Construction of gap modules

Toshio SUMI

Faculty of Arts and Science
Kyushu University

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Definition

Let G be a finite group not of prime power order.

- A real G-module means a finite dimensional real vector space with a linear G-action.
- $\pi(G)$ denotes the set of primes dividing $|G|$, the order of G.
- $\mathcal{P}(G)$ denotes the set of subgroups P with $|\pi(P)| \leq 1$
- $\mathcal{PH}(G)$ denotes the set of pairs (P, H) of subgroups of G such that $P \in \mathcal{P}(G)$ and $P < H \leq G$.

Definition

A G-module V is called a *gap G-module* if

$$\dim V^P > 2 \dim V^H$$

for all $(P, H) \in \mathcal{PH}(G)$.

Gap modules

- For a set $\mathcal{S}$ of subgroups of G, a G-module V is *$\mathcal{S}$-free* if $V^L = 0$ for all $L \in \mathcal{S}$.
- For a prime p, $O^p(G)$ denotes the minimal (normal) p-power index subgroup of G, called Dress subgroup of type p.

$$O^p(G) = \bigcap_{L \leq G, [G:L]=p^*} L$$

- $\mathcal{L}(G)$ denotes the set of subgroups of G containing some $O^p(G)$

$$G \in \mathcal{L}(G)$$

Definition

A finite group G is called a *gap group* if there exists an $\mathcal{L}(G)$-free gap G-module.

Notation

- For a G-module V, define $d_V : \mathcal{PH}(G) \to \mathbb{Z}$ by

$$d_V(P, H) = \dim V^P - 2 \dim V^H$$

- For a subset $\mathcal{S}$ of $\mathcal{PH}(G)$, a real G-module V is called *positive* on $\mathcal{S}$ if $d_V(P, H) > 0$ for any $(P, H) \in \mathcal{S}$.
- For a subset $\mathcal{S}$ of $\mathcal{PH}(G)$, a real G-module V is called *nonnegative* on $\mathcal{S}$ if $d_V(P, H) \geq 0$ for any $(P, H) \in \mathcal{S}$.

A finite group G is a gap group if there exists an $\mathcal{L}(G)$-free G-module which is positive on $\mathcal{PH}(G)$.

Oliver group

A finite group G is an *Oliver group*, if G has no series of subgroups of the form

$$P \triangleleft H \triangleleft G$$

where $|\pi(P)| \leq 1$, $|\pi(G/H)| \leq 1$ and H/P is cyclic. Particularly, each nonsolvable group is an Oliver group.

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Theorem (Oliver 1975)

A finite group G has a fixed point free smooth action on a disk if and only if G is an Oliver group.

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Theorem (Oliver 1975)

A finite group G has a fixed point free smooth action on a disk if and only if G is an Oliver group.

Theorem (Laitinen-Morimoto 1998)

A finite group G has a one fixed point smooth action on a sphere if and only if G is an Oliver group.

Surgery Theory

Theorem (Morimoto 1998, 2008)

Let G be a finite Oliver and gap group and Y a smooth G-manifold such that the underlying manifold of Y is diffeomorphic to the disk of dimension $n \geq 5$ and $Y^G \neq \emptyset$. Let $F_1, \ldots, F_t$ denote the connected components of Y^G, and let $k_1, \ldots, k_t$ be nonnegative integers. Suppose the following condition.

1. *$\pi_1(Y^P)$ is finite group of order prime to $|P|$ for any $P \in \mathcal{P}(G)$.*
2. *$k_i = k_j$ whenever some connected component Y^H_α of Y^H, $H \in \mathcal{L}(G)$, contains both F_i and F_j.*

Then there exist a gap G-module W and a G-action on the disk D such that

- *$D^G = \coprod_{i=1}^{t} \coprod_{j=1}^{k_i} F_{i,j}$ (each $F_{i,j}$ is diffeomorphic to F_i),*
- *∂D is G-diffeomorphic to $\partial(Y \times D(W))$,*
- *each normal bundle $\nu(F_{i,j}, D)$ is G-isomorphic to $\nu(F_i, Y) \oplus W$.*

Examples I

A gap group G satisfies that $\mathcal{L}(G) \cap \mathcal{P}(G) = \emptyset$.

Example

Suppose that $\mathcal{L}(G) \cap \mathcal{P}(G) = \emptyset$.

1. Any nonabelian perfect group is a gap group.
2. If $|\{p \in \pi(G) \mid p \neq 2,\ O^p(G) \neq G\}| \geq 2$, then G is a gap group. ([Laitinen-Morimoto 1998])
3. S_n ($n \geq 6$) are gap groups. ([Dovermann-Herzog 1997])
4. $S_4 \times S_5$, $S_n \times C_2$ ($n \geq 6$) and $A_n \times C_2$ ($n \geq 5$) are gap groups. ([Morimoto-S-Yanagihara 2000])
5. S_1, S_2, S_3, S_4 are not gap groups.
6. S_5 is not a gap group. ([Morimoto-Yanagihara 1996])
7. $S_5 \times C_2$ is not a gap group.

