}$$

such that EG is **contractible** (usually EG is infinite-dimensional).

Example 9.1. For $G = S^1$,

$$\begin{tikzcd} S^1 \arrow[r, hook] & S^\infty = ES^1 \arrow[d] \\ & \mathbb{C}P^\infty = BS^1 \end{tikzcd}$$

"Hopf fibration"

Definition 9.1. The Borel construction of a G-space M is defined as

$$(M \times EG)/G = M \times_G EG$$

where G acts diagonally on $M \times EG$.

NOTES .

- $M \times EG \cong M$ since EG contractible.
- G acts freely on $M \times EG$, since G acts freely on EG.

Definition 9.2. The **equivariant cohomology** of $G \curvearrowright M$ is $H_G^*(M) := H^*\left(M \times_G EG\right)$.

Special cases :

1. Suppose G acts freely on M. Then $H_G^*(M) = H^*(M \times_G EG)$ and

$$\begin{tikzcd} M \times_G EG \arrow[d] & EG \arrow[l, hook'] \\ M/G & \end{tikzcd}$$

fiber bundle over M/G

But the fiber is contractible, so $M \times_G EG \cong M/G \Longrightarrow \underbrace{H_G^*(M) \cong H^*(M/G)}_{\substack{\text{So in this case,} \\ \text{"naive" definition agrees with new definition}}}$

2. On the other extreme, suppose $G \curvearrowright \{\text{pt}\}$ trivially

$$H_G^*(\text{pt}) = H^*(\text{pt} \times_G EG) = H^*(BG).$$

In particular $H_{S^1}^*(\text{pt}) = H^*(\mathbb{C}P^\infty) = \mathbb{C}\,[u], \quad \deg u = 2$
similarly $H_{T^n}^*(\text{pt}) \cong \mathbb{C}[u_1, u_2, \ldots, u_n], \quad \deg u_i = 2.$

First properties :

- A G-equivariant map $f : M \longrightarrow N$ induces $f^* : H_G^*(N) \longrightarrow H_G^*(M)$.
- Since $M \times_G EG$ is a fiber bundle
$$\begin{tikzcd} M \arrow[r, hook] & M \times_G EG \arrow[d, "\pi"] \\ & BG \end{tikzcd},$$
so we always have a map backwards
$$\pi^* : H^*(BG) = H_G^*(\mathrm{pt}) \longrightarrow H^*(M \times_G EG) = H_G^*(M)$$
$\Longrightarrow H_G^*(M)$ is a $H_G^*(\mathrm{pt})$-module.
- For any G-equivariant homotopy $f_t : M \longrightarrow N$, $f_t^* : H_G^*(N) \longrightarrow H_G^*(M)$ are all the same ring map. In particular, H_G^* invariant under G-equivariant homotopy.

Luckily for us, there is another model (which does not directly involve the infinite-dimensional space EG and BG) for computing $H_G^*(M; \underbrace{\mathbb{R}}_{\text{because its a de Rham model}})$ called the Cartan model, an "equivariant version of the de Rham complex".

Definition 9.3. The equivariant differential forms of degree q are

$$\Omega_G^q(M) := \bigoplus_{2i+j=q} \left(S^i(\mathfrak{g}^*) \otimes \Omega^i(M)\right)^G$$

Remark : We can think of this as "G-equivariant polynomial map $\alpha : \mathfrak{g} \longrightarrow \Omega^*(M)$".

NOTE . Variables on $\mathfrak{g}$ have degree 2.

Definition 9.4. The **equivariant exterior derivative** is

$$d_G : \Omega_G^q(M) \longrightarrow \Omega_G^{q+1}(M)$$

defined by

$$(d_G\alpha)(\xi) = d(\alpha(\xi)) + i(\xi^\sharp)\alpha(\xi) \qquad {}^\forall \xi \in \mathfrak{g}.$$

[Exercise] $d_G^2 = 0$ **HINT :** for $\xi = \mathfrak{g}, \alpha \in \Omega_G^q(M), \alpha(\xi)$ is $\{\exp t\xi\}$-invariant.

[Exercise] Prove : for the case $M = \mathrm{pt}$, show that $\mathrm{Ker}\, d_G/\mathrm{Im} d_G = S(\mathfrak{g}^*)^G$. In particular, if $G = T$ a torus, since the coadjoint action is trivial,

$$\mathrm{Ker}\, d_T/\mathrm{Im} d_T = S(\mathfrak{t}^*) \cong \underbrace{\mathbb{R}[u_1, \ldots, u_n]}_{\substack{\text{agrees with } H^*((\mathbb{C}P^\infty)^n, \mathbb{R}) \\ \text{from before!}}}$$

In general, we have

Theorem 9.1. (Cartan) "Equivariant de Rham Theorem"
Let G be a compact connected Lie group, M a G-manifold. Then

$$H_G^*(M; \mathbb{R}) \cong \mathrm{Ker}\, d_G/\mathrm{Im} d_G$$

[FACT or Exercises]

1. Under appropriate identifications, use the exact sequences in the Delzant construction

$$\begin{array}{ccccccccc} \{1\} & \longrightarrow & K & \longrightarrow & T^d & \longrightarrow & T^n & \longrightarrow & \{1\} \\ 0 & \longrightarrow & k & \longrightarrow & \mathfrak{t}^d & \longrightarrow & \mathfrak{t}^n & \longrightarrow & 0 \\ 0 & \longrightarrow & (\mathfrak{t}^n)^* & \longrightarrow & (\mathfrak{t}^d)^* & \longrightarrow & k^* & \longrightarrow & 0 \end{array}$$

   to show that

$$H_K^*(\mathrm{pt}) \cong H_K^*(\mathbb{C}^d) \cong \underbrace{\mathbb{R}[u_1, \ldots, u_d]}_{H^*_{T^d}(\mathbb{C}^d)} / \mathcal{K},$$

   where $\mathcal{K}$ is exactly the ideal of linear relation from before.

2. **Recall :** Equivariant 2-form, by the definition of degree on $\Omega^q_G(M)$, must be of the form $\omega + \Phi$, where $\omega \in \Omega^2(M)^G$ and $\Phi : \mathfrak{g} \longrightarrow \Omega^0(M) = \mathcal{C}^\infty(M)$ is G-equivariant.
   Prove that $\omega + \Phi$ is equivariantly closed, that is, $d_G(\omega + \Phi) = 0$, if and only if

   - ω is closed.
   - Φ is a moment map for ω
     (i.e. it satisfies Hamiltonian equation).

Remark : Note that there is no condition on non-degeneracy. There is a whole theory of Hamiltonian actions for closed 2-forms (see Guillemin-Ginzburg-Karshon).

FACT : Let $\triangle$ be a Delzant polytope and $M_\triangle$ the associated symplectic toric manifold. Then

$$\begin{aligned} H^*_{T^n}(M_\triangle) &\cong H^*_{T^d}(\mathbb{C}^d)/\mathfrak{J} \\ &\cong \mathbb{C}[u_1, \ldots, u_d]/\mathfrak{J} \end{aligned}$$

We will see justification when we discuss Kirwan surjectivity.

$\boxed{\text{Question}}$ WHY DO WE CARE about equivariant cohomology?

[WARNING : very partial and biased answers]

- It is still related to ordinary cohomology by the diagram
$$\begin{array}{ccc} M & \hookrightarrow & M \times_G EG \\ & & \downarrow \\ & & BG \end{array}$$
  since the inclusion $M \hookrightarrow M \times_G EG$ induces a map $H^*_G(M) \longrightarrow H^*(M)$.

- It turns out to be (in many situations) easier to compute, with intimate relations to combinations.

- In particular, e.g. for a symplectic quotient $\Phi^{-1}(0)/G$ ($G \curvearrowright \Phi^{-1}(0)$ freely) instead of computing $H^*\left(\Phi^{-1}(0)/G\right)$ directly, can ask instead to compute $H^*_G(\Phi^{-1}(0))$.

- Many "linear-algebraic objects" (e.g. flag manifolds $\mathrm{GL}(n;\mathbb{C})/B$, quiver varieties and analogues) have natural group actions on them, and relationships to representation theory. So, for example, [just a sample of the literature!]

