The Atiyah-Jones type problem for the space of holomorphic maps on a certain toric variety

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1.1. Motivation: Segal's theorem (1/5)

§1. Introduction

1.1. Motivation: Segal's theorem

In this talk, we shall consider the generalization of Segal's result concerning to the space

$$\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^{n-1})$$

of based holomorphic maps from $\mathbb{C}\mathrm{P}^1$ to $\mathbb{C}\mathrm{P}^{n-1}$ of degree d.
More precisely, we would like to study the inclusion map

$$i_D : \mathrm{Hol}_D^*(\mathbb{C}\mathrm{P}^m, X_\Sigma) \to \mathrm{Map}_D^*(\mathbb{C}\mathrm{P}^m, X_\Sigma)$$

for a toric variety X_Σ associated to a fan Σ.

1.1. Motivation: Segal's theorem (2/5)

Definition Let X and Y be a connected spaces.

1. Let $\mathrm{Map}^*(X,Y)$ denote the space of all base point preserving maps $f:(X,*)\to(Y,*)$ with the compact open topology.
2. For a homotopy class $D\in\pi_0(\mathrm{Map}^*(X,Y))=[X,Y]$, let

$$\mathrm{Map}^*_D(X,Y)\subset\mathrm{Map}^*(X,Y)$$

   denote the path component containing the homotopy class D.
3. When X and Y are complex manifolds, we denote by

$$\mathrm{Hol}^*_D(X,Y)$$

   the subspace of $\mathrm{Map}^*_D(X,Y)$ consisting of all based holomorphic maps $f:X\to Y$ with $[f]=D$.

1.1. Motivation: Segal's theorem (3/5)

Definition Let z denote the complex variable.

Let $\mathrm{P}^d = \mathrm{P}^d(\mathbb{C})$ denote the space of all monic polynomials

$$f(z) = z^d + a_{d-1}z^{d-1} + a_{d-2}z^{d-2} + \cdots + a_1 z + a_0 \in \mathbb{C}[z]$$

of degree d. (So there is a homeomorphism $\mathrm{P}^d(\mathbb{C}) \cong \mathbb{C}^n$)

1.1. Motivation: Segal's theorem (4/5)

Remark Let $f : X \to Y$ be a map and $N \geq 1$ be an integer.

1. A map f is called *a homotopy equivalence up to dimension N* (reps. *a homology equivalence up to dimension N*) if

$$f_* : \pi_k(X) \to \pi_k(Y) \quad (\text{resp. } f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z}))$$

   is an isomorphism for any $k < N$ and an epimorphism for $k = N$.

2. A map f is called *a homotopy equivalence through dimension N* (resp. *a homology equivalence through dimension N*) if

$$f_* : \pi_k(X) \to \pi_k(Y) \quad (\text{resp. } f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z}))$$

   is an isomorphism for any $k \leq N$.

1.1. Motivation: Segal's theorem (5/5)

Consider the space $\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^{n-1})$ of all based holomorphic maps $f : (\mathbb{C}\mathrm{P}^1, *) \to (\mathbb{C}\mathrm{P}^{n-1}, [1 : \cdots : 1])$ of degree d. If we identify $\mathbb{C}\mathrm{P}^1 = S^2 = \mathbb{C} \cup \infty$, we can identify

$$\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^{n-1}) = \{(f_1(z), \cdots, f_n(z)) \in (\mathrm{P}^d)^n : (*)_{\mathbb{C}}\},$$

$(*)_{\mathbb{C}}$ $\quad (f_1(\alpha), \cdots, f_n(\alpha)) \neq (0, 0, \cdots, 0)$ for any $\alpha \in \mathbb{C}$.
(i.e. $f_1(z), \cdots, f_n(z)$ have no common root.)

Theorem (G. Segal; [Se] (1979))

If $n \geq 2$, the inclusion map

$$i_d : \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^{n-1}) \xrightarrow{\subset} \mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^{n-1}) \simeq \Omega^2 S^{2n-1}$$

is a homotopy equivalence up to dimension $(2n-3)d$. □

1.2. Generalizations of Segal's result (1/9)

1.2. Generalizations of Segal's result

In this section we recall several results concerning the generalizations of Segal's result.

Because we shall consider the generalization of the Segal's result for the case of toric varieties, from now on recall several basic facts concerning toric varieties.

1.2. Generalizations of Segal's result (2/9))

1.2.1. Cones, fans and toric varieties

Definition Let $\{\mathbf{u}_k\}_{k=1}^m \subset \mathbb{Z}^n$.

1. *A rational (convex) polyhedral cone* $\sigma \subset \mathbb{R}^n$ is the subset of the form

$$\sigma = \text{Cone}(\{\mathbf{u}_k\}_{k=1}^m) := \sum_{k=1}^m \mathbb{R}_{\geq 0} \cdot \mathbf{u}_k = \{\sum_{k=1}^m \lambda_k \mathbf{u}_k : \lambda_k \geq 0\}.$$

2. σ is called *strongly covex* if $\sigma \cap (-\sigma) = \{\mathbf{0}\}$.
3. *A face of σ* is the intersection $\{L = 0\} \cap \sigma$, where L is a linear form such that $L \geq 0$ on σ.

1.2. Generalizations of Segal's result (3/9)

Definition Let Σ be a finite collection of cones in $\mathbb{R}^n$.