Examples II

Example (S)

Suppose that $\mathcal{L}(G) \cap \mathcal{P}(G) = \varnothing$.

1. If G is a generalized quaternion group Q_{4n} of order $4n$,
$$\langle x,\ y \mid x^{2n} = 1,\ y^2 = x^n,\ y^{-1}xy = x^{-1} \rangle$$
G is not a gap group but $G \times C_p$ is for all odd prime p.
2. $G \times D_{2n}$ is a gap group if and only if G is.
3. For a 2-group K, $G \times K$ is a gap group if and only if G is.
4. A finite group which has a quotient gap group is a gap group.

Examples III

Theorem (S)

A nonsolvable general linear group $GL(n, q)$ is a gap group. A nonsolvable projective linear group $PGL(n, q)$ is a gap group if and only if $(n, q) \neq (2, 5), (2, 7), (2, 9), (2, 17)$.

Theorem (S)

The automorphism group of any sporadic group is a gap group.

Criterion to be a gap group

Let K be an index 2 subgroup of G.
For an element x of G, we set

- $\varphi(x) = \{q\text{: odd prime} \mid x \in {}^{\exists}N \leq G,\ O^q(N) \neq N\}$
- $E_2(G, K) = \{x \in G \setminus K \mid |x| = 2, |\varphi(x)| > 1 \text{ or } O^2(C_G(x)) \notin \mathcal{P}(G)\}$
- $E_4(G, K) = \{x \in G \setminus K \mid |x| = 2^* \geq 4, |\varphi(x)| > 0\}$
- $E(G, K) = E_2(G, K) \cup E_4(G, K)$

$$E_2^o(G, K) \subseteq E_2(G, K),\ E_4^o(G, K) \subseteq E_4(G, K),\ E^o(G, K) \subseteq E(G, K)$$

Criterion to be a gap group

Let K be an index 2 subgroup of G.
For an element x of G, we set

- $\varphi(x) = \{q\text{: odd prime} \mid x \in {}^{\exists}N \le G,\ O^q(N) \ne N\}$
- $E_2(G,K) = \{x \in G \smallsetminus K \mid |x| = 2, |\varphi(x)| > 1 \text{ or } O^2(C_G(x)) \notin \mathcal{P}(G)\}$
- $E_2^o(G,K) = \{x \in G \smallsetminus K \mid |x| = 2, O^2(C_G(x))) \notin \mathcal{P}(G)\}$
- $E_4(G,K) = \{x \in G \smallsetminus K \mid |x| = 2^* \ge 4, |\varphi(x)| > 0\}$
- $E_4^o(G,K) = \{x \in G \smallsetminus K \mid |x| = 2^* \ge 4, C_G(x) \notin \mathcal{P}(G)\}$
- $E(G,K) = E_2(G,K) \cup E_4(G,K)$
- $E^o(G,K) = E_2^o(G,K) \cup E_4^o(G,K)$

$$E_2^o(G,K) \subseteq E_2(G,K),\ E_4^o(G,K) \subseteq E_4(G,K),\ E^o(G,K) \subseteq E(G,K)$$

Criterion to be a gap group I

Definition

A finite group G not of prime power order is called an *almost gap group* if there exists an $\mathcal{L}(G)$-free module which is positive on $\{(P,H) \in \mathcal{PH}(G) \mid P \notin \mathcal{L}(G)\}$.

Theorem

Let G be a finite group with $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$ and let K be a index 2 subgroup of G. Suppose that K is an almost gap group. G is a gap group if and only if $E^o(G,K)$ is not empty.

Criterion to be a gap group II

Proposition

Suppose that $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$.
$\exists \{O^2(G)\}$*-free gap* G*-module iff* $\exists\, \mathcal{L}(G)$*-free gap* G*-module*

Theorem (Morimoto-S-Yanagihara, 2000)

Let G *be a finite group with* $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$ *and let* L *be subgroup of* G *with* $L \geq O^2(G)$. *If* L *is not an almost gap group, then* G *is not a gap group.*

Proof of examples, I

Theorem

Let G be a finite group with $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$ and let K be a 2 power index subgroup of G. Suppose that K is an almost gap group. G is a gap group if and only if $E^o(G, K)$ is not empty.

Example (Dovermann-Herzog, 1997 and Morimoto-S-Yanagihara, 2000)

S_n for $n \geq 7$ is a gap group.

Proof.