- In Schubert calculus, computations in equivariant cohomology have interpretations in terms of multiplicities of representations in tensor products.

- Can explicitly build Weyl group representation on equivariant cohomology of flag variaties (and subvarieties called Hessenberg varieties).

- Explicitly build representations of Kac-Moody Lie algebras on equivariant cohomology.

References :

(a) Allen Knutson, expository article on ArXiv keywords: symplectic, algebraic, Horn.

(b) Julianna Tymoczko, expository article on ArXiv keyword: permutation representations, equivariant cohomology.

(c) Nakajima "Varieties associated to quivers".

10 Survey of Localization Techniques

Reference : T.Holm, ArXiv, "Act globally, compute locally ..."

$\boxed{\text{Question}}$ What can you do with equivariant cohomology that you can't do with ordinary cohomology.

$\boxed{\text{THEME}}$ "First, localize"

WARNING / History : Localization in equivariant topology is very old idea, not at all symplectic geometry.

- In our context, by **localization** we will mean a result which relates the G-equivariant topology of a G-space X to the topology of the G-fixed points.
- Until stated otherwise, we work exclusively with $\mathbb{R}$ or $\mathbb{C}$ coefficients and $G = T$ a compact torus.

10.1 Injectivity into fixed points

Suppose M is a compact T-manifold. First observe that there is a T-equivariant embedding $M^T \xhookrightarrow{i} M$. This induces a ring map

$$i^* : H_T^*(M;\mathbb{C}) \longrightarrow H_T^*(M^T;\mathbb{C}).$$

The first localization theorem says,

Theorem 10.1. (Borel) With assumptions as above, the kernel and cokernel of i^* are torsion $H_T^*(\mathrm{pt};\mathbb{C})$-modules.

Recall : An element $x \in H_T^*(M;\mathbb{C})$ is $H_T^*(\mathrm{pt};\mathbb{C})$-torsion if there exists $0 \neq r \in H_T^*(\mathrm{pt};\mathbb{C})$ such that $rx = 0$. So in particular, if $H_T^*(M;\mathbb{C})$ is a **free** $H_T^*(\mathrm{pt};\mathbb{C})$-module, then i^* is injective.

NOTE . If the T-action is free on M, then $H_T^*(M;\mathbb{C}) \cong H^*(M/T;\mathbb{C})$ and $H^*(M/T;\mathbb{C})$ is entirely $H_T^*(\mathrm{pt};\mathbb{C}) \cong \mathbb{C}[u_1,\ldots,u_{\dim T}]$-torsion, because degrees are bounded above. So in this case it makes that $\operatorname{Ker} i^* = H_T^*(M;\mathbb{C})$, because $M^T = \emptyset$. In the case of Hamiltonian T-space, we're at the opposite extreme.

Theorem 10.2. Suppose (M,ω,T,Φ) compact Hamiltonian. Then the above map

$$i^* : H_T^*(M;\mathbb{C}) \longrightarrow H_T^*(M^T;\mathbb{C})$$

is injective.

NOTES .

- Indeed, one can show that $H_T^*(M;\mathbb{C})$ is a **free** $H_T^*(\mathrm{pt};\mathbb{C})$-module.
- **THE POINT :** The T-action on M^T is trivial, so the fiber bundle
$$\begin{tikzcd} M^T \arrow[r, hook] & M^T \times_T ET \arrow[d] \\ & BT \end{tikzcd}$$
is trivial and $H_T^*(M^T;\mathbb{C}) \cong H^*(M^T;\mathbb{C}) \otimes H_T^*(\mathrm{pt};\mathbb{C})$.
- Such a theorem cannot hold in ordinary cohomology! e.g. if $M^T =$ isolated, then $H^*(M^T;\mathbb{C}) \cong \mathbb{C}^{\sharp(M^T)}$, all in degree 0. So any element of positive degree in $H^*(M;\mathbb{C})$ would be in $\operatorname{Ker}(i^*_{\text{ordinary}})$. To take advantage of this theorem we must identify $\operatorname{Im}(i^*)$.

10.2 Goresky-Kottwitz-MacPherson theory [GKM]

Reference : Julianna S. Tymoczko. An introduction to equivariant cohomology and homology, following Goresky, Kottwitz, and MacPherson.

Suppose $T \curvearrowright M$ a compact T-manifold.

Definition 10.1. The equivariant i-skeleton of M is defined as

$$M_i := \{x \in M \mid \dim(T \cdot x) \leq i\}$$

Supposing $\dim M = n$, this gives us a filtration which interpolates between M^T and M

$$M^T = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

THE POINT : For compact Hamiltonian T-space, it turns out that to understand $\mathrm{im}(i^*)$, it suffices to understand M_1!!

NOTE . $M^T \xhookrightarrow{i} M$ and $M^T \xhookrightarrow{j} M_1$ induces a commutative diagram

$$\begin{tikzcd} H_T^*(M_1;\mathbb{C}) \arrow[d, "j^*"'] & H_T^*(M;\mathbb{C}) \arrow[l] \arrow[dl, "i^*"] \\ H_T^*(M^T;\mathbb{C}) & \end{tikzcd} \tag{3}$$

Theorem 10.3. (Tolman-Weitsman)
Suppose M is a compact Hamiltonian T-space. Then, in (3), $\mathrm{Im}(i^*) = \mathrm{Im}(j^*)$ in $H_T^*(M^T;\mathbb{C})$.

Historical remarks : There is a lot of related and previous work in more general contexts, e.g. Atiyah, Chang-Skjelbred, Franz-Puppe, etc ....

Now it suffices to understand $\mathrm{Im}(j^*)$. To do so, we make some simplifying assumptions.

$\boxed{\textbf{ASSUMPTION 1}}$ The fixed point set M^T is isolated, i.e.

$$M^T = \{p_1, p_2, \ldots, p_m\}.$$

THE POINT : This simplifies $H_T^*(M^T;\mathbb{C})$. Indeed,

$$H_T^*(M^T;\mathbb{C}) \cong \bigoplus_{i=1}^m \mathbb{C}[u_1, \ldots, u_d]$$

where $d = \dim T$. This is a restrictive condition, but there are many examples that satisfy it, e.g., toric varieties and (partial) flag varieties. Now observe that at every fixed point $p \in M^T$, there is a linear T-action $T \curvearrowright T_pM$. Choose T-invariant complex structure on T_pM, we may consider $T \curvearrowright T_pM$ on a $\mathbb{C}$-linear representation and decompose

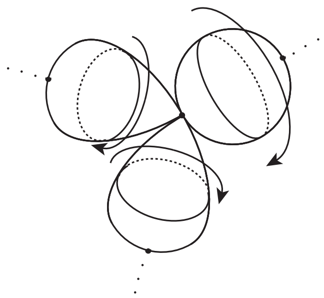
$$T_pM \cong \mathbb{C}_{\alpha_{1,p}} \oplus \cdots \oplus \mathbb{C}_{\alpha_{k,p}}$$

into T-weight spaces ($k = \frac{1}{2}\dim M$). We call $\alpha_{i,p} \in \mathrm{Lie}(T^d)^*_{\mathbb{Z}}$ **the isotropy weights at** p.

NOTE . By assumption, all $\alpha_{i,p}$ are all not zero at all $p \in M^T$.

$\boxed{\textbf{ASSUMPTION 2}}$ At every $p \in M^T$, the isotropy weights $\{\alpha_{i,p}\}_{i=1}^k$ are pairwise independent in $(\mathfrak{t}^n)^*$.

TERMINOLOGY : If $T \curvearrowright (M, \omega)$ as above and satisfies Assumptions 1 and 2, then we say the action is **GKM**. Another interpretation, or consequence, of Assumptions 1 and 2, the equivariant 1-skeleton M_1 is 2-dimensional. That is, M_1 is a union of S^2's, equipped with a T-action given by a non-zero weight $\alpha \in \mathfrak{t}^*_{\mathbb{Z}}$ (i.e. a homomorphism $T^d \longrightarrow S^1 \underset{\text{standard}}{\curvearrowright} S^2$), intersecting at the fixed points:



M is a “wedge of balloons”

Again, this is a restrictive condition but symplectic toric manifolds (with its full torus action) and coadjoint orbits of Lie groups (with the maximal torus action) satisfy Assumptions 1 and 2.