Then Σ is called *a fan* if the following 3 conditions hold:

1. Each $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.
2. If $\sigma \in \Sigma$ and $\tau \subset \sigma$ is a face of σ, then $\tau \in \Sigma$.
3. If $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2$ is a face of σ_k for each $k = 1, 2$. (Hence, $\sigma_1 \cap \sigma_2 \in \Sigma$.)

1.2. Generalizations of Segal's result (4/9)

Remark

If σ is a strongly convex rational polyhedral cone, one can define the affine variety U_σ by using its dual cone $\sigma^\vee$.
If Σ is a fan in $\mathbb{R}^n$, the toric variety X_Σ (associated to Σ) can be given as

$$X_\Sigma = \bigcup_{\sigma \in \Sigma} U_\sigma$$

by gluing together U_σ and U_τ along their common open subset $U_{\sigma \cap \tau}$ for all $\sigma, \tau \in \Sigma$.

1.2. Generalizations of Segal's result (5/9)

Definition Let Σ be a fan in $\mathbb{R}^n$ and let $\Sigma(1) = \{\rho_k\}_{k=0}^{r-1}$ be the set of all one dimensional cones in Σ.

1. For each $0 \leq k \leq r-1$, let $\mathbf{n}_k \in \mathbb{Z}^n$ denote *the primitive generator* of ρ_k such that $\mathbb{Z}_{\geq 0} \cdot \mathbf{n}_k = \rho_k \cap \mathbb{Z}^n$.
2. The subset $\mathbf{n} = \{\mathbf{n}_{i_1}, \cdots, \mathbf{n}_{i_s}\} \subset \Sigma(1)$ is call *primitive* if the whole set $\mathbf{n}$ does not span a cone in Σ but any of its proper subsets spans a cone in Σ.
3. Define the integer $2 \leq r_{\min}(\Sigma) \leq r$ by

$$r_{\min}(\Sigma) = \min\{s : \{\mathbf{n}_{i_1}, \cdots, \mathbf{n}_{i_s}\} \text{ is primitive}\}.$$

1.2. Generalizations of Segal's result (6/9)

1.2.2. Results of Guest and Mostovy-Miranueva

Theorem (M. Guest (1994))

Let Σ be a fan and let X_Σ denote the toric variety associated to Σ. If X_Σ is a compact smooth toric variety and $D=(d_0,\cdots,d_{r-1})\in(\mathbb{Z}_{\geq 0})^r$ such that $\sum_{k=0}^{r-1} d_k\mathbf{n}_k=\mathbf{0}$, then the inclusion map

$$i_D:\mathrm{Hol}_D^*(S^2,X_\Sigma)\to\Omega^2_DX_\Sigma$$

is a homotopy equivalence up to dimension $n(D)$, where $n(D):=\min\{d_0,\cdots,d_{r-1}\}$. □

1.2. Generalizations of Segal's result (7/9)

Theorem (Mostovoy-Miranueva (2013))

Let Σ be a fan and let X_Σ denote the toric variety associated to Σ. Let $D=(d_0,\cdots,d_{r-1})\in(\mathbb{Z}_{\geq 1})^r$ be r-tuple of integers such that $\sum_{k=0}^{r-1} d_k\mathbf{n}_k=\mathbf{0}$. Then if X_Σ is a compact smooth toric variety, the inclusion map

$$i_D:\mathrm{Hol}_D^*(\mathbb{C}\mathrm{P}^m,X_\Sigma)\to \mathrm{Map}_D^*(\mathbb{C}\mathrm{P}^m,X_\Sigma)$$

is a homology equivalence through dimension $N(D,\Sigma)$, where

$$N(D,\Sigma):=\ (2r_{\min}(\Sigma)-2m-1)\min\{d_0,\cdots,d_{r-1}\}-2.\qquad\square$$

1.2. Generalizations of Segal's result (8/9)

Conjecture A

Let Σ be a fan in $\mathbb{R}^n$ such that $\Sigma(1) = \{\rho_0, \cdots, \rho_{r-1}\}$, let X_Σ denote the toric variety associated to Σ, and let $D = (d_0, \cdots, d_{r-1}) \in (\mathbb{Z}_{\geq 1})^r$ such that $\sum_{k=0}^{r-1} d_k \mathbf{n}_k = \mathbf{0}$ with $\mathbb{Z}_{\geq 0} \cdot \mathbf{n}_k = \rho_k \cap \mathbb{R}^n$.

Then, even if X_Σ is a *non-compact* smooth toric variety, when $r_{\min}(\Sigma) > m$ and $\Sigma(1)$ spans $\mathbb{R}^n$, is the inclusion map

$$i_D : \mathrm{Hol}_D^*(\mathbb{C}\mathrm{P}^m, X_\Sigma) \to \mathrm{Map}_D^*(\mathbb{C}\mathrm{P}^m, X_\Sigma)$$

a homology equivalence through dimension $N(D, \Sigma)$? □

1.2. Generalizations of Segal's result (9/9)

Remark When $X_\Sigma = \mathbb{T}^r = (\mathbb{C}^*)^r$ is an algebraic torus, there is no (non-trivial) holomorphic map $f : \mathbb{C}\mathrm{P}^1 \to X_\Sigma$ except constant maps. However, because $r_{\min}(\Sigma) = 1$ for $X_\Sigma = \mathbb{T}^r$, $2r_{\min}(\Sigma) = 2 < 2m + 1$ for any $m \geq 1$ and and Conjecture A is correct for $X_\Sigma = \mathbb{T}^r$!