Since $(1, 2, 3, 4)(5, 6, 7) \in S_n$, $(1, 2, 3, 4) \in E^o(S_n, A_n)$. □

$$E^o_4(G, K) = \{x \in G \smallsetminus K \mid |x| = 2^* \geq 4, C_G(x) \notin \mathcal{P}(G)\}$$
$$E^o_2(G, K) = \{x \in G \smallsetminus K \mid |x| = 2, O^2(C_G(x))) \notin \mathcal{P}(G)\}$$

Proof of examples, I

Theorem

Let G be a finite group with $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$ and let K be a 2 power index subgroup of G. Suppose that K is an almost gap group. G is a gap group if and only if $E^o(G,K)$ is not empty.

Example

$A_n \times C_2$ for $n \geq 5$ is a gap group.

Proof.

We see $A_n \times C_2$ is a subgroup of S_{n+2} such that $C_2 = \{(), (n+1, n+2)\}$. Since $(1,2,3)(n+1,n+2), (1,2,3,4,5)(n+1,n+2) \in S_n$, $(n+1,n+2) \in E_2^o(A_n \times C_2, A_n)$. □

$$E_4^o(G,K) = \{x \in G \smallsetminus K \mid |x| = 2^* \geq 4, C_G(x) \notin \mathcal{P}(G)\}$$
$$E_2^o(G,K) = \{x \in G \smallsetminus K \mid |x| = 2, O^2(C_G(x))) \notin \mathcal{P}(G)\}$$

Proof of examples, I

Theorem

Let G be a finite group with $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$ and let K be a 2 power index subgroup of G. Suppose that K is an almost gap group. G is a gap group if and only if $E^o(G, K)$ is not empty.

Example

S_5 is not a gap group.

Proof.

$C_{S_5}((4,5)) = \langle (1,2), (1,3), (4,5) \rangle \cong S_3 \times C_2$. $O^2(C_{S_5}((4,5)) \cong C_3$.
$C_{S_5}((1,2,3,4)) = \langle (1,2,3,4) \rangle \cong C_4$. Then $E^o(S_5, A_5) = \emptyset$. □

$$E_4^o(G,K) = \{x \in G \setminus K \mid |x| = 2^* \geq 4, C_G(x) \notin \mathcal{P}(G)\}$$
$$E_2^o(G,K) = \{x \in G \setminus K \mid |x| = 2, O^2(C_G(x))) \notin \mathcal{P}(G)\}$$

Proof of examples, II

Example (Dovermann-Herzog, 1997 and Morimoto-S-Yanagihara, 2000)

S_6 is a gap group.

Proof.

$$C_G((1,2)) = \langle (1,2),(3,6),(4,6),(5,6)\rangle$$

$$O^2(C_G((1,2))) = \langle (3,4,5),(4,5,6)\rangle \cong A_4.$$

Then $(1,2) \in E^o(S_6, A_6)$. □

$C_{S_6}((1,2,3,4)) \cong C_4 \times C_2$ and $C_{S_6}((1,2)(3,4)(5,6)) \cong C_{S_6}((1,2))$.

Proof of examples, III

Proposition

Let G be a finite group with $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$. Let $G_{\{2\}}$ be a a Sylow 2-subgroup of G. If $|\pi(N_G(G_{\{2\}})/G_{\{2\}})| \geq 2$, then G is a gap group. In particular, if $G_{\{2\}}$ is normal, then G is a gap group.

Proof.

Let p and q be distinct primes of $\pi(N_G(G_{\{2\}})/G_{\{2\}})$. Take elements x and y of $\pi(N_G(G_{\{2\}})$ of order p and q respectively. Consider the subgroups $N_p = \langle x \rangle G_{\{2\}}$, $N_q = \langle y \rangle G_{\{2\}}$. Then $\mathrm{Ind}_{N_p}^G V(N_p) \oplus \mathrm{Ind}_{N_q}^G V(N_q) \oplus V(G)$ is an $\mathcal{L}(G)$-free gap G-module. □

Construction of gap modules – ideas – I

For a set $\mathcal{S}$ of subgroups of G, $\mathrm{RO}(G)^{\mathcal{S}}$ denotes the set of all $\mathcal{S}$-free real G-modules.

Proposition

We can construct a gap G-module of form $\sum_C \mathrm{Ind}_C^G V_C$, where V_C is a C-module. Here C runs over representatives of conjugacy classes of cyclic subgroups of G.

$$\mathcal{L}(G) \cap K = \{L \cap K \mid L \in \mathcal{L}(G)\}$$

$$RO(G) \otimes \mathbb{Q} = \textstyle\sum_C \mathrm{Ind}_C^G RO(C) \otimes \mathbb{Q}$$

$$RO(G)^{\mathcal{L}(G)} \otimes \mathbb{Q} = \textstyle\sum_C \mathrm{Ind}_C^G RO(C)^{\mathcal{L}(G)\cap C} \otimes \mathbb{Q}$$