Define the following combinatorial object associated to M_1.

Definition 10.2. The **GKM graph** of $T \curvearrowright M$ [assume M is GKM] is a labeled graph $\mathit{\Gamma} = (V, E, \alpha)$, where

$$\begin{aligned} V &= M^T = \text{fixed points} \\ E &= \left\{ (p,q) \in M^T \times M^T \,\middle|\, {}^{\exists}\text{an embedded } S^2 \subseteq M_1, (S^2)^T = \{p,q\} \right\}. \end{aligned}$$

We also label each edge with the weight $\alpha_{(p,q)} \in \mathfrak{t}^*_{\mathbb{Z}}$ specifying the action of T^d on the copy of S^2 corresponding to (p,q) [1].

In particular, $\mathit{\Gamma}$ encodes the T-equivariant topology of M_1.

Recall : The isotropy weights $\alpha_{(p,q)}$ can be viewed as elements of $H^*_T(\mathrm{pt};\mathbb{C}) \cong S(\mathfrak{t}^*)$.

Theorem 10.4. (after **Goresky-Kottwitz-MacPherson**)
Let M be compact Hamiltonian T-space satisfying Assumptions 1 and 2. Then the image of i^* is

$$\mathrm{Im}(i^*) \cong \left\{ (f_p) \in \bigoplus_{p \in M^T} \mathbb{C}[u_1, \dots, u_d] \cong H^*_T(M^T;\mathbb{C}) \,\middle|\, \begin{array}{l} \alpha_{(p,q)} \mid f_p - f_q \ {}^{\forall}(p,q) \in E \\ \text{in } H^*_T(\mathrm{pt};\mathbb{C}) = \mathbb{C}[u_1, \dots, u_d] \end{array} \right\}$$

Example 10.1. (1) $S^1 \underset{\text{standrd}}{\curvearrowright} S^2$. $M^{S^1} = (S^2)^{S^1} = \{N, S\}$. The S^1-isotropy action at $T_N S^2 \cong \mathbb{C}$ is weight -1, so $\alpha = -1 \cdot u \in \mathbb{C}[u] = H_{S^1}(\mathrm{pt})$.

The GKM graph is
$$\begin{array}{ll} \bullet & f(u) = f_N(u) \\ \big| & \\ \bullet & g(u) = f_S(u) \end{array}$$

so the image of i^* is

$$\left\{ \begin{array}{ll} \bullet & f(u) \in \mathbb{C}[u] \\ \big| & \\ \bullet & g(u) \in \mathbb{C}[u] \end{array} \,\middle|\, \begin{array}{l} u \mid f(u) - g(u) \\ \text{i.e. } f \text{ and } g \text{ have the same} \\ \text{constant term.} \end{array} \right\}$$

$\Longrightarrow$ as a $H^*_{S^1}(\mathrm{pt};\mathbb{C})$-module, $H^*_{S^1}(S^2;\mathbb{C})$ has as basis

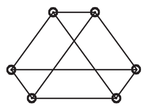
$$\begin{array}{ll} \bullet & 1 \\ \big| & \\ \bullet & 1 \end{array} \qquad\qquad \begin{array}{ll} \bullet & u \\ \big| & \\ \bullet & 0 \end{array}$$

[1] There is a sign ambiguity on the choice of labeling but the combinational description of $\mathrm{im}(i^*)$ is independent of this choice

(2)

$$T^2 \curvearrowright \big(M = \mathfrak{Fl}(\mathbb{C}^3)\big) = \text{coadjoint orbit of } SU(3)$$
$$= SO(3) \cdot \begin{bmatrix} i\lambda_1 & & \\ & i\lambda_2 & \\ & & i\lambda_3 \end{bmatrix} (\lambda_1 > \lambda_2 > \lambda_3,\ \lambda_1 + \lambda_2 + \lambda_3 = 0)$$

The GKM graph is of the form $M^T = \text{orbit of } \begin{bmatrix} i\lambda_1 & & \\ & i\lambda_2 & \\ & & i\lambda_3 \end{bmatrix}$ under the Weyl group S_3

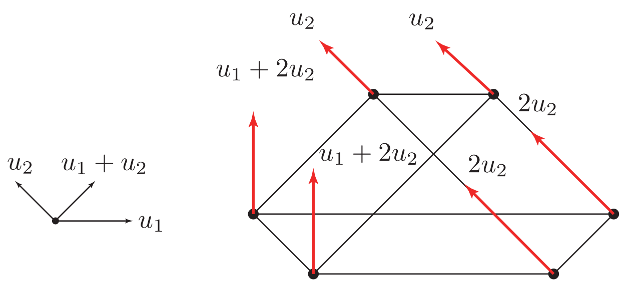


$\subseteq (\mathfrak{k}^2)^*$

NOTES .

- This is actually the image under the moment map Φ of the equivariant 1-skeleton.
- From the local normal form near a T-fixed point $p \in M^T$ and the general formula for a T-moment map for a linear T-action, we see that the isotropy weight data on each edge is already encoded by taking the (primitive) vector in the direction of the corresponding $\Phi(S^2_{(p,q)})$.
- We may depict a degree 2 element in the equivariant cohomology (i.e. linear in the u_i so an element of $\mathfrak{k}^*$) by drawing an arrow with vector at every point.

Example 10.2.



This satisfies GKM conditions, so is an element of $\mathrm{im}(i^*)$

THE POINT : As can be seen in the above examples, GKM spaces have **explicit**, **computable** $H_T^*(M;\mathbb{C})$.

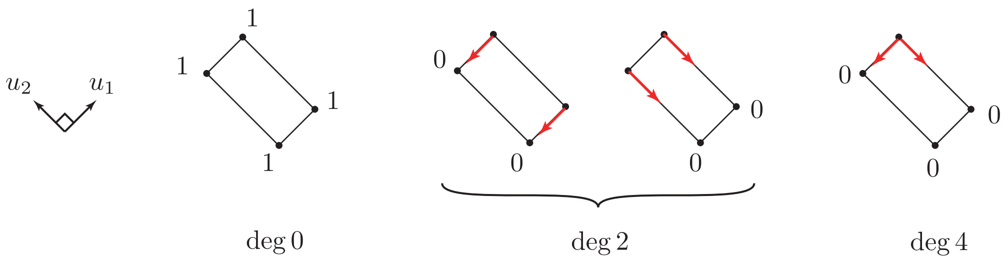
10.3 Some applications and generalizations of GKM theory

(**WARING AGAIN :** very partial sample!)

Reference : J. Tymoczko, "Permutation actions on equivariant cohomology" ArXiv : 0706.0460

1. Building $H_T^*(\mathrm{pt};\mathbb{C})$-module generators:
   The explicit description allows us to inductively build computationally convenient module generators.

   Example 10.3. $T^2 = S^1 \times S^1 \curvearrowright S^2 \times S^2$



this is built so that "below a critical point p_i" all vertices have 0 polynomials. Such module generators are, for example, used extensively in Schubert calculus (i.e. computations in $H_T^*(G/P)$.

2. Building representation on $H_T^*(M;\mathbb{C})$. In the case $M = \mathfrak{Fl}(\mathbb{C}^n)$, we have seen already that $M^T \cong W \cong S_n$, the group of permutations on n elements.

$$(\spadesuit)\begin{cases}\text{Since } S_n \text{ is a group, it acts on itself by (left or right) multiplication.}\\ \text{It also acts on } \mathbb{C}[u_1,\ldots,u_n] \cong H_T(\mathrm{pt};\mathbb{C}) \text{ by permuting variables.}\end{cases}$$

$\rightsquigarrow$ since $H_T^*(M;\mathbb{C})$ can be described as functions

$$f : S_n \cong M^T \longrightarrow \mathbb{C}[u_1,\ldots,u_n]$$

can define using ($\spadesuit$) different actions on cohomology.