1.3. Generalizations of Segal's result (non-compact case) (1/7)

1.3. Generalizations of Segal's result (non-compact case)

We would like to consider the inclusion map

$$i_D : \mathrm{Hol}_D^*(\mathbb{C}\mathrm{P}^m, X_\Sigma) \to \mathrm{Map}_D^*(\mathbb{C}\mathrm{P}^m, X_\Sigma)$$

for a non-compact smooth toric variety X_Σ and study whether the result of Mosotovy-Varanueva holds or not for this case.

1.3. Generalizations of Segal's result (non-compact case) (2/7)

Remark (Atiyah-Jones-Segal type problem)

For a complex manifold (or variety) $X \subset \mathbb{C}\mathrm{P}^l$, does there exist an integer $N(D)$ such that the inclusion map

$$i_D : \mathrm{Hol}^*_D(\mathbb{C}\mathrm{P}^m, X) \to \mathrm{Map}^*_D(\mathbb{C}\mathrm{P}^m, X)$$

is a homology equivalence through dimension $N(D)$ and that $\lim_{D\to\infty} N(D) = \infty$? □

The above problem is called the Aiyah-Jones-Segal type conjecture.

1.3. Generalizations of Segal's result (non-compact case) (3/7)

Definition Let $n \geq 3$ and let I be collection of subsets of $[n] = \{0, 1, 2, \cdots, n-1\}$ such that card$(\sigma) \geq 2$ for any $\sigma \in I$.

1. For $\sigma = \{i_1, \cdots, i_s\} \in I$, let
$$L_\sigma := \{(x_0, \cdots, x_{n-1}) \in \mathbb{C}^n : x_{i_1} = \cdots = x_{i_s} = 0\}.$$
2. Let Y_I denote the subspace of $\mathbb{C}^n$ given by $Y_I := \mathbb{C}^n \setminus \bigcup_{\sigma \in I} L_\sigma$.
3. Define the subspace $X_I \subset \mathbb{C}\mathrm{P}^{n-1}$ by
$$X_I := Y_I/\mathbb{C}^* = (\mathbb{C}^n \setminus \bigcup_{\sigma \in I} L_\sigma)/\mathbb{C}^*, \quad \text{where}$$
$\mathbb{C}^*$ acts on Y_I by $\alpha \cdot (x_0, \cdots, x_{n-1}) := (\alpha x_0, \cdots, \alpha x_{n-1})$.

1.3. Generalizations of Segal's result (non-compact case) (4/7)

Example Let $n \geq 3$ and let I be collection of subsets of $[n] = \{0, 1, 2, \cdots, n-1\}$ such that card$(\sigma) \geq 2$ for any $\sigma \in I$.

1. If $I = I(n) = \{\{0, 1, \cdots, n-1\}\} = \{[n]\}$, $L_{[n]} = \{\mathbf{0}\}$ and

$$X_I = X_{I(n)} = (\mathbb{C}^n \setminus L_{[n]})/\mathbb{C}^* = (\mathbb{C}^n \setminus \{\mathbf{0}\})/\mathbb{C}^* = \mathbb{C}\mathrm{P}^{n-1}.$$

2. If $I = J(n) = \{\{i, j\} : 0 \leq i < j\}$, then

$$\begin{aligned} X_I &= X_{J(n)} = (\mathbb{C}^n \setminus \bigcup_{0 \leq i < j \leq n-1} L_{\{i,j\}})/\mathbb{C}^* \\ &= \mathbb{C}\mathrm{P}^{n-1} \setminus \bigcup_{0 \leq i < j \leq n-1} H_{i,j}, \qquad \text{where} \\ H_{i,j} &= \{[x_0 : \cdots : x_{n-1}] \in \mathbb{C}\mathrm{P}^{n-1} : x_i = x_j = 0\}. \end{aligned}$$

1.3. Generalizations of Segal's result (non-compact case) (5/7)

Remark Let $n \geq 3$ and let I be collection of subsets of $[n] = \{0, 1, 2, \cdots, n-1\}$ such that card$(\sigma) \geq 2$ for any $\sigma \in I$.

(i) In general, X_I has the following form:

$$X_I = \mathbb{C}\mathrm{P}^{n-1} \setminus \bigcup_{\sigma \in I} H_\sigma, \text{ where}$$
$$H_\sigma = \{[x_0 : \cdots : x_{n-1}] \in \mathbb{C}\mathrm{P}^{n-1} : x_j = 0 \text{ if } i \in \sigma\}.$$

(ii) The space X_I is a smooth toric variety and X_I is a *non-compact* toric variety if $I \neq I(n)$.

(iii) It is known that X_I is simply connected and $\pi_2(X_I) = \mathbb{Z}$.

Remark Note that in this case we can take $r = n$ and $d_0 = d_1 = \cdots = d_{n-1} = d$ (so $D = (d, d, \cdots, d)$).

1.3. Generalizations of Segal's result (non-compact case) (6/7)

Remark Let $n \geq 3$ and let I be collection of subsets of $[n] = \{0, 1, 2, \cdots, n-1\}$ such that card$(\sigma) \geq 2$ for any $\sigma \in I$.

If we identify $S^2 = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ and choose the points ∞ and $[1 : \cdots : 1] \in X_I$ as the corresponding base-points, we can identify

$$\mathrm{Hol}_d^*(S^2, X_I) = \{(f_0(z), \cdots, f_{n-1}(z) \in \mathrm{P}^d(\mathbb{C})^n : (*)_I\},$$

where

$(*)_I$ The polynomials $f_{i_1}(z), \cdots, f_{i_s}(z)$ have no common root for any $\sigma = \{i_1, \cdots, i_s\} \in I$.

