Homeomorphism groups of non-compact surfaces endowed with the Whitney topology

Tatsuhiko Yagasaki (Kyoto Institute of Technology, Japan)

Coauthors :

Taras Banakh (Ivan Franko National University of Lviv, Ukraine)
Kotaro Mine (University of Tokyo, Japan)
Katsuro Sakai (Kanagawa University, Japan)

The 41 th Symposium on Transformation Groups
Nov. 15, 2014, Gamagori, Aichi Pref.

§1. Homeomorphism Groups with Whitney Topology

M : a connected n-manifold (possibly with boundary)

— separable, metrizable

$\mathcal{H}(M)$: Homeomorphism Group of M

Whitney Topology : $(h \in \mathcal{H}(M),\ \mathcal{U} \in \mathrm{cov}(M))$

Basic Open sets : $\mathcal{O}(h,\mathcal{U}) := \left\{g \in \mathcal{H}(M) \mid g : \mathcal{U}\text{-close to } h\right\}$

$*$ $g : \mathcal{U}$-close to $h \iff \forall\, x \in M \quad \exists\, U \in \mathcal{U}$ s.t. $g(x), h(x) \in U$

— Top group

$K \subset M$

$\mathcal{H}(M;K) = \{h \in \mathcal{H}(M) : h|_K = \mathrm{id}_K\} < \mathcal{H}(M)$ (Whitney Topology)

$\mathcal{H}(M;K)_0$: the identity connected component of $\mathcal{H}(M;K)$

$\mathcal{H}_c(M;K) \subset \mathcal{H}(M;K)$: Subgroup of Homeo's with compact support

Problem. Topological Properties of $\mathcal{H}(M;K)$ and $\mathcal{H}_c(M;K)$

Local Models for $\mathcal{H}(M,K)$ and $\mathcal{H}_c(M,K)$

$\mathcal{H}(M,K)$ and $\mathcal{H}_c(M,K)$: Homogeneous, Infinite-dimensional

We can expect that $\mathcal{H}(M,K)$ and $\mathcal{H}_c(M,K)$

are Top manifolds modeled on some typical infinite-dim spaces.

Test case — $\mathcal{H}(\mathbb{R})$ and $\mathcal{H}_c(\mathbb{R})$

(1) Models for Compact-Open Topology $\mathcal{H}(\mathbb{R})_{co}$: (R. D. Anderson, et al)

$\mathcal{H}_+(\mathbb{R})_{co} \approx \ell_2 \approx \prod^\omega \ell_2$ (Tychonoff Product of ℓ_2)

$\mathcal{H}_c(\mathbb{R})_{co} \approx (\prod^\omega \ell_2)_f$ (finite sequences) (Weak Product of ℓ_2)

$(\mathcal{H}_+(\mathbb{R})_{co}, \mathcal{H}_c(\mathbb{R})_{co}) \approx (\prod^\omega \ell_2, (\prod^\omega \ell_2)_f) \approx (\ell_2 \times \ell_2, \ell_2 \times \ell_2^f)$

(2) Models for Uniform Topology $\mathcal{H}^u(\mathbb{R})_u$: (MSYY, 2011)

$(\mathcal{H}^u(\mathbb{R})_u)_0 = \mathcal{H}^u_b(\mathbb{R})_u \approx \ell_\infty$

$\mathcal{H}_c(\mathbb{R})_u \approx \ell_2 \times \ell_2^f$

$(\mathcal{H}^u_b(\mathbb{R})_u, \mathcal{H}_c(\mathbb{R})_u) \approx (\ell_\infty \times \ell_2 \times \ell_2, \{0\} \times \ell_2 \times \ell_2^f)$

(3) Models for Whitney Topology $\mathcal{H}(\mathbb{R})$: (BMS, 2011)

$\mathcal{H}(\mathbb{R}) \approx \mathcal{H}_+(\mathbb{R}) \approx \square^\omega \ell_2$ (Box product of ℓ_2)