- can build explicitly S_n-representations on $H_T^*(M;\mathbb{C})$ or on $H_T(Y;\mathbb{C})$ for certain subvarieties $Y \subseteq M$.
- derive "divided difference operators" in Schubert calculus.

3. Explicit computations of ordinary cohomology of symplectic quotients. Given (M,ω,T,Φ), [with some technical conditions] first compute $H_T^*(M;\mathbb{C})$ to try to understand

$$\underbrace{H^*\left(M \mathbin{/\!\!/}_{0} T;\mathbb{C}\right) \cong H_T^*\left(\Phi^{-1}(0);\mathbb{C}\right).}_{\text{Much more on this later}}$$

More recent developments in GKM-type theory

(**WARNING:** assumptions are subtly different in all references! for details you must check each paper.)

- Work in the algebraic category : (T = algebraic torus)
  - similar results for equivariant Chow rings, $T \curvearrowright$ projective nonsingular varieties [Brion]
  - equivariant intersection cohomology, $T \curvearrowright$ singular projective varieties [Braden-MacPherson]
  - equivariant algebraic K-theory [Vezzosi-Vistoli '03]
- Symplectic category, weakened assumptions: $H_T^*(-;\mathbb{C})$.
  - some non-compact cases: [Harada-Holm]
  - Assumption 2 only: [Guillemin-Holm]
  - $\dim(M_1) \leq 4$: [Goldin-Holm]
- Symplectic category, topological equivariant integral K-theory

- for K-theory over $\mathbb{C}$: [Kutson-Rosu]
- similar module generators: [Guillemin-Kogan]
- some non-compact cases, integral K-theory: [Harada-Landweber]

$$\vdots$$

(lots more!)

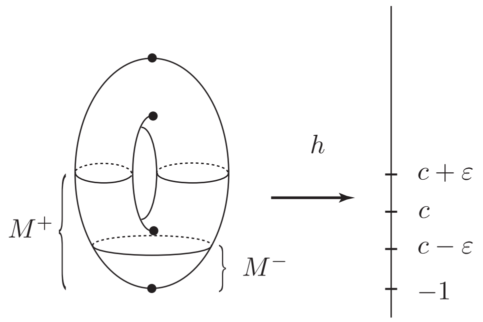
11 Morse theory and moment maps

("How to prove §10 in the symplectic category")

$\boxed{\text{THEME}}$ "To compute ordinary / equivariant topology of Hamiltonian G-space and their symplectic quotients, use Morse theory associated to the moment map".

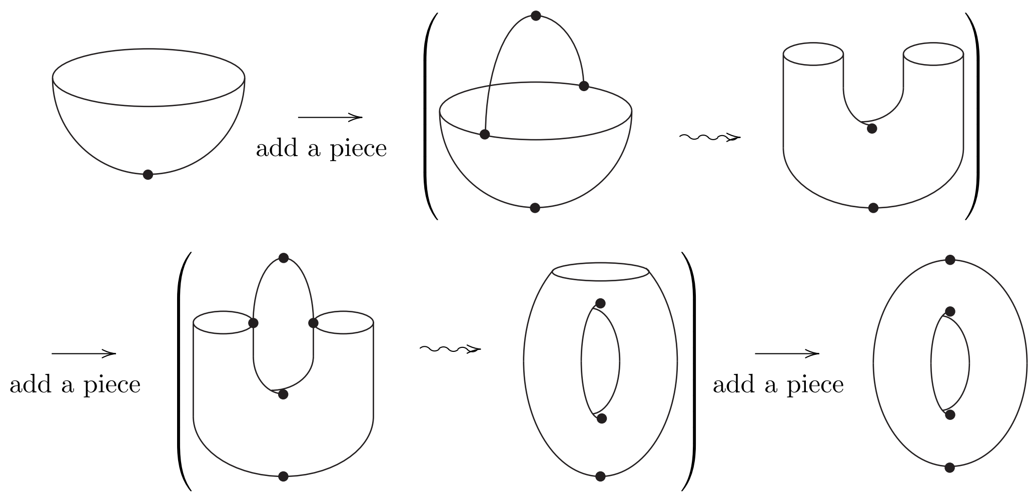
11.1 QUICK review of Morse theory in pictures

Illustration by example, M = 2-dimensional torus, $h : M \longrightarrow \mathbb{C}$ height function



$$M^+ := h^{-1}\left((\infty, c+\varepsilon]\right)$$
$$M^- := h^{-1}\left((\infty, c-\varepsilon]\right)$$

Then one can compute the topology of M "piece by piece", with one piece for each critical point, starting at the bottom.



THE RECIPE for the "pieces" : At a critical p_c, compute the dimension of the negative eignspace of the Hessian of h at p_c (=: index of h at p_c), call it λ_{p_c} or λ_c. Let

$$D^{\lambda_c} := \text{unit disc in } \mathbb{R}^{\lambda_c}$$
$$S^{\lambda_c - 1} := \text{unit sphere} \subseteq D^{\lambda_c}$$

Then

$$M^+ \sim D^{\lambda_c} \cup_{S^{\lambda_c-1}} M^-,$$

where we haven't specified the attaching map.

Remark : In classical Morse theory critical points are isolated, but Morse-Bott theory allows us to deal with some cases where critical points are non-isolated. In that case, data to go from M^-

to M^+ also involves the topology of the critical set. Now general principles tell us, at each stage, that we have a LES of the pair (M^+, M^-):

$$\cdots \longrightarrow H^*(M^+, M^-) \longrightarrow H^*(M^+) \longrightarrow H^*(M^-) \longrightarrow \cdots$$

So, if we know something about the "ends", we can say something about $H^*(M^+)$ (Also work equivariantly.)
$\Longrightarrow$ inductive arguments for: Poincaré polynomials, injectivity, GKM, $\cdots$

11.2 Moment maps as Morse functions

Suppose (M, ω, T, Φ) is a (compact) connected Hamiltonian T-space. We wish to do Morse theory with $\Phi : M \longrightarrow \mathfrak{t}^*$ but Φ is $\mathfrak{t}^*$-valued, not necessarily $\mathbb{R}$-valued. One solution to do this is the following.

FACT : For each $X \in \mathrm{Lie}(T)$, consider $\Phi^X := \langle \Phi, X\rangle : M \longrightarrow \mathbb{R}$ the **component of Φ along X**. Then this is a **Morse-Bott function**, and

$$\mathrm{Crit}(\Phi^X) = \bigcap_{t \in \overline{\{\exp sX\}} \subseteq T} M^t$$

where $M^t := \{p \in M \mid t \cdot p = p\}$ and $\overline{\{\exp sX\}}$ denotes the closure in T of the 1-parameter subgroup generated by X, In particular, for generic X, then

$$\mathrm{Crit}(\Phi^X) = M^T$$

Definition 11.1. A Morse-Bott function on a compact Riemannian manifold is a function $f : M \longrightarrow \mathbb{R}$ such that

- $\mathrm{Crit}(f) \leq M$ is a submanifold
- ${}^\forall p \in \mathrm{Crit}(f), T_p\mathrm{Crit}(f) = \mathrm{Ker}\,(\mathrm{Hess}(f)_p)$.

Here, the analogy of $(D^\lambda, S^\lambda - 1)$ becomes a (disc, sphere)-bundle of a vector bundle over (a connected component of) $\mathrm{Crit}(f)$. Let's examine more closely the relationship between the Morse theory and the T-action. Assume X is generic. Let $p \in M^T$. Then locally near p, the T-action looks like $T \curvearrowright (\mathbb{C}^n, \omega_0)$ **linear**, so

$$T \curvearrowright \mathbb{C}^n \cong \bigoplus_{i=1}^n \mathbb{C}_{\alpha_{i,p}} \left.\vphantom{\bigoplus_{i=1}^n}\right\} \begin{array}{l}\text{decomposition}\\ \text{into weight spaces}\end{array}$$

and locally

$$\Phi(z_1, \ldots, z_n) = -\frac{1}{2}\sum_{i=1}^n \alpha_{i,p}\|z_i\|^2,$$

so

$$\langle \Phi, X\rangle(z_1, \ldots, z_n) = -\frac{1}{2}\sum_{i=1}^n \langle \alpha_{i,p}, X\rangle \|z_i\|^2.$$

¿From this explicit description, we see

(1) $T_p(\mathrm{Crit}(f)) = \bigoplus_{i:\alpha_{i,p}=0} \mathbb{C}_{\alpha_{i,p}}$.