1.3. Generalizations of Segal's result (non-compact case) (7/7)

Recall the following classical result:

Theorem (M. Guest, A. Kozlowski, KY (1994))

Let $n \geq 3$ and I denote the collection of the subsets of $[n] = \{0, 1, 2, \cdots, n-1\}$ such that $\mathrm{card}(\sigma) \geq 2$ for any $\sigma \in I$. If $d \geq 1$, the inclusion map

$$i_d : \mathrm{Hol}_d^*(S^2, X_I) \to \Omega_d^2 X_I$$

is a homotopy equivalence up to dimension d. □

1.4. The main result of this talk (1/6)

1.4. The main result

Let $d \geq 1$ be an integer and we would like to consider the Atiyah-Jones-Segal type result for the the inclusion map

$$i_d : \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^m, X_I) \to \mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^m, X_I)$$

when I is a collection of subsets of $[n] = \{0, 1, \cdots, n-1\}$ such that $\mathrm{card}(\sigma) \geq 2$ for any $\sigma \in I$.

Definition Define the positive integer $d(I)$ by

$$d(I) := \min\{\mathrm{card}(\sigma) : \sigma \in I\}.$$

1.4. The main result (2/6)

Theorem I (The case $m = 1$; A. Kozlowski, KY)

If $n \geq 3$ and $d(I) \geq 3$, the inclusion map

$$i_D : \mathrm{Hol}^*_d(S^2, X_I) \to \Omega^2_d X_I$$

is a homotopy equivalence through dimension $N(d,I)$, where

$$N(d,I) := (2d(I) - 3)d - 2. \quad \square$$

1.4. The main result (3/6)

Remark Let Σ_I denote the fan of the toric variety X_I. Then we can show that

$$\Sigma_I(1) = \{\mathbb{R}_{\geq 0} \cdot \mathbf{e}_0, \mathbb{R}_{\geq 0} \cdot \mathbf{e}_1, \cdots, \mathbb{R}_{\geq 0} \cdot \mathbf{e}_{n-1}\},$$

where $\{\mathbf{e}_k\}_{k=1}^{n-1}$ denotes the standard basis of $\mathbb{R}^{n-1}$ and $\mathbf{e}_0 = -\sum_{k=1}^{n-1} \mathbf{e}_k$.
So $r = n$ and $\mathbf{n}_k = \mathbf{e}_k$ for $0 \leq k \leq n-1$. Moreover, one can also show that

$$r_{\min}(\Sigma_I) = d(I)$$

and $d_k = d$ for all $0 \leq k \leq n-1$.

1.4. The main result (4/6)

Hence, we have:

Corollary II (A. Kozlowski, KY)

Conjecture A is true for a non-complact smooth toric variety $X = X_I$ *when* $m = 1$ □

Conjecture B

Is Conjecure true for $X = X_I$ even if $m \geq 2$?, i.e.
For $m \geq 2$, is the inclusion map

$$i_d : \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^m, X_I) \xrightarrow{\subset} \mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^m, X_I)$$

a homology equivalence through dimension

$$N(d,m) := (2d(I) - 2m - 1)d - 2? \qquad \square$$

1.4. The main result (5/6)

Theorem III (Some improvement of Segal's result)

If $n \geq 3$ *and* $I = I(n)$, $X_I = \mathbb{C}\mathrm{P}^{n-1}$ *and the inclusion map*

$$i_d : \mathrm{Hol}_d^*(S^2, \mathbb{C}\mathrm{P}^{n-1}) \to \Omega_d^2 \mathbb{C}\mathrm{P}^{n-1}$$

is a homotopy equivalence through dimension $N(d,n)$, *where* $N(d,n) := (2n-3)(d+1) - 1$. □

Remark The above result can also be proved by using the result due to [C²M²].

1.4. The main result (6/6)

Remark

Mostovy-Varanueva [MV] uses *the Stone-Weerstass Theorem* for vector bundles. So if a toric variety X_Σ is non-compact, it is impossible to use their method.

However, if $m = 1$ and $X_\Sigma = X_I$, then we can probe it by using *the scanning maps*.

2.1. (i) The arrangement of coordinate subspaces (1/4)

§2. The topology of X_I

2.1. Coordinate subspaces and polyhedral products

Definition Let K be a collection of some subsets of $[n] = \{0, 1, \cdots, n-1\}$.

Then K is called *a simplicial complex on the index set $[n]$* if the following condition holds:

$$(*) \qquad \tau \subset \sigma \text{ and } \sigma \in K \Rightarrow \tau \in K$$

Remark In this talk, a simplicial complex K means *an abstract simplicial complex* and assume that it always contains the empty set $\emptyset$.

2.1. (i) The arrangement of coordinate subspaces (2/4)

Definition Let K be a simplicial complex on the index set $[n] = \{0, 1, 2, \cdots, n-1\}$.