$\mathcal{H}(\mathbb{R})_0 = \mathcal{H}_c(\mathbb{R}) \approx \boxdot^\omega \ell_2 \approx \ell_2 \times \mathbb{R}^\infty$ (Small box product of ℓ_2)

$(\mathcal{H}(\mathbb{R}), \mathcal{H}_c(\mathbb{R})) \approx (\mathcal{H}_+(\mathbb{R}), \mathcal{H}_c(\mathbb{R})) \approx (\square^\omega \ell_2, \boxdot^\omega \ell_2)$

Box products :

— $\square^\omega \ell_2 = \left(\prod^\omega \ell_2, \text{ Box Top} \right)$ Basic open subsets : $\prod_{i=0}^\infty U_i$ ($U_i \subset \ell_2$: open)

$\boxdot^\omega \ell_2 \subset \square^\omega \ell_2$ (finite sequences) $\square_n X_n$ $\boxdot_n (X_n, *_n)$

— (P. Mankiewicz, 1974) Classification of Top. Type of LF spaces

$$\boxdot^\omega \mathbb{R} \approx \mathbb{R}^\infty \equiv \operatorname{dir\,lim} \{\mathbb{R}^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \cdots\}$$

$$\boxdot^\omega \ell_2 \approx \ell_2 \times \mathbb{R}^\infty$$

Expectation. When $cl_M(M - K)$: non-compact (and $K \subset M$: good)

$$(\mathcal{H}(M,K), \mathcal{H}_c(M,K)) \underset{\text{local}}{\approx} (\square^\omega \ell_2, \boxdot^\omega \ell_2)$$

— $\mathcal{H}_c(M,K)$: a paracompact $(\ell_2 \times \mathbb{R}^\infty)$-manifold

In this talk we consider the 2-dim case.

M : a connected 2-manifold

$K \subset M$: a subpolyhedron (in some triangulation of M)

§2. Previous Results in Compact cases

[1] Homeomorphism Groups in Compact cases :

"$cl_M(M - K)$ is compact"

$\mathcal{H}(M;K) = \mathcal{H}_c(M;K)$: Whitney Top = Compact-Open Top.

(1) $\mathcal{H}_c(M;K)$: a metrizable ℓ_2-manifold

(R.Luke - W.K.Mason (1972), et al. + Theory of top ℓ_2-manifolds)

(2) Classification of Homotopy type of $\mathcal{H}_c(M;K)_0$

(M.E. Hamstrom (1966), et al.)

$\mathcal{H}_c(M;K)_0 \simeq *$ with several exceptional cases

$(\mathcal{H}_c(M;K)_0 \approx \ell_2)$ $(\mathcal{H}_c(M;K)_0 \simeq P \implies \mathcal{H}_c(M;K)_0 \approx P \times \ell_2)$

(3) Mapping class group

$\mathcal{H}(M;K)_0 = \mathcal{H}_c(M;K)_0 \subset \mathcal{H}_c(M;K)$: Open normal subgroup

$\mathcal{M}_c(M;K) = \mathcal{H}_c(M;K)/\mathcal{H}(M;K)_0$

[2] Spaces of Embeddings and Bundle Theorem in dim 2

(R.Luke - W.K.Mason (1972), Yagasaki (2000))

$L \subset N$: subpolyhedra of M s.t. $\mathrm{cl}_M(N - L)$ is compact

$\mathcal{E}^*_L(N, M)$: the space of proper embeddings $f : N \to M$ s.t. $f|_L = \mathrm{id}_L$

Compact - Open topology

$R : \mathcal{H}(M, L) \to \mathcal{E}^*_L(N, M)$, $R(h) = h|_N$: the restriction map

(1) $\mathcal{E}^*_L(N, M)$: an ℓ_2-manifold if $\dim(N - L) \geq 1$.

(2) The map R has a local section at id_N.