Construction of gap modules – ideas – II

A G-module V regards as a vector space with a G-invariant inner product. For a G-invariant subspace U of V (that is U is a submodule of V), we denote by $V - U$ the orthogonal complement subspace of U in V. Laitinen and Morimoto used the G-module

$$V(G) = (\mathbb{R}[G] - \mathbb{R}) - \bigoplus_{p \in \pi(G)} (\mathbb{R}[G] - \mathbb{R})^{O^p(G)}$$

to show that G is Oliver iff $^{\exists}$one fixed point action on a sphere. This module is the maximal $\mathcal{L}(G)$-free submodule of $\mathbb{R}[G]$.

$$\dim V(G)^H = (|G/H| - 1) - \sum_{p \in \pi(G)} (|G/O^p(G)H| - 1)$$

Construction of gap modules – ideas – III

- $\mathcal{PH}^2(G)$: the subset of $\mathcal{PH}(G)$ containing (P, H) such that

$$[H : P] = [O^2(G)H : O^2(G)P] = 2, O^q(G)P = G$$

Proposition (Laitinen-Morimoto 1998)

1. *$V(G)$ is $\mathcal{L}(G)$-free and nonnegative.*
2. *$d_{V(G)}(P, H) = 0$ if and only if $P \in \mathcal{L}(G)$ or $(P, H) \in \mathcal{PH}^2(G)$*

For a G-module V, put

$$V_{\mathcal{L}(G)} = (V - V^G) - \bigoplus_{p \in \pi(G)} (V^{O^p(G)} - V^G)$$

which is the maximal $\mathcal{L}(G)$-free G-submodule of V.

Construction of gap modules – ideas – IV

$$V(G) = \mathbb{R}[G]_{\mathcal{L}(G)}$$

Proposition

Let G be a group satisfying that $O^2(G) = G$ or $\pi(G/[G,G])$ contains two odd primes. If $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$, then $V(G)$ is a gap G-module.

Proposition (Nonnegative + Positive = Positive)

Suppose that $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$. If there exists an $\mathcal{L}(G)$-free module which is positive for any $(P,H) \in \mathcal{PH}^2(G)$, then $W \oplus V(G)^{\oplus \dim W + 1}$ is a gap G-module and G is a gap group.

Construction of gap modules – ideas – V

Remark

1. *If V is a gap module, then $V^{\oplus m}$ is also for $m \in \mathbb{N}$.*
2. *For a G-module V and a nonnegative G-module W, it holds that $d_V \le d_{V\oplus W}$, that is, $d_V(P,H) \le d_{V\oplus W}(P,H)$ for all $(P,H) \in \mathcal{PH}(G)$.*
3. *Let K be a subgroup of G. For a nonnegative K-module W, if $W^{K\cap O^2(G)} = 0$, then $(\mathrm{Ind}_K^G W)_{\mathcal{L}(G)} \oplus V(G)^{\oplus n}$ is nonnegative where $n = \min(-\min d_{(\mathrm{Ind}_K^G W)_{\mathcal{L}(G)}}, 0)$.*

Necessary condition to be a gap group I

- $\Lambda = \{L \le G \mid O^2(G) < L\}$
- $\Lambda_0 = \{L \in \Lambda \mid L/O^2(G) \text{ is cyclic}\}$
- $\Lambda_1 = \{L \in \Lambda_0 \mid L < {}^{\nexists}K < G\}$

Theorem (S)

Let G be a finite group with $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$. TFAE.

1. *G is a gap group.*
2. *For any $L \in \Lambda$, there exists an $(\mathcal{L}(G) \cap L)$-free L-module W_L such that $d_{W_L}(P,H) > 0$ for $(P,H) \in \mathcal{PH}(L)$.*
3. *For any $L \in \Lambda_0$, there exists an $(\mathcal{L}(G) \cap L)$-free L-module W_L such that $d_{W_L}(P,H) > 0$ for $(P,H) \in \mathcal{PH}(L)$.*
4. *For any $L \in \Lambda_1$, there exists an $(\mathcal{L}(G) \cap L)$-free L-module W_L such that $d_{W_L}(P,H) > 0$ for $(P,H) \in \mathcal{PH}(L)$.*

Necessary condition to be a gap group II

Let G be a finite group such that $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ and $G/O^2(G)$ is nontrivial cyclic.

Theorem

If there exists an element $x \in G$ such that $G = O^2(G)\langle x \rangle$ and $\langle x \rangle \cap O^2(G) \notin \mathcal{P}(G)$, then G is a gap group.