(2) the positive eigenspace of $\mathrm{Hess}(f)_p = \bigoplus_{i:\langle \alpha_{i,p}, X\rangle < 0} \mathbb{C}_{\alpha_{i,p}}$.

(3) the negative eigenspace of $\mathrm{Hess}(f)_p = \displaystyle\bigoplus_{i:\langle \alpha_{i,p}, X\rangle > 0} \mathbb{C}_{\alpha_{i,p}}$ and in particular

$$\text{index at } p \text{ of } \Phi^X = \underbrace{2 \cdot \sharp \{\alpha_{i,p} : \langle \alpha_{i,p}, X\rangle > 0\}}_{\text{this is computable by local } T\text{-isotropy data!}}$$

11.3 Equivariant Euler class

With this in mind, analyze the LES, but this time in equivariant cohomology.

$$(\diamondsuit) \qquad \begin{tikzcd} \cdots \arrow[r] & H_T^*(M^+, M^-; \mathbb{C}) \arrow[r, "(\clubsuit)"] \arrow[d, "\cong"', "\substack{(M^+,M^-) \text{ retracts to} \\ (D^{\lambda_c}, S^{\lambda_c - 1})}"] & H_T^*(M^+; \mathbb{C}) \arrow[r] \arrow[d] & H_T^*(M^-; \mathbb{C}) \arrow[r] & \cdots \\ & H_T^*(D^{\lambda_c}, S^{\lambda_c - 1}; \mathbb{C}) \arrow[r] \arrow[d, "\cong"', "\text{Thom isom}"] & H_T^*(D^{\lambda_c}; \mathbb{C}) & & \\ & H_T^{*-\lambda_c}(D^{\lambda_c}; \mathbb{C}) \arrow[ur, "\cup e_T(\nu_c^-)"'] & & & \end{tikzcd}$$

multiplication by the equivariant Euler class of $e_T(\nu_c^-)$ the negative normal bundle ν_c^-

$\boxed{\text{KEY POINTS}}$

- For every inductive argument the first step is to prove $(\diamondsuit)$ splits into SES.
- To show $(\diamondsuit)$ splits, it suffices to show $(\clubsuit)$ is injective.
- To show $(\clubsuit)$ injective, it suffices to show $e_T(\nu_c^-)$ is not a zero divisor in $H_T^*(D^{\lambda_c}; \mathbb{C}) \cong H_T^*(\underbrace{\Sigma_c}_{\text{component of } \mathrm{Crit}(\Phi^X)}; \mathbb{C})$

THEME : Atiyah-Bott lemma exactly implies that $e_T(\nu^-)$ is not a zero divisor.

NOTE (once again)**.** It is crucial that we are working in equivariant cohomology, because an ordinary Euler class $e(\nu_c^-) \in H^*(\Sigma_c; \mathbb{C})$ is always a zero divisor.

DIGRESSION on equivariant characteristic classes :
A G-equivariant bundle over a G-space M is a bundle $E \xrightarrow{\pi} M$ and a lift of the G-action on M to an action on E by bundle maps. Given an equivariant bundle $G \curvearrowright \begin{pmatrix} E \\ \downarrow \\ M \end{pmatrix}$, we automatically get an ordinary bundle over the Borel construction

$$\begin{array}{c} E \times_G EG \\ \big\downarrow \\ M \times_G EG \end{array}$$

and define

$$c_G(E) := c \begin{pmatrix} E \times_G EG \\ \downarrow \\ M \times_G EG \end{pmatrix} \in H^*(M \times_G EG) = H_G^*(M)$$

for any characteristic class c.

Example 11.1. Consider the S^1-equivariant line bundle $\mathbb{C}_\lambda \longrightarrow \mathrm{pt}$, where $S^1 \curvearrowright \mathbb{C}_\lambda$ with weight $\lambda \neq 0$. Then the equivariant Euler class $e_{S^1}\begin{pmatrix}\mathbb{C}_\lambda\\ \downarrow\\ \mathrm{pt}\end{pmatrix} = \lambda \cdot u \in \mathbb{C}[u] = H^*_{S^1}(\mathrm{pt})$ is not a zero divisor in $\mathbb{C}[u]$.

In general, if $T^n = (S^1)^n \curvearrowright \begin{pmatrix}\mathbb{C}_\alpha\\ \downarrow\\ \mathrm{pt}\end{pmatrix}$ with weight $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \cong \mathrm{Lie}(T)^*_{\mathbb{Z}}$

$$[\text{so } T^n \longrightarrow S^1 \curvearrowright \mathbb{C} \text{ via } (t_1, \ldots, t_n) \longmapsto t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_n^{\alpha_n} \in S^1]$$

Then $e_{T^n}\begin{pmatrix}\mathbb{C}_\alpha\\ \downarrow\\ \mathrm{pt}\end{pmatrix} = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n \in \mathbb{C}[u_1, \ldots, u_n] \cong H^*_{T^n}(\mathrm{pt})$, again not a zero divisor if $\alpha \neq 0$.

Back to Morse theory :

Suppose $\langle \Phi, X \rangle$ is Morse, so critical points are isolated, so the negative normal bundle is just

$$T \curvearrowright \begin{pmatrix} \displaystyle\bigoplus_{i:\langle \alpha_{i,p}, X\rangle > 0} \mathbb{C}_{\alpha_{i,p}} \\ \downarrow \\ \mathrm{pt} \end{pmatrix}$$

In particular all $\alpha_{i,p}$ in the sum above are non-zero. Then

$$e_T\left(\bigoplus_{i:\langle \alpha_{i,p}, X\rangle > 0} \mathbb{C}_{\alpha_{i,p}}\right) = \prod_{i:\langle \alpha_{i,p}, X\rangle > 0} e_T\left(\mathbb{C}_{\alpha_{i,p}}\right)$$

a product of non-zero linear polynomials, in particular not a zero divisor in $\mathbb{C}[u_1, \ldots, u_n]$.

$\boxed{\textbf{KEY LEMMA}}$ (state a special case), for general case (original version: Atiyah-Bott 1982)

Let B be a compact oriented manifold with a T-action (T = compact torus). Suppose $E \longrightarrow B$ is T-equivariant finite-dimensional complex vector bundle. Suppose further that there exists $S^1 \subseteq T$ such that $E^{S^1} = B$. Then the T-equivariant Euler class $e_T(E) \in H^*_T(B; \mathbb{C})$ is not a zero divisor.

Remark : There are many variations and generalizations of this Lemma.

Example 11.2.

- over $H^*(-;\mathbb{Z})$, stated in [Harada-Holm]
- in algebraic geometry, for algebraic K-theory [Vezzosi-Vistoli 2003]
- in symplectic geometry, for topological integral K-theory [Harada-Landweber]

$\rightsquigarrow$ Since the Atyiah-Bott lemma is theoretical ingredient in Morse-theoretic arguments using Φ, once the A-B lemma generalizes, expect many other results to generalize.

11.4 Sample Morse argument : injectivity

Recall : Suppose (M, ω, T, Φ) compact Hamiltonian. Then

$$i : M^T \hookrightarrow M$$

is a T-equivariant embedding, inducing

$$i^* : H_T^*(M; \mathbb{C}) \longrightarrow H_T^*(M^T; \mathbb{C})$$

Theorem 11.1. i^* is injective.