$R : \mathcal{H}(M, L) \to \mathrm{Im}\, R$: a principal $\mathcal{H}(M, N)$-bundle

- $\mathrm{Im}\, R$: an open neighborhood of id_N in $\mathcal{E}^*_L(N, M)$

∗ R.Luke - W.K.Mason (1972)

— N = a proper arc, an orientation-preserving circle, $L = \emptyset$

— Conformal mapping theorem

∗ Yagasaki (2000) — General case

§3. Non-Compact case "$cl_M(M-K)$ is non-compact"

[BMSY, 2011] (BMSY [arXiv:0802.0337v1])

(1) $(\mathcal{H}(M,K), \mathcal{H}_c(M,K)) \underset{\text{local}}{\approx} (\square^\omega \ell_2, \boxdot^\omega \ell_2)$

(2) $\mathcal{H}_c(M;K)$: a paracompact $(\ell_2 \times \mathbb{R}^\infty)$-manifold

(3) $\mathcal{H}(M;K)_0 = \mathcal{H}_c(M;K)_0 \subset \mathcal{H}_c(M;K)$: Open normal subgroup

$\mathcal{M}_c(M;K) = \mathcal{H}_c(M;K)/\mathcal{H}(M;K)_0$: Mapping Class Group

(4) $M_i \subset M$ $(i \in \mathbb{N})$: Compact s.t. $M_i \subset \text{Int}_M M_{i+1}$, $M = \cup_i M_i$

$\mathcal{H}_c(M;K) = \text{Dir Lim}\, \mathcal{H}(M; K \cup (M - M_i))$ in Category of Top Groups

~~Top Spaces~~

[BMSY, 2014]

(1) $\mathcal{H}(M;K)_0 \approx \ell_2 \times \mathbb{R}^\infty$

(2) M : Non-Compact $\Longrightarrow$ $\#\mathcal{M}_c(M) = \begin{cases} 1 & \text{in the exceptional cases} \\ \aleph_0 & \text{in all other cases} \end{cases}$

Exceptional cases : $M = X - K$

X = Annulus, Disk or Möbius band,

K = Non-empty compact subset of one boundary circle of X

Comparison with Compact - Open Topology

(1) Whitney Topology : $\mathcal{H}(M;K)_0 = \mathcal{H}_c(M;K)_0 \approx \ell_2 \times \mathbb{R}^\infty \simeq *$

(2) Compact - Open Topology (Yagasaki, 2000, 2004)

$$(\mathcal{H}(M;K)_{co})_0 \approx \begin{cases} \mathbb{S}^1 \times \ell_2 \simeq \mathbb{S}^1 & \text{if } (M,K) = (\mathbb{R}^2,\emptyset),\ (\mathbb{R}^2, 1\text{pt}), \\ & \quad (\mathbb{S}^1 \times \mathbb{R}, \emptyset),\ (\mathbb{S}^1 \times [0,\infty), \emptyset), \\ & \quad (\mathbb{M} - \partial\mathbb{M}, \emptyset) \\ \ell_2 \simeq * & \text{in all other cases.} \end{cases}$$

$\cup$ h.e.

$(\mathcal{H}_c(M;K)_{co})_0$

Remark. $\mathcal{H}_c(\mathbb{R}^2)_{co} \simeq \mathbb{S}^1$

(i) The contraction of $\mathcal{H}_c(\mathbb{R}^2)_{co}$ induced by the Alexander trick is **not continuous.**

(ii) We can directly construct an essential loop in $\mathcal{H}_c(\mathbb{R}^2)_{co}$ (some kind of rotation)

since Compact-Open Top does not impose enough control on the end of $\mathbb{R}^2$.

Idea of Proof. $\mathcal{H}(M;K)_0 \approx \ell_2 \times \mathbb{R}^\infty$ in Non-Compact case

M : Non-Compact

$M = \cup_{n=0}^{\infty} M_n$: M_n : Compact 2-submanifolds of M s.t. $M_n \subset \mathrm{Int}_M M_{n+1}$

$\Downarrow$

$\mathcal{H}_c(M;K) = \cup_n \mathcal{H}(M; K \cup (M - M_n))$ (a tower of closed subgroups)

each $\mathcal{H}(M; K \cup (M - M_n)$: Compact Case

Top Group   Tower of Closed subgroups

$G \quad \Longleftarrow \quad G_n \ (n \in \omega)$ $(\omega = \{0, 1, 2, \cdots\})$