Let $\pi(\langle x \rangle) = \{q_1, q_2, \dots, q_t\}$, C_j a Sylow q_j-subgroup of $\langle x \rangle$ and η_j an irreducible complex C_j-module such that $\eta_j^P = 0$ for any nontrivial subgroup P of C_j. Set

$$U = (\mathbb{C} - \eta_1) \cdots (\mathbb{C} - \eta_t) \in R(\langle G_1 \rangle) \otimes \cdots \otimes R(\langle G_t \rangle) \cong R(\langle x \rangle)$$

and let V be a real $\langle x \rangle$-module which is direct sum of the realification of the module U and $V(\langle x \rangle)^{\oplus n}$ for sufficiently large n. Then $(\mathrm{Ind}_{\langle x \rangle}^{G} V)_{\mathcal{L}(G)}$ is a gap G-module.

Construction of nonnegative modules – $E(G,K)$ –

For an element x of G, we denote by $\varphi(x)$ the set of odd primes q such that $x \in N_q$ and $O^q(N_q) \neq N_q$ for some subgroup N_q of G, and by $\psi(x)$ the number of elements of the set $\varphi(x)$.

$E_4(G, O^2(G)) = \{x \in G \smallsetminus O^2(G) \mid |x| = 2^* \geq 4,\ \psi(x) > 0\}$

Proposition

Let G be a finite group. For an element x of $E_4(G, O^2(G))$, the G-module

$$W_x = \sum_{q \in \varphi(x)} \mathrm{Ind}_{N_q}^G V(N_q)$$

is nonnegative and $\mathcal{L}(G)$-free such that

- *$d_{W_x}(P,H) > 0$ for any $(P,H) \in \mathcal{PH}^2(G)$ with $(x) \cap (H \smallsetminus P) \neq \emptyset$ and $P \notin \mathcal{L}(G)$.*

Construction of nonnegative modules – $E(G, K)$ –

For an element x of G, we denote by $\varphi(x)$ the set of odd primes q such that $x \in N_q$ and $O^q(N_q) \neq N_q$ for some subgroup N_q of G, and by $\psi(x)$ the number of elements of the set $\varphi(x)$.

$E_2(G, O^2(G)) = \{x \in G \smallsetminus O^2(G) \mid |x| = 2,\ \psi(x) > 1 \text{ or } O^2(C_G(x)) \notin \mathcal{P}(G)\}$

Proposition

Let G be a finite group. For an element x of $E_2(G, O^2(G))$, the G-module

$$W_x = \bigoplus_{q \in \varphi(x)} \mathrm{Ind}_{N_q}^G V(N_q) \oplus \bigoplus_{q \in \pi(C_G(x)) \smallsetminus \{2\}} \mathrm{Ind}_{M_q}^G V(M_q),$$

where S_q is a Sylow q-subgroup of $C_G(x)$ and $M_q = \langle x \rangle \times S_q$, is nonnegative and $\mathcal{L}(G)$-free such that $d_{W_x}(P, H) > 0$ for any $(P, H) \in \mathcal{PH}^2(G)$ with $\langle x \rangle \cap (H \smallsetminus P) \neq \emptyset$ and $P \notin \mathcal{L}(G)$.

Construction of nonnegative modules – $E^o(G,K)$ –

For $E^o_4(G,K)$ and $E^o_2(G,K)$,

Proposition

Let G be a finite group. For an element x of $E^o_4(G,O^2(G))$, the G-module

$$W_x = \sum_{q\in\varphi(x)} \mathrm{Ind}_{M_q}^G V(M_q),$$

where S_q is a Sylow q-subgroup of $C_G(x)$ and $M_q = \langle x\rangle \times S_q$, is nonnegative and $\mathcal{L}(G)$-free such that

- *$d_{W_x}(P,H) > 0$ for any $(P,H) \in \mathcal{PH}^2(G)$ with $(x) \cap (H \smallsetminus P) \neq \emptyset$ and $P \notin \mathcal{L}(G)$.*

$$E^o_4(G,O^2(G)) = \{x \in G \smallsetminus O^2(G) \mid |x| = 2^* \geq 4,\ |\pi(C_G(x))| > 1\}$$

Construction of nonnegative modules – $E^o(G,K)$ –

For $E^o_4(G,K)$ and $E^o_2(G,K)$,

Proposition

Let G be a finite group. For an element x of $E^o_2(G,O^2(G))$, the G-module

$$W_x = \bigoplus_{q\in\pi(C_G(x))\setminus\{2\}} \mathrm{Ind}_{M_q}^G V(M_q),$$

where S_q is a Sylow q-subgroup of $C_G(x)$ and $M_q = \langle x\rangle \times S_q$, is nonnegative and $\mathcal{L}(G)$-free such that

- $d_{W_x}(P,H) > 0$ *for any* $(P,H) \in \mathcal{PH}^2(G)$ *with* $(x) \cap (H \setminus P) \neq \emptyset$ *and* $P \notin \mathcal{L}(G)$.

$$E^o_2(G,O^2(G)) = \{x \in G \setminus O^2(G) \mid |x| = 2,\ O^2(C_G(x)) \notin \mathcal{P}(G)\}$$

Construction of nonnegative modules

Theorem

Let G be a finite group with $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$. There exists an $\mathcal{L}(G)$-free nonnegative G-module W such that $d_W(P, H) > 0$ if $E(G, O^2(G)) \cap (H \setminus P) \neq \emptyset$ for any $(P, H) \in \mathcal{PH}^2(G)$.