Let $X \in \mathfrak{k}$ generic, so $\mathrm{Crit}(\Phi) = M^T =: \Sigma$. Let $c_1 < c_2 < \cdots < c_m$ be the set of critical values and let $\Sigma_{c_1}, \Sigma_{c_2}, \ldots, \Sigma_{c_m}$ be the associated components of Σ (without loss generality, each Σ_{c_i} connected). For small enough $\varepsilon > 0$, let

$$\begin{aligned} M_{c_i}^{\pm} &:= \left(\Phi^X\right)^{-1}((-\infty, c_i \pm \varepsilon]) \\ \Sigma_{c_i}^{\pm} &:= M_{c_i}^{\pm} \cap \Sigma \end{aligned}$$

Proof. **Base case**. $M_{c_1}^+$ is equivariantly homotopic to $\Sigma_{c_1} = \left(M_{c_1}^+\right)^T$, the **minimal** critical component, so certainly $i^* : H_T^*(M_{c_1}^+; \mathbb{C}) \longrightarrow H_T^*(\Sigma_{c_i}; \mathbb{C})$ is an isomorphism, so in particular injective.
Inductive step. $M_{c_i}^+$ is equivariantly homotopic to $M_{c_{i-1}}^+$. Assume the result for $M_{c_{i-1}}^+$ and prove for $M_{c_i}^+$. By the A-B lemma, the LES for $(M_{c_i}^+, M_{c_{i-1}}^+)$ splits:

$$\begin{tikzcd} 0 \arrow[r] & H_T^*(M_{c_i}^+, M_{c_i}^-; \mathbb{C}) \arrow[r] \arrow[d, "(\flat)"] & H_T^*(M_{c_i}^+; \mathbb{C}) \arrow[r] \arrow[d, "(\natural)"] & H_T^*(M_{c_i}^-; \mathbb{C}) \arrow[r] \arrow[d, "(\sharp)"] & 0 \\ 0 \arrow[r] & H_T^*(\Sigma_{c_i}; \mathbb{C}) \arrow[r] & H_T^*(\Sigma_{c_i}^+; \mathbb{C}) \arrow[r] & H_T^*(\Sigma_{c_i}^-; \mathbb{C}) \arrow[r] & 0 \end{tikzcd}$$

By induction, $(\sharp)$ is injective, and $(\flat)$ is injective by the Thom isomorphism, so by the Five-Lemma $(\natural)$ is injective. $\square$

Remarks :
This is the most straightforward of these types of arguments, but similar techniques are used to prove

- [Tolman-Weitsman] For $j : M^T \hookrightarrow M_1$, $i : M^T \hookrightarrow M$, then $\mathrm{im}(j^*) = \mathrm{im}(i^*)$.
- If $T \curvearrowright (M, \omega)$ is GKM, then

$$H_T^*(M; \mathbb{C}) \cong \left\{ (f_p) \in \bigoplus_{p \in M^T} H_T^*(p; \mathbb{C}) \,\middle|\, \alpha_{(p,q)} | f_p - f_q{}^{\forall}(p, q) \right\}$$

12 The topology of symplectic quotients

12.1 An introduction to "Kirwan method"

$\boxed{\text{Question}}$ Compute, i.e. give nice description of topological invariants of symplectic quotients using equivariant symplectic data from the original Hamiltonian G-space.

$\boxed{\text{Some answer}}$ The Kirwan method

ⓐ "Kirwan surjectivity" (1984)

Theorem 12.1. Suppose $G \curvearrowright (M,\omega) \xrightarrow{\Phi} \mathfrak{g}^*$, Hamiltonian G-space. (Let G compact connected Lie group.) Assume Φ is proper and G acts freely on $\Phi^{-1}(0)$. Then the inclusion $i : \Phi^{-1}(0) \hookrightarrow M$ induces a ring homomorphism

$$\begin{tikzcd} H^*_G(M;\mathbb{C}) \arrow[r, "i^*"] \arrow[dr, "\mathcal{K}"'] & H^*_G(\Phi^{-1}(0);\mathbb{C}) \arrow[d, "\cong"] \\ & H^*(M /\!\!/_0 G;\mathbb{C}) \end{tikzcd}$$

which is **surjective**.

$\boxed{\text{SO WHAT?}}$ To compute $H^*(M /\!\!/_0 G;\mathbb{C})$, Kirwan surjectivity says that it suffices to compute

$$\left.\begin{array}{ll} ⓑ & H^*_G(M;\mathbb{C}) \\ ⓒ & \operatorname{Ker}\mathcal{K}. \end{array}\right\} \text{Here, we can use equivariant techniques}$$

One-line proof of ⓐ : Use equivariant Morse theory of $f := \|\Phi\|^2 : M \longrightarrow \mathbb{R}$. [we will come back to this]

Remark : There are also equivariant version of Kirwan surjectivity. i.e.

$$H^*_{G\times K}(M;\mathbb{C}) \longrightarrow H^*_K(M /\!\!/_0 G;\mathbb{C})$$

Now solve ⓑ and ⓒ.
ⓑ For $G = T$, we have already seen many techniques!
ⓒ Compute Ker $(\mathcal{K})$: some history.

- The Jeffrey-Kirwan residue formula
  - compute, in principle, Ker $(\mathcal{K})$ by giving a formula for $\int_{[M /\!\!/ G]} \mathcal{K}(\alpha)$ for all α.
    $\rightsquigarrow$ re-derivation of Witten's nonabelian localization formula.
  - in particular computes $\underbrace{H^*((n,d))}_{\substack{\text{moduli space of rank } n \text{ degree } d \text{ semistable holo-} \\ \text{morphic vector bundles over a Riemann surface with} \\ (n,d)=1 \text{ and with fixed determinant.}}}$ by the residue theorem.

    $\rightsquigarrow$ provides mathematical proof of formula for intersection pairings found by Witten.

    NOTE . For this gauge theory example, generators already found by Atiyah-Bott.
- Tolman-Weitsman:

- **direct** computation of $\mathrm{Ker}\,(\mathcal{K})$ using M^T data
- **Example application**: re-derives the Stanley-Reisner ideal for H^*(symplectic toric manifolds)

- Rebecca Goldin:
  - a refinement of Tolman-Weitsman
    $\rightsquigarrow$ produces efficient and visibly finite algorithm for computing $\mathrm{Ker}\,(\mathcal{K})$
  - **Example application**: [Goldin, Goldin-Mare]
    Let $T \subseteq G$ maximal torus, and let $\lambda \in (\mathcal{K}^*)_{\mathbb{Z}} \subseteq \mathfrak{g}^*$.
    Consider the Hamiltonian T-space

$$T \curvearrowright G \cdot \lambda \cong G/G_\lambda \hookrightarrow \mathfrak{g}^* \longrightarrow \mathfrak{k}^* \ .$$

Borel-Weil $\Longrightarrow Q(G/G_\lambda) \underset{G\text{-rep}}{\cong} V_\lambda$

$$\text{Quantization commutes with reduction} \Longrightarrow Q\left(G/G_\lambda \underset{\mu}{/\!\!/} T\right) = Q\,(G/G_\lambda)^{\mu\text{-weight space}}$$
$$= (V_\lambda)^{\mu\text{-weight space}}$$

[Goldin, Goldin-Mare] : applies Kirwan method and Schubert calculus techniques to compute

$$H^*\left(G/G_\lambda \underset{\mu}{/\!\!/} T\right)$$

- S. Martin (ArXiv): "reduce from G to T".
  Many techniques exist for T a torus, not for general (in many cases)

$$\mathrm{Ker}\left(\mathcal{K}_G : H_G^*(M;\mathbb{C}) \longrightarrow H_G^*(\Phi_G^{-1}(0);\mathbb{C})\right)$$

using knowledge of

$$\mathrm{Ker}\left(\mathcal{K}_T : H_T^*(M;\mathbb{C}) \longrightarrow H_T^*(\Phi_T^{-1}(0);\mathbb{C})\right)$$

where $T \subseteq G$ maximal torus.

Remark about proof : Analyze the geometric relationship between $M /\!\!/ G$ and $M /\!\!/ T$.
Notice

$$\begin{tikzcd} M \arrow[r, "\Phi_G"] \arrow[dr, "\Phi_T = \pi \circ \Phi_G"'] & \mathfrak{g}^* \arrow[d, "\pi"] \\ & \mathcal{K}^* \end{tikzcd}$$

so $\Phi_G^{-1}(0) \hookrightarrow \Phi_T^{-1}(0)$.