$(\, G_n \subset G_{n+1}, \ G = \bigcup_n G_n \,)$

§4. Results on Top Groups and Towers of Subgroups

G : Top group (e : the identity element of G)

G_n $(n \in \omega)$: Tower of Closed subgroups of G

$$p : \boxdot_n (G_n, e) \longrightarrow G \ : \quad p(x_0, x_1, \dots, x_k, e, e, \dots) = x_k \cdots x_1 x_0$$

[1] (BMSY [arXiv:0802.0337v1], 2011) (0) p : continuous, surjective

(1) p : open at $(e)_n \implies G = \mathrm{Dir\,Lim}\, G_n$ (in Category of Top Groups)

(2) p has a local section at e
each G_n : Locally contractible $\implies G$: Locally contractible
($\mathcal{H}_c(M^n)$: Locally contractible $\forall\, n$)

[2] (2007 - 2008) (BMSY [arXiv:0802.0337v1])

(#) (i) $p : \boxdot_n G_n \to G$: open (ii) $G_n \to G_n / G_{n-1}$
admits a global section s_n.

$$\implies \qquad s = \boxdot_n s_n : \boxdot_n (G_n/G_{n-1}) \longrightarrow \boxdot_n G_n, \quad p : \boxdot_n G_n \longrightarrow G, \quad ps : \boxdot_n (G_n/G_{n-1}) \xrightarrow[\approx]{ps} G$$

(#) + Results in Compact Case (§2) $\implies \mathcal{H}(M;K)_0 \approx \boxdot^\omega \ell_2 \approx \ell_2 \times \mathbb{R}^\infty$

[3] (2009 -) T. Banakh - D. Repovš — Series of papers

Study of Top LF-manifolds and Direct limit of Uniform spaces

Sufficient Condition that Top Group $\approx \ell_2 \times \mathbb{R}^\infty$ (BMRSY, 2013)

(i) G : Non-metrizable (ii) $G_n \approx \ell_2$

(iii) $p : \boxdot_n G_n \to G$: open

(iv) $G_{n+1} \to G_{n+1}/G_n$ has a local section

(v) each Z-point of G_{n+1}/G_n is a strong Z-point.

(for example, G_{n+1}/G_n is an ℓ_2-manifold.)

$\Longrightarrow \quad G \approx \ell_2 \times \mathbb{R}^\infty$

($\star$) Criterion of $\ell_2 \times \mathbb{R}^\infty$

+ Results in Compact Case (§2) $\Longrightarrow \quad \mathcal{H}(M;K)_0 \approx \ell_2 \times \mathbb{R}^\infty$

Below we give **Sketch of** ($\star$)

Notations.

(1) M : a connected 2-manifold

$K \subset M$: a subpolyhedron "$cl_M(M - K)$ is non-compact"

(2) We can represent $M = \bigcup_{n\in\omega} M_n$, where

M_n : a compact subpolyhedron of M, $M_n \subset \mathrm{Int}_M M_{n+1}$, $\mathrm{Int}_M M_n \not\subset K$.

$K_n = K \cup (M - \mathrm{Int}_M M_n) \quad (n \in \omega)$

(3) Consider Subgroup and Tower of subgroups :

$G = \mathcal{H}(M; K)_0 \qquad G_n = \mathcal{H}(M; K_n)_0 \quad (n \in \omega)$

We shall show that G and G_n $(n \in \omega)$ satisfy the next conditions :

[1] G : Non-metrizable **[2]** $G_n \approx \ell_2$

[3] $p : \boxdot_n G_n \to G$: open

[4] $\pi : G_{n+1} \to G_{n+1}/G_n$ admits a local section

[5] G_{n+1}/G_n : an ℓ_2-manifold

$\rightsquigarrow$ each point of G_{n+1}/G_n is a strong Z-point.

Then, Criterion of $\ell_2 \times \mathbb{R}^\infty$ implies that $G \approx \ell_2 \times \mathbb{R}^\infty$.