Necessary condition to be a gap group

Theorem (S)

Let G be a finite group with $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$. TFAE.

1. *G is a gap group.*
2. *For any $x \in G \smallsetminus O^2(G)$, there exists an $(\mathcal{L}(G) \cap O^2(G)\langle x \rangle)$-free $O^2(G)\langle x \rangle$-module U_x such that $d_{U_x}(P, H) > 0$ for $(P, H) \in \mathcal{PH}(O^2(G)\langle x \rangle)$.*

❶ ⇒ ❷:

For a gap G-module V,

$$U_x = (\mathrm{Res}^G_{O^2(G)\langle x \rangle} V)_{\mathcal{L}(G) \cap O^2(G)\langle x \rangle}$$

is a required module.

Necessary condition to be a gap group

Theorem (S)

Let G be a finite group with $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$. TFAE.

1. *G is a gap group.*
2. *For any $x \in G \smallsetminus O^2(G)$, there exists an $(\mathcal{L}(G) \cap O^2(G)\langle x \rangle)$-free $O^2(G)\langle x \rangle$-module U_x such that $d_{U_x}(P, H) > 0$ for $(P, H) \in \mathcal{PH}(O^2(G)\langle x \rangle)$.*

❶ $\Leftarrow$ ❷:

$$\bigoplus_{(x)^{\pm} \subset G \smallsetminus O^2(G)} \mathrm{Ind}_{O^2(G)\langle x \rangle}^{G} U_x$$

is a gap G-module.

Construction of gap modules I

Let G be a finite group such that $G/O^2(G)$ is a nontrivial cyclic group and let K be an index 2 subgroup of G.
Note that $E_2(G,K) = \emptyset$ if $K \neq O^2(G)$. In the previous argument, we see that there exists an $\mathcal{L}(G)$-free nonnegative G-module $W(G)$ such that $d_{W(G)}(P,H)$ is positive if $(P,H) \in \mathcal{PH}(G) \smallsetminus \mathcal{PH}^2(G)$ or $(H \smallsetminus P) \cap E(G,K) \neq \emptyset$.

Construction of gap modules II

Let $\{C_j \mid j \in J\}$ be a complete set of representatives of all conjugacy classes in G of cyclic subgroups C with $C \not\leq K$.

$$J(2) = \{j \in J \mid C_j \in \mathcal{P}(G)\}$$

$$s_j = |N_G(C_j)/C_j|$$

for $j \in J$.

Proposition

$$\sum_{j \in J(2)} s_j^{-1} \leq 1$$

$$\sum_{j \in J(2)} s_j^{-1} = 1 \Leftrightarrow J(2) = J$$

Construction of gap modules III

$$m = \mathrm{LCM}\{s_j \mid j \in J(2)\}$$

$$U = \sum_{j \in J(2)} \left((\mathrm{Ind}_{C_j}^{G}(\mathbb{R}[C_j] - \mathbb{R}))_{\mathcal{L}(G)} \right)^{\oplus m s_j^{-1}}$$

$$n = \min(-\min d_U - 1, 0), \quad 0 \le n \le \dim U + 1$$

$$U(K) := U \oplus (W(G) \oplus V(G))^{\oplus n}$$

$$U(K; G) := U \oplus W(G)^{\oplus n}$$

Theorem

Let G be a finite group such that $G/O^2(G)$ is a nontrivial cyclic group and let K be an index 2 subgroup of G. If $E^o(G,K) \neq \emptyset$, then $U(K)$ is nonnegative and $\mathcal{L}(G)$-free, and $d_{U(K)}(P,H) > 0$ for any $H \not\leq K$.

Construction of gap modules IV

Let G be a finite group such that $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$ and $G/O^2(G)$ is cyclic. Let consider the sequence of index 2 subgroups of G

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_{t-1} \triangleright G_t = O^2(G), \ [G_k : G_{k+1}] = 2$$

Theorem

If $E^o(G_k, G_{k+1}) \neq \emptyset$ for any $0 \leq k < t$, then $\bigoplus_{0 \leq k < t} U(G_k; G) \oplus nV(G)$ is an $\mathcal{L}(G)$-free gap G-module (for sufficient large n) and in partiular G is a gap group.