1. For each $\sigma = \{i_1, \cdots, i_k\} \subset [n]$, define

$$L_\sigma = \{(x_0, \cdots, x_{n-1}) \in \mathbb{C}^n : x_{i_1} = \cdots = x_{i_k} = 0\}.$$

2. Define the complement $U(K)$ of *the coordinate subspace arrangement* by

$$U(K) := \mathbb{C}^n \setminus \bigcup_{\sigma \notin K, \sigma \subset [n]} L_\sigma .$$

2.1. (i) The arrangement of coordinate subspaces (3/4)

Remark *Let I be a collection of some subsets of $[n]$ and set*

$$K(I)=\{\sigma\subset[n]:L_\sigma\not\subset\bigcup_{\tau\in I}L_\tau\}.$$

1. *Then $K(I)$ is a simplicial complex on the index set $[n]$.*
2. $U(K(I))=\mathbb{C}^n\setminus\bigcup_{\tau\in I}L_\tau$ *and* $\bigcup_{\sigma\notin K(I)}L_\sigma=\bigcup_{\tau\in I}L_\tau.$

Therefore,

$$\begin{aligned} Y_I &= \mathbb{C}^n\setminus\bigcup_{\sigma\in I}L_\sigma=U(K(I)),\\ X_I &= Y_I/\mathbb{C}^*=U(K(I))/\mathbb{C}^*. \end{aligned}$$

2.1. (i) The arrangement of coordinate subspaces (4/4)

Example Let K be a simplicial complex on the index set $[n]$.

1. If $K = \partial\Delta^{n-1} = \{\sigma \subset [n] : \sigma \neq [n]\}$,

$$U(K) = \mathbb{C}^n \setminus \{z_0 = \cdots = z_{n-1} = 0\} = \mathbb{C}^n \setminus \{\mathbf{0}\}.$$

2. If K is a simplicial complex on the index set $[n]$ given by $K = \{\phi, \{0\}, \{1\}, \cdots, \{n-1\}\}$, then

$$U(K) = \mathbb{C}^n \setminus \bigcup_{0 \leq i < j \leq n-1} \{(x_0, \cdots, x_{n-1}) \in \mathbb{C}^n : x_i = x_j = 0\}.$$

2.1. (ii) The polyhedral product (1/4)

Definition Let K be a simplicial complex on the index set $[n] = \{0, 1, \cdots, n-1\}$ and let

$$(\underline{X}, \underline{A}) = \{(X_0, A_0), \cdots, (X_{n-1}, A_{n-1})\} \quad (A_i \subset X_i)$$

Define *the polyhedral product* $\mathcal{Z}_K(\underline{X}, \underline{A})$ of $(\underline{X}, \underline{A})$ w.r.t. K by

$$\begin{cases} \mathcal{Z}_K(\underline{X}, \underline{A}) & := \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma, \qquad \text{where} \\ (\underline{X}, \underline{A})^\sigma & := \{(x_0, \cdots, x_{n-1}) \in \prod_{k=0}^{n-1} X_k : x_k \in A_k \text{ for } k \notin \sigma\} \\ & = \prod_{k \in \sigma} X_k \times \prod_{k \notin \sigma} A_k. \end{cases}$$

2.1. (ii) The polyhedral product (2/4)

Definition Let K be a simplicial complex on $[n]$.
If $(X, A) = (X_k, A_k)$ for all k, we write

$$\begin{aligned} \mathcal{Z}_K(\underline{X}, \underline{A}) &= \mathcal{Z}_K(X, A) = \bigcup_{\sigma \in K} (X, A)^\sigma, \quad \text{where} \\ (X, A)^\sigma &= \{(x_0, \cdots, x_{n-1}) \in X^n : x_k \in A \text{ if } k \notin \sigma\} \\ &\cong X^{\mathrm{card}(\sigma)} \times A^{n-\mathrm{card}(\sigma)}. \end{aligned}$$

2.1. (ii) The polyhedral product (3/4)

Example Let K be a simplicial complex on the index set $[n]$.

(i) $\mathcal{Z}_K(D^2, S^1) = \mathcal{Z}_K$ (*the moment angle complex of* K)

(ii) $U(K) = \mathbb{C}^n \setminus \bigcup_{\sigma \notin K} L_\sigma = \mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)$.

Remark (Buchstaber-Panov, [BP]) *If K is a simplicial complex on the index set $[n]$ and set $\mathbb{T}^n := (\mathbb{C}^*)^n$, there is $\mathbb{T}^n$-equivariant deformation retraction*

$$\begin{array}{ccc} U(K) & \xrightarrow[\simeq]{r} & \mathcal{Z}_K \\ \| & & \| \\ \mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*) & \xrightarrow[\simeq]{r} & \mathcal{Z}_K(D^2, S^1) \end{array}$$

2.1. (ii) The polyhedral product (4/4)

Lemma ([BP]) *Let K be a simplicial complex on the vertex set $[r]$, and let $DJ(K)$ denote the Davis-Januszkiewicz space of K defined by*

$$DJ(K) := \mathcal{Z}_K(\mathbb{C}\mathrm{P}^\infty, *) \subset (\mathbb{C}\mathrm{P}^\infty)^r = B\mathbb{T}^r.$$

Then $\mathcal{Z}_K = \mathcal{Z}_K(D^2, S^1)$ is the homotopy fibre of the inclusion map

$$DJ(K) \xrightarrow{\subset} (\mathbb{C}\mathrm{P}^\infty)^r = B\mathbb{T}^r.$$

Thus, there is a fibration sequence (up to homotopy)

$$\mathcal{Z}_K \longrightarrow DJ(K) \xrightarrow{\subset} (\mathbb{C}\mathrm{P}^\infty)^r = B\mathbb{T}^r. \qquad \square$$

2.2. The toric variety X_I (1/5)

2.2. The topology of the toric variety X_I

Remark Let I be a collection of subsets of $[n] = \{0, 1, \cdots, n-1\}$ such that $\mathrm{card}(\sigma) \geq 2$ for any $\sigma \in I$. Recall that

$$X_I = (\mathbb{C}^n \setminus \bigcup_{\sigma \in I} L_\sigma)/\mathbb{C}^* = U(K(I))/\mathbb{C}^*.$$

Then one can define the $\mathbb{T}^{n-1}$-action on X_I by

$$(t_1, \cdots, t_{n-1}) \cdot [x_0 : \cdots : x_{n-1}] := [x_0 : t_1 x_1 : \cdots : t_{n-1} x_{n-1}]$$

and it is easy to see that X_I is a toric subvariety of $\mathbb{C}\mathrm{P}^{n-1}$.