[1] Whitney Topology + Diagonal argument $\Longrightarrow$ G is not 1st countable

[2] Compact Case (§2) $\rightsquigarrow$ G_n : an ℓ_2-manifold, $G_n \simeq *$ $\quad \therefore\ G_n \approx \ell_2$

[4], [5]

(1) First consider the groups $\quad H_n = \mathcal{H}(M; K_n) \quad (n \in \omega)$

$$\begin{array}{ccccc} & & H_n & & \\ & \swarrow^{\pi} & & \searrow^{R} & \\ H_n/H_m & & \xrightarrow[\text{homeo}]{\varphi} & & \operatorname{Im} R \ \overset{\text{open}}{\subset}\ \mathcal{E}^*_{K_n}(K_m, M) \end{array} \qquad (m \leq n)$$

Compact Case (§2) $\rightsquigarrow$ R has a local section, $\mathcal{E}^*_{K_n}(K_m, M)$: an ℓ_2-manifold

$\therefore\ \pi : H_n \to H_n/H_m$ has a local section, H_n/H_m : an ℓ_2-manifold

(2) $G_m \subset H_m$: open $\quad \therefore\ H_m/G_m$: discrete

$\pi : H_n/G_m \to H_n/H_m$: a locally trivial bundle with fiber H_m/G_m

$\rightsquigarrow$ $\pi : G_{n+1} \to G_{n+1}/G_n$ has a local section, G_{n+1}/G_n : an ℓ_2-manifold

[3] $p : \boxdot_n G_n \to G$: open

(1) Compact Case (§2) $\rightsquigarrow$ $R_n : G_n \to \mathcal{E}^*_{K_n}(K_{n-1}, M)$ has a local section

$s_n : (V_n, \mathrm{id}_{K_{n-1}}) \to (G_n, \mathrm{id}_M)$ at $\mathrm{id}_{K_{n-1}}$.

— s_n $(n \in \omega)$ $\rightsquigarrow$ a local section s of p $\quad\therefore$ p : open

(2) (a direct argiment to show that p is open)

Suppose U_n is a symmetric open nbd of id_M in G_n $(n \in \omega)$

$\rightsquigarrow$ We have to show that $p(\boxdot_n U_n)$ is a nbd of id_M in G

(3) (Notations) $\qquad \mathcal{U} \in \mathrm{cov}(M)$

$A \subset M \quad \mathcal{S}t(A, \mathcal{U}) = \cup\{U \in \mathcal{U} : A \cap U \neq \emptyset\} \qquad \mathcal{S}t(\mathcal{U}) = \{\mathcal{S}t(U, \mathcal{U}) : U \in \mathcal{U}\}$

(4) Inductively we can find $\mathcal{U}_n, \mathcal{V}_n \in \mathrm{cov}(M)$ $(n \in \omega)$ such that

(i) (a) $\mathcal{S}t(\mathcal{U}_n) \prec \mathcal{V}_{n-1} \qquad (\mathcal{V}_{-1} = \{M\})$

(b) $h \in \mathcal{H}(M; K_n)$, h : $\mathcal{U}_n$-close to $\mathrm{id}_M \implies h \in U_n$

(ii) (a) $\mathcal{S}t(\mathcal{V}_n) \prec \mathcal{U}_n \qquad$ (b) $f \in \mathcal{E}^*_{K_n}(K_{n-1}, M)$, f : $\mathcal{S}t(\mathcal{V}_n)$-close to $\mathrm{id}_{K_{n-1}}$

$\implies f \in V_n$, $s_n(f)$: $\mathcal{U}_n$-close to id_M

(5) $\exists\, \mathcal{V} \in \mathrm{cov}(M)$ s.t. $\{\mathcal{S}t(x, \mathcal{V}) \mid x \in M - \mathrm{Int}_M M_{n-1}\} \prec \mathcal{V}_n$ $(n \in \omega)$

$\mathcal{O}(\mathrm{id}_M, \mathcal{V}) \subset p(\boxdot_n U_n)$

Mapping class groups of non-compact surfaces

M : a non-compact connected 2-manifold (possibly with boundary)