Construction of gap modules V

Let $G_1, \ldots, G_s$ be complete representatives of subgroups of conjugacy classes of G such that G/G_i is cyclic and there is no subgroup K of G such that G/K is cyclic and $K > G_i$. Let $G_{i,1}, \ldots, G_{i,k_i}$ be subgroups of $G_{i,0} := G_i$ such that

$$[G_{i,0} : G_{i,1}] = [G_{i,1} : G_{i,2}] = \cdots = [G_{i,k_i-1} : G_{i,k_i}] = 2.$$

Put $\mathcal{S} = \{(G_{i,j-1}, G_{i,j}) \mid 1 \le i \le s, 1 \le j \le j_i\}$.

$$\bigoplus_{(H,H')\in\mathcal{S}} U(H;G) \oplus V(G)^{\oplus n}$$

is a gap G-module for sufficient large n.

Construction of gap modules VI

For each $j \in J$, put

$$(P_j, H_j) = \begin{cases} (O^2(C_G(C_j))(C_j \cap K), O^2(C_G(C_j))C_j), & H_j \in G_{t-1} \\ (C_j \cap K, C_j), & otherwize \end{cases}$$

$$t_j = \begin{cases} |N_{G_{\{2\}}}(C_j)/C_j|, & H_j \in G_{t-1} \\ s_j = |N_G(C_j)/C_j|, & otherwize \end{cases}.$$

Theorem

If $J = J(2)$, then

$$\sum_{j \in J} t_j^{-1} d_V(P_j, H_j) = 0$$

which implies that $d_V(P_j, H_j) = 0$ for any an $\mathcal{L}(G)$-free nonnegative G-module V.

Construction of gap modules VII

We summarize that

Theorem

Let G be a finite group such that $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$. Let Γ be the set of all representatives of conjugacy classes of 2-power index subgroups L of G with $[G : L] = 2$ or $[L : O^2(G)] = 2$.

1. *If $E^o(L, O^2(G)) = \emptyset$ for some $L \in \Gamma$, then G is not a gap group.*
2. *If $E^o(L, O^2(G)) \neq \emptyset$ for all $L \in \Gamma$, then G is a gap group.*

Corollary

If there is an element x of G such that $G = \langle x \rangle O^2(G)$ and $|\pi(\langle x \rangle)| \geq 3$, then G is a gap group.

Construction of gap modules VIII

S: a noncomplete sporadic group

$G = Aut(S) \cong S \rtimes C_2$.

S		$\lvert C_G(x)\rvert$	$E^o(G,S)$
M_{12}	empty		$2C, 4C, 4D$
HN	empty		$2C, 4D, 4E, 4F, 8C, 8E, 8D, 8F$
J_2	$8C$	2^5	$2C, 4B, 4C, 8B$
J_3	$8C$	2^5	$2B, 4B, 8B$
M^cL	$8C$	2^5	$2B, 4B, 8B$
$O'N$	$8E$	2^5	$2B, 8C, 8D$
Fi_{22}	$16C$	2^5	$2D, 2E, 2F, 4F, 4G, 4H, 4I, 4J, 8E, 8F, 8G, 8H$
Fi'_{24}	$16B$	2^6	$2C, 2D, 4D, 4E, 4F, 4G, 8D, 8E, 8F$
He	$16A, 16B*$	$2^4, 2^4$	$2C, 4D, 8B, 8C*$
M_{22}	$4D, 8B$	$2^6, 2^4$	$2B, 2C, 4C$
Suz	$8G, 16A$	$2^8, 2^4$	$2C, 2D, 4E, 4F, 8D, 8E, 8F, 8H$
HS	$8D, 8E$	$2^6, 2^6$	$2C, 2D, 4D, 4E, 4F$

Construction of gap modules IX

Theorem

The automorphism group of a sporadic group is a gap group.

Lemma (Morimoto-S-Yanagihara, 2000)

If K is a subgroup of G with odd index possessing an $(\mathcal{L}(G) \cap K)$-free positive K-module V, then $\mathrm{Ind}_K^G V$ is a gap G-module.

$$Aut(M_{22}) \overset{77}{>} K \twoheadrightarrow S_6, \quad Aut(Suz) \overset{405405}{>} K' \twoheadrightarrow S_6,$$

$$Aut(HS) \overset{1100}{>} S_8 \times C_2,$$

$$HS \cap (S_8 \times C_2) = S_8$$

Sufficient condition I

Theorem

Let

$$G_{\{2\}} \smallsetminus O^2(G) = \amalg_{x \in \Psi} (x)^{\pm}$$

where $G_{\{2\}}$ *is a Sylow 2-subgroup of* G*. Suppose that* $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$, $[G : O^2(G)] = 2$ *and that there is* $x \in \Psi$ *such that*

$$\Psi \smallsetminus E(G, O^2(G)) \subset \langle x \rangle,$$

that is, for any $y \in \Psi$, $(y) \cap \langle x \rangle = \emptyset$*, there exists an* $\mathcal{L}(G)$*-free nonnegative* G*-module* W_y *such that* $d_{W_y}(P, H) > 0$ *for* $(P, H) \in \mathcal{PH}^2(G)$ *with* $(y) \cap H \neq \emptyset$*. Then*