One has to carefully analyze the diagram

$$\begin{tikzcd} \Phi_G^{-1}(0)/T \arrow[r, hook] \arrow[d] & \Phi_T^{-1}(0)/T =: M /\!\!/ T \\ \Phi_G^{-1}(0)/G & \end{tikzcd}$$

12.2 Proof of Kirwan surjectivity : Morse theory of $\|\Phi\|^2$

Here G compact connected, (M,ω) Hamiltonian G-space with proper Φ.

BASIC IDEA :

Consider $M \xrightarrow{\Phi} \mathfrak{g}^* \xrightarrow{\|\cdot\|^2} \mathbb{R}$

- Note $f = \|\Phi\|^2 \geq 0$.
- Think of this as a Morse function.
- The minimum is $f^{-1}(0) = \Phi^{-1}(0)$. So if we can build M by "Morse strata", then if we can prove at every stage
$$H_G^*(M^+) \twoheadrightarrow H_G\left(\Phi^{-1}(0)\right)$$
we are done.

RATHER MAJOR TECHNICAL PROBLEM :
f is NOT Morse, NOT Morse-Bott!
$\rightsquigarrow$ Kirwan famously resolves this problem by showing that f has enough good properties to make the philosophy go through. [we'll sidestep this issue!]
Build M as usual using Morse strata, with

$$S_\beta = \text{Morse stratum corresponding to a component } C_\beta \text{ of } \mathrm{Crit}(f).$$

Adding one stratum at a time, we have the LES

$$(\Diamond) \cdots \longrightarrow H_G^*\left(\sqcup_{\gamma\leq\beta} S_\gamma, \sqcup_{\gamma<\beta} S_\gamma; \mathbb{C}\right) \xrightarrow{(\flat')} H_G^*(\sqcup_{\gamma\leq\beta} S_\gamma; \mathbb{C}) \xrightarrow{(\natural')} H_G^*(\sqcup_{\gamma<\beta} S_\gamma; \mathbb{C}) \longrightarrow \cdots$$

Base case : $S_0 \cong \Phi^{-1}(0)$. So certainly $H_G^*(S_0) \twoheadrightarrow H_G^*\left(\Phi^{-1}(0)\right)$. Since $(\natural')$ surjective is implied by $(\flat')$ injective, the same argument as yesterday, if $(\Diamond)$ splits into SES at every stage, then we're done. Kirwan shows that a different LES exists, on which we can apply A-B, and its splitting implies the splitting of $(\Diamond)$.

12.3 Computing $\mathrm{Ker}\,\mathcal{K}$

© [Tolman-Weitsman] Here $G = T$.

BASIC IDEA : Use both $\|\Phi\|^2$ and $\langle \Phi, X\rangle$

Remarks

- $\mathrm{Crit}\|\Phi\|^2 \supseteq M^T$
- Since $H_T^*(M) \hookrightarrow H_T^*(M^T)$, to understand $\mathrm{Ker}\,(\mathcal{K})$, suffices to understand $i^*(\mathrm{Ker}\,(\mathcal{K}))$ and suffices to understand how $\mathrm{Ker}\,(\mathcal{K})$ restricts to $\mathrm{Crit}\|\Phi\|^2$.
- For any $\beta \in \mathcal{K}^*$ image of critical point of $\|\Phi\|^2$, C_β corresponding component of $\mathrm{Crit}\|\Phi\|^2$, then $C_\beta \subseteq \mathrm{Crit}\langle\Phi,\beta\rangle$.

Theorem 12.2. [Tolman-Weitsman]
Let $Y \subseteq \mathcal{K}^*$ be the set of images under Φ of a $\mathrm{Crit}\|\Phi\|^2$ (Y is finite). Then $\mathrm{Ker}\,(\mathcal{K}_T)$ is the sum of ideals
$$\sum_{\xi\in Y} \mathcal{K}_\xi$$

where

$$M_\xi := \langle \Phi, \xi \rangle^{-1}(-\infty, 0]$$
$$\mathcal{K}_\xi := \left\{ \alpha \in H_T^*(M) \mid \alpha|_{M_\xi} = 0 \right\}$$

Remark : If $T \curvearrowright (M, \omega)$ GKM, then the condition for $\mathcal{K}_\xi$ places strong conditions on the cohomology class, because of the GKM compatibility conditions.

A progress report : the Kirwan method in other contexts
Recall :

ⓐ "Kirwan surjectivity"

$$\mathcal{K} : \ H_G^*(M) \twoheadrightarrow H_G^*\left(\Phi^{-1}(0)\right)$$

ⓑ Compute $H_G^*(M)$.
ⓒ Compute $\mathrm{Ker}\,(\mathcal{K})$.

$\boxed{\textbf{GOAL}}$ Attempt ⓐ – ⓒ in other settings.

VARIATIONS :

- symplectic $\rightsquigarrow$ hyperkähler [more in a moment]
- Borel-equivariant or ordinary cohomology $\rightsquigarrow$ (equivariant or ordinary) topological integral K-theory.

Remark : The **essential** step is to prove a version of a K-theoretic Atiyah-Bott lemma: [see Harada-Landweber, ArXiv: 0503609, Lemma 2.3]

Historical note : in algebraic K-theory [2003. Vezzosi-Vistoli]

- There are essential difference between A-B lemma in K-theory versus $H^*(-;\mathbb{Z})$, for our purposes, Euler classes behave better in K-theory.
  References, Harada-Landweber math SG/0503609 ⓐ, math SG/0612660 ⓑ and ⓒ.
- **Singular** GIT quotients, and the case of moduli space. [Jeffrey, Kirwan, Kiemn, Woolf]
  By using partial desingularization, get surjection

$$H_G^*(M) \twoheadrightarrow \underbrace{IH^*(M /\!/ G)}_{\text{intersection cohomology}}$$

  Recent announcement : similar computation $IH^*(\mathcal{M}(n,d))$, when n, d not coprime, so $\mathcal{M}(n,d)$ singular.

- infinite-dimensional version : G compact connected Lie $\rightsquigarrow$ LG loop group.

There exist a theory of Hamiltonian LG-spaces in analogy with our previous discussions, and an example of a Hamiltonian LG-quotient is a moduli space of flat connections on a principal G-bundle over a Riemann surface with one boundary component.

Theorem 12.3. [Bott, Tolman, Weitsman]
Suppose $LG \curvearrowright (\mathcal{M}, \omega) \xrightarrow[\Phi]{} L\mathfrak{g}^*$, Φ is proper. Then $\Phi^{-1}(0) \hookrightarrow \mathcal{M}$ induces a surjection

$$\mathcal{K} : H_G^*(M) \twoheadrightarrow H_G^*(\Phi(0)) \cong H^*\left(\Phi^{-1}(0)/G\right)$$

PRELIMINARY REPORT/ WORK IN PROGRESS [Harada-Paul Selick] Same holds in K-theory

$$\mathcal{K} : K_G^*(\mathcal{M}) \twoheadrightarrow K_G^*\left(\Phi^{-1}(0)\right) .$$

- ordinary cohomology $\rightsquigarrow$ orbifold cohomology
  Suppose that 0 is a regular value, so $M /\!/ G$ orbifold.

We want to compute $H^*_{\mathrm{orb}}([M /\!/ G])$
[Goldin-Holm-Knutson] Case $G = T$
Define the notion of **inertial cohomology** of a Hamiltonian T-space,

$$NH_T^*(M) \underset{\text{as a vector space}}{:=} \bigoplus_{t \in T} H_T^*(M^t)$$

They prove that $\Phi^{-1}(0) \hookrightarrow M$ induces a surjection

$$\mathcal{K}_{\mathrm{orb}} : NH_T^*(M) \twoheadrightarrow H^*_{\mathrm{orb}}([M /\!/ T])$$

Examples :

1. orbitfold toric manifolds ($\supseteq$ weighted projective spaces)
2. orbifold weight varieties (in terms of GKM theory)

13 The topology of hyperkähler quotients: a progress report

Definition 13.1. A hyperkähler manifold is a (smooth) manifold equipped with 3 symplectic structures $\omega_I, \omega_J, \omega_K$ with compatible complex structures I, J, K all with respect to the same metric g (called the hyperkähler metric).

Moreover, I, J, K interact like the quaternions.

THINK make the T_pM a quaternionic-hermitian space.