2.2. The toric variety X_I (2/5)

Definition Let $\{\mathbf{e}_k\}_{k=1}^{n-1} \in \mathbb{R}^{n-1}$ denote the standard basis of $\mathbb{R}^{n-1}$ and let $\mathbf{e}_0 := -\sum_{k=1}^{n-1} \mathbf{e}_k$, where

$$\begin{cases} \mathbf{e}_1 & = (1,0,0,\cdots,0,0) \\ \mathbf{e}_2 & = (0,1,0,\cdots,0,0) \\ \quad\vdots & \qquad\vdots \qquad \vdots \\ \mathbf{e}_{n-1} & = (0,0,0,\cdots,0,1) \end{cases}$$

For each subset $\sigma \subsetneqq [n] = \{0,1,\cdots,n-1\}$, define *the strongly convex rational polyhedral cone* Cone_σ in $\mathbb{R}^{n-1}$ by

$$\mathrm{Cone}_\sigma := \begin{cases} \left\{ \sum_{k=1}^{s} a_k \mathbf{e}_{i_k} : a_k \geq 0 \right\} & \text{if } \sigma = \{i_1,\cdots,i_s\}, \\ \{\mathbf{0}\} & \text{if } \sigma = \emptyset. \end{cases}$$

2.2. The toric variety X_I (3/5)

Lemma *Let I be a collection of the subsets of $[n] = \{0, 1, \cdots, n-1\}$ such that $\mathrm{card}(\sigma) \geq 2$ for any $\sigma \in I$.*

1. *There is the Segal type fibration sequence (up to homotopy)*

$$X_I \xrightarrow{q_I} \mathcal{Z}_{K(I)}(\mathbb{C}\mathrm{P}^\infty, *) = DJ(K(I)) \xrightarrow{\subset} (\mathbb{C}\mathrm{P}^\infty)^{n-1}.$$

2. *If K is a simplicial complex on the index set $[n]$, the set $\Sigma(K) := \left\{\mathrm{Cone}_\sigma : \sigma \in K\right\}$ is a fan in $\mathbb{R}^{n-1}$.*
3. *In particular, if $K = K(I)$, the set $\Sigma_I := \Sigma(K(I))$ is the fan associated to the toric variety X_I and $\Sigma_I(1) = \{\mathbb{R}_{\geq 0} \cdot \mathbf{e}_k\}_{k=0}^{n-1}$.*

2.2. The toric variety X_I (4/5)

Example Let $n \geq 3$ and set $[n] = \{0, 1, \cdots, n-1\}$ as before.

1. If $I = J(n) = \{\{i,j\} : 0 \leq i < j \leq n-1\}$,

$$\begin{cases} K(J(n)) & = \{\emptyset, \{k\} : 0 \leq k \leq n-1\} \\ \Sigma_{J(n)} & = \{\{\mathbf{0}\}, \mathbb{R}_{\geq 0} \cdot \mathbf{e}_k : 0 \leq k \leq n-1\} \\ X_{J(n)} & = \mathbb{C}\mathrm{P}^{n-1} \setminus \bigcup_{0 \leq i < j \leq n} H_{i,j} \end{cases}$$

where $H_{i,j} := \{[x_0 : \cdots : x_{n-1}] \in \mathbb{C}\mathrm{P}^{n-1} : x_i = x_j = 0\}$.

2. If $I = I(n) := \{[n]\}$,

$$\begin{cases} K(I(n)) & = \{\sigma \subset [n] : \sigma \neq [n]\} \\ \Sigma_{I(n)} & = \{\mathrm{Cone}_\sigma : \sigma \subsetneqq [n]\} \\ X_{I(n)} & = (\mathbb{C}^n \setminus L_{[n]})/\mathbb{C}^* = (\mathbb{C}^n \setminus \{\mathbf{0}\})/\mathbb{C}^* = \mathbb{C}\mathrm{P}^{n-1} \end{cases}$$

2.2. The toric variety X_I (5/5)

Remark (Homogenous representation) Let Σ be a fan in $\mathbb{R}^m$, $\Sigma(1)=\{\rho_k\}_{k=0}^{r-1}$, and $\mathbf{n}_k\in\mathbb{Z}^m$ the primitive generator of ρ_k such that $\rho_k\cap\mathbb{Z}^m=\mathbb{Z}_{\geq 0}\cdot\mathbf{n}_k$ for each $0\leq k<r$. *Then if* $\{\mathbf{n}_0,\cdots,\mathbf{n}_{r-1}\}$ *spans* $\mathbb{R}^m$, there is an isomorphism

$$X_\Sigma\cong U(\mathcal{K}_\Sigma)/G_\Sigma,$$

where $\mathcal{K}_\Sigma$ and G_Σ denote the simplicial complex on the index set $[r]=\{0,1,\cdots,r-1\}$ and the subgroup of $\mathbb{T}^r=(\mathbb{C}^*)^r$ defined by

$$\begin{cases}\mathcal{K}_\Sigma=\{\{i_1,\cdots,i_s\}\subset[r]:\{\mathbf{n}_{i_1},\cdots,\mathbf{n}_{i_s}\}\text{ spans a cone in }\Sigma\},\\ G_\Sigma=\{(\mu_0,\cdots,\mu_{r-1})\in\mathbb{T}^r:\prod_{k=0}^{r-1}\mu_k^{\langle\mathbf{n}_k,\mathbf{e}_j\rangle}=1\text{ for }1\leq\forall j\leq m\}.\end{cases}$$