$$(\mathrm{Ind}_{\langle x \rangle}^{G}(\mathbb{R}[\langle x \rangle] - \mathbb{R}))_{\mathcal{L}(G)} \oplus (V(G) \oplus \bigoplus_{y \in \Psi, y \neq x} W_y)^{\oplus n}$$

is a gap G*-module for a sufficiently large integer* n*.*

Sufficient condition II

Recall that if there are two distinct odd primes r such that $O^r(G) \neq G$, then G is a gap group.
Let consinder a finite group G such that $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$, $O^2(G) \neq G$, and $O^q(G) \neq G$ for a unique odd prime q.

Let S be a complete set of representatives of conjugacy classes of G represented by elements of order 2 which does not lie in $E(G, O^2(G)) \cup O^2(G)$. Fix a Sylow 2-subgroup $G_{\{2\}}$ of G. (We can assume that x belongs to $G_{\{2\}}$ for any $x \in S$ without loss of generality.) Let $S = \{x_1, \ldots, x_r\}$ and s_j denotes the order of $C_{G_{\{2\}}}(x_j)/\langle x_j \rangle$ for $1 \leq j \leq r$.

Sufficient condition III

Theorem

Let G be a finite group such that $O^q(G) \neq G$ for some unique odd prime q, $[G : O^2(G)] = 2$ and $\mathcal{L}(G) \cap \mathcal{P}(G) = \emptyset$. TFAE.

1. *G is a gap group.*
2. *$E(G, O^2(G))$ is not empty.*
3. $\displaystyle\sum_{j=1}^{r} s_j^{-1} \neq 1.$
4. *There are two elements of $G_{\{2\}}$ of order 2 which are conjugate in G but not conjugate in $G_{\{2\}}$.*

Sufficient condition IV

Theorem

Let G be a finite nongap group such that $[G : O^2(G)] = 2$ and $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$. If the abelian group $(G_{\{2\}} \cap O^2(G_{\{2\}}))/[G_{\{2\}}, G_{\{2\}}]$ is generated by xy for involutions x, y of $G_{\{2\}} \smallsetminus O^2(G)$ which are conjugate in G, then $O^2(G)$ is of odd order.

Theorem

Let G be a finite group with $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$ and $O^q(G) \neq G$ for an odd prime q. If $O^2(G)$ is of even order (eg. nonsolvable group), then G is a gap group.

Group having nontrivial center

Proposition

Let G be a finite group with $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$. Suppose that the center $Z(G)$ of G is not a 2-group (also not trivial). If $O^2(G)$ is of even order then G is a gap group.

Remark

Note that if $G/Z(G)$ is gap then so is G. The converse is not true in general: It is not true that G is gap implies that $G/Z(G)$ is gap. For a nonabelian q-group P,

- $G = Q_{4n} \times P$ *is gap if* $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$,
- $G = D_{2n} \times P/Z(P)$ *is not gap.*

Application I

Let $Sm(G)$ be a set, called the Smith set, consisting of all differences $[T_x(\Sigma)] - [T_y(\Sigma)]$ of $RO(G)$ for a smooth G-action of a homotopy sphere Σ with $\Sigma^G = \{x, y\}$.

Theorem

Let G be a finite Oliver group with $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$ and $O^q(G) \neq G$ for an odd prime q. Then

$$RO(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \subseteq Sm(G).$$

Application II

Conjecture

Let K be a finite group with $[K : O^2(K)] = 2$. Suppose that $\mathcal{P}(K) \cap \mathcal{L}(K) = \varnothing$ and $E^o(K, O^2(K)) = \varnothing$. Then it seems that elements of $K \smallsetminus O^2(K)$ of order 2 are conjugate in K.

Theorem

If K is an Oliver group satisfying the property of the above conjecture, then

$$RO(K)^{\mathcal{L}(K)}_{\mathcal{P}(K)} \subseteq Sm(K).$$

Dimension

We might want to know a gap module with smaller dimension as possible. To find a gap module with smallest dimension, we consider the integer linear programming. For a matrix

$$A = \begin{bmatrix} & \vdots & \\ \cdots & d_V(P,H) & \cdots \\ & \vdots & \end{bmatrix},$$

where (P,H) runs over $\mathcal{PH}(G)$ on rows and V runs over $\mathcal{L}(G)$-free irreducible G-modules on columns.

$$\begin{aligned} &\text{minimize} && [\cdots, \dim V, \cdots]x \\ &\text{subject to} && Ax \geq \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, x \geq 0, x \in \mathbb{Z}^{|\mathrm{Irr}(G)|} \end{aligned}$$

Thank you for your attention!