$\mathcal{M}_c(M) = \mathcal{H}_c(M)/\mathcal{H}(M)_0$

Theorem. The following conditions are equivalent:

[**1**] $\mathcal{M}_c(M)$: trivial

[**2**] $\mathcal{M}_c(M)$ is a torsion group (i.e., each element has finite order)

[**3**] M : exceptional i.e., $M \approx X - K$:

X = Annulus $\mathbb{A}$, Disk $\mathbb{D}$ or Möbius band $\mathbb{M}$,

K = Non-empty compact subset of one boundary circle of X

Sketch of Proof : [**1**] $\Rightarrow$ [**2**] $\Rightarrow$ [**3**] $\Rightarrow$ [**1**]

[**2**] $\Rightarrow$ [**3**] **:**

Lemma 1.

(1) Every boundary circle C of M is a retract of M.

(2) $h \in \mathcal{H}(M)$ $h \in \mathcal{H}(M)_0 \iff \exists$ an isotopy from h to id_M with compact support

Lemma 2. $\mathcal{M}_c(M) \supset \mathbb{Z}$ in each of the following cases:

(1) M contains a handle;

(2) M contains at least two disjoint Möbius bands;

(3) M contains at least two boundary circles;

(4) M contains a Möbius band and a boundary circle;

(5) M is separated by a circle $C \subset \operatorname{Int} M$

into two non-compact connected subsurfaces L_1 and L_2.

($\because$) (5) $h_n :=$ the n-fold Dehn twist along C $(n \in \mathbb{Z})$

Claim : $h_n \in \mathcal{H}(M)_0 \iff n = 0$

(i) Suppose $h_n \in \mathcal{H}(M)_0$.

$\exists$ an isotopy $h_n \simeq \mathrm{id}_M$ with a compact support K.

(ii) $\exists$ a path ℓ in M which

connects a point in $L_1 \setminus K$ with a point of $L_2 \setminus K$ and crosses C once.

(iii) $h_n\ell \simeq \ell$ in M rel. end points $\therefore$ $C^n \simeq (h_n\ell)\ell^{-1} \simeq *$ in M.

(iv) M retracts onto C $\therefore$ $C^n \simeq *$ in C $\therefore$ $n = 0$

Proof of [2] ⇒ [3]. Suppose $\mathcal{M}_c(M)$: a torsion group.

(1) By Lemma 2, M contains

(i) at most one Möbius band
(ii) at most one boundary circle
} not simultaneous

(iii) no handle and

(iv) no circle separating M into two non-compact connected subsurfaces.

(2) $M = \bigcup_{n\in\omega} M_n$, where

(i) M_n : a compact connected subsurfaces of M,

(ii) $M_n \subset \mathrm{int}_M M_{n+1}$, (iii) if $\partial M \neq \emptyset$ then $M_0 \cap \partial M \neq \emptyset$,

(iv) if L is a connected component of $M - \mathrm{Int}_M M_n$,

then L is non-compact and $L \cap M_{n+1}$ is connected.

(3) Every M_n has exactly one boundary circle meeting $M - \mathrm{Int}_M M_n$.

(4) Three possible cases :

Case (i): M contains no boundary circle and no Möbius band.

$M \approx \mathbb{D} - K \quad (K \subset \partial\mathbb{D} : \text{ compact}, \neq \emptyset)$ $\qquad$ ($\forall\ M_n$: a disk)

Case (ii): M contains a Möbius band. $\qquad$ ($\forall'\ M_n$: a Möbius band)

$M \approx \mathbb{M} - K \quad (K \subset \partial\mathbb{M} : \text{ compact}, \neq \emptyset)$

Case (iii): M contains a boundary circle C. $\qquad$ ($\forall'\ M_n$: an annulus)

$(M, C) \approx (\mathbb{A} - K, C_1) \quad (\partial\mathbb{A} = C_1 \cup C_2, \quad K \subset C_2 : \text{ compact}, \neq \emptyset)$

[End of Talk]

Thank you very much for your attention !