3.1. Stabilization (1/11)

§3. Stabilization maps and the idea of the proof

3.1. Stability result (scanning maps)

In this section, we shall study the stabilization map

$$s_d : \mathrm{Hol}_d^*(S^2, X_I) \to \mathrm{Hol}_{d+1}^*(S^2, X_I)$$

and prove the stabilization theorem by using *the scanning map*,

$$S : \lim_{d\to\infty} \mathrm{Hol}_d^*(S^2, X_I) \to \Omega_0^2 \mathcal{Z}_{K(I)}(\mathbb{C}\mathrm{P}^\infty, *).$$

Next we shall show that the stabilization map s_d is a homology equivalence through dimension $n(d, I)$ by using the Vassiliev type spectral sequence and prove the main result (Theorem I).

3.1. Stabilization (2/11)

Definition Let $d \geq 1$, X be a connected based space and S_d the the symmetric group of d-letters.

1. Note that S_d acts X^d by the coordinate permutations. Let $\mathrm{SP}^d(X)$ denote *the d-th symmetric product* given by the orbit space, $\mathrm{SP}^d(X) := X^d/S_d$
2. Each element $\alpha \in \mathrm{SP}^d(X)$ may be represented as a finite formal sum

$$\alpha = \sum_{k=1}^{s} d_k x_k$$

$$(x_k \in X,\ d_k \in \mathbb{Z}_{\geq 1},\ x_i \neq x_j \text{ if } i \neq j,\ \sum_{k=1}^{s} d_k = d)$$

3.1. Stabilization (3/11)

Definition Let (X, A) be a pair of connected based space.

1. If $* \in A \subset X$ is the base-point, one has the inclusion $\mathrm{SP}^d(X) \subset \mathrm{SP}^{d+1}(X)$ by $\alpha \mapsto \alpha + *$.

   Let $\mathrm{SP}^\infty(X)$ denote the union $\mathrm{SP}^\infty(X) := \bigcup_{d=1}^{\infty} \mathrm{SP}^d(X)$.

2. Define the equivalence relation $\sim$ on $\mathrm{SP}^d(X)$ by

$$\alpha \sim \beta \Leftrightarrow \alpha \cap (X \setminus A) = \beta \cap (X \setminus A) \quad \text{for } \alpha, \beta \in \mathrm{SP}^d(X).$$

3. Let $\mathrm{SP}^d(X, A)$ and $\mathrm{SP}(X, A)$ be the quotient spaces

$$\mathrm{SP}^d(X, A) := \mathrm{SP}^d(X)/\sim, \quad \mathrm{SP}(X, A) := \bigcup_{d=1}^{\infty} \mathrm{SP}^d(X, A).$$

3.1. Stabilization (4/11)

Definition (i) Define the space $E^d_I(X)$ by

$$E^d_I(X)=\big\{(\xi_0,\cdots,\xi_{n-1})\in \mathrm{SP}^d(X)^n:\cap_{j\in\sigma}\xi_j=\emptyset \text{ for } \forall\sigma\in I\big\}.$$

Note that $\exists$ a natural homeomorphism $\mathrm{Hol}^*_d(S^2,X_I)\cong E^d_I(\mathbb{C})$.

(ii) Let $s_d:\mathrm{Hol}^*_d(S^2,X_I)\to \mathrm{Hol}^*_{d+1}(S^2,X_I)$ denote *the stabilization map* given by the composite of maps

$$\mathrm{Hol}^*_d(S^2,X_I)\cong E^d_I(\mathbb{C})\xrightarrow{s'_d}E^{d+1}_I(\mathbb{C})\cong \mathrm{Hol}^*_{d+1}(S^2,X_I),$$

where s'_d denotes the map given by $(d<|w_0|<d+1)$

$$\begin{array}{ccc} E^d_I(\mathbb{C})\cong E^d_I(\{w\in\mathbb{C}:|w|<d\}) & \longrightarrow & E^{d+1}_I(\mathbb{C}) \\ \xi & \longrightarrow & \xi+w_0 \end{array}$$

3.1. Stabilization (5/11)

Remark Because there is a homotopy commutative diagram

$$\begin{array}{ccc} \mathrm{Hol}^*_d(S^2, X_I) & \xrightarrow{s_d} & \mathrm{Hol}^*_{d+1}(S^2, X_I) \\ i_d \downarrow \cap & & i_{d+1} \downarrow \cap \\ \Omega^2_d X_I & \xrightarrow[\simeq]{} & \Omega^2_{d+1} X_I \end{array}$$

we obtain the map

$$i_\infty = \lim_{d\to\infty} : \lim_{d\to\infty} \mathrm{Hol}_d(S^2, X_I) \to \lim_{d\to\infty} \Omega^2_d X_I \simeq \Omega^2_0 X_I$$

where the colimit $\lim_{d\to\infty} \mathrm{Hol}_d(S^2, X_I)$ is taken from the stabilization maps s_d's.