FIRST EXAMPLE :

$$\mathbb{H}^n \cong T^*\mathbb{C}^n$$

In analogy with what we've seen,

Definition 13.2. A hyperhamiltion G-space is a hyperkähler manifold $(M, \omega_I, \omega_J, \omega_K)$ with a G-action which is Hamiltonian with respect to all 3 symplectic structures

$$G \curvearrowright M \xrightarrow[\Phi_H \oplus \Phi_J \oplus \Phi_K]{} \mathfrak{g}^* \oplus \mathfrak{g}^* \oplus \mathfrak{g}^*$$

Definition 13.3. A hyperkähler quotient is defined as

$$M \underset{(\alpha_1,\alpha_2,\alpha_3)}{/\!/\!/\!/} G := \Phi_I^{-1}(\alpha_1) \cap \Phi_J^{-1}(\alpha_2) \cap \Phi_K^{-1}(\alpha_3)/G$$

Example 13.1. All in analogy with symplectic quotients

1. Symplectic toric manifolds $\mathbb{C} /\!/ T^k$

$$\rightsquigarrow \text{hypertoric manifold } \mathbb{H}^n /\!/\!/\!/ T^k \cong T^*\mathbb{C}^N /\!/\!/\!/ T^k$$

$$\rightsquigarrow \text{rational affine hyperplane arrangements}$$

2. Nakajima quiver varieties

$$\bigoplus_{i \to j} \operatorname{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j}) /\!/ \prod_i U(n_i)$$

$$\rightsquigarrow \bigoplus_{i \to j} \operatorname{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j}) \oplus \operatorname{Hom}(\mathbb{C}^{n_j}, \mathbb{C}^{n_i}) /\!/\!/\!/ \prod_i U(n_i)$$

[There's also versions with "framings"]

3. **Gauge theory :**

moduli of holomorphic bundles over Riemann surface $\mathcal{A} /\!/ \mathcal{G}$	$\rightsquigarrow$	moduli space of Higgs bundles over Riemann surface $T^*\mathcal{A} /\!/\!/\!/ \mathcal{G}$

4. **Recall :** GIT quotient $\longleftrightarrow$symplectic quotients
We can think of (Φ_I, Φ_J, Φ_K) as taking values in $\mathfrak{g}^* \oplus \mathfrak{g}^*_{\mathbb{C}}$, so

$$(\Phi_{\mathbb{R}}, \Phi_{\mathbb{C}}) : M \longrightarrow \mathfrak{g}^* \oplus \mathfrak{g}^*_{\mathbb{C}}$$

$\Phi_{\mathbb{C}}$ is holomorphic with respect to the I complex structure, and in fact is a holomorphic moment map with respect to $\omega_{\mathbb{C}} = \omega_J \pm I\omega_K$.

$$\begin{aligned} M \underset{(\alpha,0)}{/\!/\!/\!/} G &= \Phi_{\mathbb{C}}^{-1}(0) \underset{\alpha}{/\!/} G \\ &= \text{GIT quotient of } \Phi(0) \\ &= \Phi_{\mathbb{C}}^{-1}(0)^{\alpha\text{-st}}/G_{\mathbb{C}} \end{aligned}$$

PROGRESS REPORT

Kirwan method for hyperkähler quotients

More specifically, is there a Kirwan surjectivity theorem for hyperkähler quotients? i.e. Does

$$\Phi_{\mathbb{R}}^{-1}(\alpha) \cap \Phi_{\mathbb{C}}^{-1}(0) \hookrightarrow M$$

induce a surjection

$$H_G^*(M) \twoheadrightarrow H_G^*\left(\Phi_{\mathbb{R}}^{-1}(\alpha) \cap \Phi_{\mathbb{C}}^{-1}(0)\right)?$$

$\boxed{\textbf{ANSWER}}$ (as of 2007)
NOT much is known!

- Known examples
  - hypertoric varieties $T^*\mathbb{C}^n /\!/\!/\!/ T^k$
  - hyper polygon space $T^*\mathbb{C}^{2n} /\!/\!/\!/ SU(2) \times T^k$ (special case of a Nakajima quiver)
  - moduli space of rank 2, degree 0 Higgs bundle, non-fixed determinant over a compact Riemann surface
    [recent $\sim$ 2005-06 Daskalopoulos, Weitsman, Wilkin]

  a surjective, using equivariant Morse theory.
  [much more subtle, because we're considering $\Phi_{\mathbb{R}}^{-1}(\alpha) \cap \Phi_{\mathbb{C}}^{-1}(0)$, not just $\Phi_{\mathbb{R}}^{-1}(\alpha)$]

Related question : As we have seen, many examples of interest arise as follows.

Symplectic : $G \curvearrowright \mathbb{C}^N$ linearly, take $X := \mathbb{C}^N /\!/ G$

"hyperkähler analogue"

$$\begin{aligned} G \curvearrowright T^*_{\text{hol}}\mathbb{C}^N &\cong \mathbb{H}^N \text{ by lifting action} \\ &\cong \mathbb{C}^N_{\text{base}} \oplus \mathbb{C}^N_{\text{fiber}} = \left\{(z_i, w_i)_{i=1}^N\right\} \end{aligned}$$

hyperkähler structure on $T^*\mathbb{C}^N$:

$$\omega_{\mathbb{R}} = \omega_{\text{std},\mathbb{C}^{2N}} = \frac{\sqrt{-1}}{2}\left(\sum_l dz_l \wedge d\bar{z}_l + \sum_l dw_l \wedge d\bar{w}_l\right)$$

$$\omega_{\mathbb{C}} = \text{standard holomorphic structure on a holomorphic cotangent space} = \sum_l dz_l \wedge dw_l$$

FACT : Such a lifted linear action is always hyperhamiltonian.

$$\text{“hyperkähler analogue of } X\text{''} = \underbrace{T^*\mathbb{C}^N \underset{(\alpha,0)}{/\!/\!/\!/} G}_{} =: M$$

WARNING This is NOT necessarily

$$T^*\left(\mathbb{C}^N \underset{\alpha}{/\!/} G\right)$$

NOTE . In this case, there exists a commuting S^1-action "rotates fibers" on $T^*\mathbb{C}^N$, i.e. $\lambda \in S^1$

$$\lambda(z, w) = (z, \lambda w)$$

FACTS :

- This S^1-action is Hamiltonian with respect to $\omega_{\mathbb{R}}$ with moment map

$$\Psi(z,w) = -\frac{1}{2}\|w\|^2.$$

- Descends to Hamilton S^1-action on the hyperkähler quotient with respect to $(\omega_{\mathbb{R}})_{\mathrm{red}}$.

FACT : If original X is compact, then Ψ on M is proper and bounded below.

$$\rightsquigarrow S^1\text{-equivariant Morse theory on } M$$

Example 13.2.

1. [Hitchin 1987] Using Ψ, computes Betti numbers of moduli space of rank 2 Higgs bundles over Riemann surfaces.
2. [Nakajima, Nakajima-Kronheimer early 1990] Use Ψ to compute Betti numbers of moduli spaces of ASD connections over ALE spaces.
3. [Hausel-Proudfoot 2003] Using integration over components of $\mathrm{Crit}(\Psi)$, they obtain via localization a hyperkähler S^1-equivariant analogue of Shaun Martin's theorem, relating $H^*_{S^1}(M /\!/\!/\!/ G)$ with $H^*_{S^1}(M /\!/\!/\!/ T)$ where $T =$ maximal torus.
4. [Harada-Holm 2004] For GKM theory, need pairwise linear independence of T-weights at $T_pM (p \in M^T)$.

FACT : For standard $T^n = T^N/T^k \curvearrowright T^*\mathbb{C}^N /\!/\!/\!/ T^k$ "GKM conditions" NOT statisfied!
$\Longrightarrow$ but $T^n \times S^1$-action is **GKM**
$\Longrightarrow$ GKM combinational description of $T^n \times S^1$-equivariant cohomology of hypertoric varieties.

Example 13.3. "Kirwan method" is useful organizing principle for questions regarding the topology of quotients
- computations very explicit
- many interactions with other fields
- many more exciting directions and possibilities!