3.1. Stabilization (6/11)

Definition Let $\epsilon > 0$ be a fixed sufficiently small real number. Let $\xi = (\xi_0, \cdots, \xi_{n-1}) \in E^d_I(\mathbb{C})$. For each $w \in \mathbb{C}$, let U_w denote the open disk of radius ϵ with the center w,

$$U_w = \{x \in \mathbb{C} : |x - w| < \epsilon\}.$$

Then consider the element $S'_d(w, \xi) \in E_I(D^2, S^1)$ given by

$$\begin{aligned} S'_d(w, \xi) &= (\xi_0 \cap U_w, \cdots, \xi_{n-1} \cap U_w) \\ &\in E_I(\overline{U}_w, \partial \overline{U}_w) \cong E_I(D^2, S^1) \end{aligned}$$

This induces the map $S'_d : \mathbb{C} \times E^d_I(\mathbb{C}) \to E_I(D^2, S^1)$ and its adjoint gives the map

$$S_d : E^d_I(\mathbb{C}) \to \mathrm{Map}(\mathbb{C}, E_I(D^2, S^1)).$$

3.1. Stabilization (7/11)

Because $\lim_{w\to\infty} S_d(w) = (\emptyset, \cdots, \emptyset)$, if we choose the point $(\emptyset, \cdots, \emptyset)$ as the base-point of $E_I(D^2, S^1)$, we obtain the map

$$S_d : E_I^d(\mathbb{C}) \to \mathrm{Map}^*(\mathbb{C} \cup \infty, E_I(D^2, S^1)) = \Omega^2 E_I(D^2, S^1).$$

Since the space $E_I^d(\mathbb{C})$ is connected, the image of S_d is contained in some component of $\Omega^2 E_I(D^2, S^1)$, which is denoted by $\Omega_d^2 E_I(D^2, S^1)$. Thus, we have the map

$$S_d : E_I^d(\mathbb{C}) \to \Omega_d^2 E_I(D^2, S^1),$$

and this induces the map

$$S = \lim_{d\to\infty} S_d : \lim_{d\to\infty} E_I^d(\mathbb{C}) \to \lim_{d\to\infty} \Omega_d^2 E_I(D^2, S^1) \simeq \Omega_0^2 E_I(D^2, S^1).$$

3.1. Stabilization (8/11)

Definition (continued) If we identify $\mathrm{Hol}_d^*(S^2, X_I) = E_I^d(\mathbb{C})$, we obtain the map

$$S : \lim_{d\to\infty} \mathrm{Hol}_d^*(S^2, X_I) \to \Omega_0^2 E_I(D^2, S^1).$$

This map S is called *the scanning map*.

Theorem A (Guest, Kozlowski, KY (1994))

1. $E_I(D^2, S^1) \simeq \mathcal{Z}_{K(I)}(\mathbb{C}\mathrm{P}^\infty, *)$ (*homotopy equivalence*).
2. $S : \lim_{d\to\infty} \mathrm{Hol}_d^*(S^2, X_I) \xrightarrow{\simeq} \Omega_0^2 E_I(D^2, S^1)$ *is a homotopy equivalence.* □

3.1. Stabilization (9/11)

If we recall the fibration sequence

$$X_I \xrightarrow{q_I} \mathcal{Z}_{K(I)}(\mathbb{C}\mathrm{P}^\infty, *) \to (\mathbb{C}\mathrm{P}^\infty)^{n-1} = B\mathbb{T}^{n-1}$$

we have the homotopy equivalence

$$\Omega_0^2 X_I \xrightarrow[\simeq]{\Omega^2 q_I} \Omega_0^2 \mathcal{Z}_{K(I)}(\mathbb{C}\mathrm{P}^\infty, *).$$

Then by using Theorem A and some digram chasing, we have:

Theorem AI *If $d(I) \geq 2$, the map*

$$i_\infty = \lim_d i_d : \lim_{d\to\infty} \mathrm{Hol}_d^*(S^2, X_I) \xrightarrow{\simeq} \Omega_0^2 X_I$$

is a homotopy equivalence. □

3.1. Stabilization (10/11)

Theorem B *If $n \geq 3$ and $d(I) \geq 3$, the stabilization map*

$$i_d : \mathrm{Hol}_d^*(S^2, X_I) \to \mathrm{Hol}_{d+1}^*(S^2, X_I)$$

is a homology equivalence through dimension $N(d, I)$, where

$$N(d, I) := (2d(I) - 3)d - 2.$$

Remark

Theorem B can be obtained by using the Vassiliev type spectral sequence. □

3.1. Stabilization (the idea of the proof) (11/11)

Theorem I (The case $m = 1$; A. Kozlowski, KY)

If $n \geq 3$ and $d(I) \geq 3$, the inclusion map

$$i_D : \mathrm{Hol}_d^*(S^2, X_I) \to \Omega_d^2 X_I$$

is a homotopy equivalence through dimension $N(d, I)$, where

$$N(d, I) := (2d(I) - 3)d - 2.$$

Proof of Theorem I

If $d(I) = \min\{\mathrm{card}(\sigma) : \sigma \in I\} \geq 3$, we can show that the two spaces $\mathrm{Hol}_d^*(S^2, X_I)$ and $\Omega_d^2 X_I$ are simply connected. Then Theorem I follows from Theorem AI and Theorem B. □