On bi-isovariantly equivalent representations

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Today's talk

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1. Isovariant maps and isovariant Borsuk-Ulam type theorems.

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2. Bi-isovariantly equivalent representations.

(I) Isovariant maps and isovariant Borsuk-Ulam type theorems

Let G be a compact Lie group and X, Y G-spaces. All maps between spaces are assumed to be continuous.

Definition

A map $f : X \to Y$ is called a *G-isovariant* map if f is G-equivariant and preserves the isotropy subgroups, i.e., $G_{f(x)} = G_x$ for all $x \in X$.

Note. If f is a G-map, then $G_x \leq G_{f(x)}$.

The notion of an isovariant map was introduced by Palais in order to classify orbit maps $p : X \to X/G$. After Palais, isovariant maps are used to study a classification problem of G-manifolds, especially in isovariant or stratified surgery theory.

A simple but important fact is that an isovariant map preserves the orbit structures.

Proposition

Let $f : X \to Y$ be a G-map. The following are equivalent.

1. $f : X \to Y$ *is G-isovariant.*
2. $f_{|G(x)} : G(x) \to G(f(x)) \subset Y$ *is bijective for any $x \in X$, where* $G(x) = \{gx \mid g \in G\}$ *is the orbit of x.*

Definition (Representation space)

1. Let $\rho : G \to O(n)$ be a representation homomorphism. Then a G-representation (space) V $(= \mathbb{R}^n)$ is defined by $gx = \rho(g)x$, $x \in V$.
2. Let denote by SV the unit sphere of V, called a representation sphere.

In this talk, we focus on representations or representation spheres, because representations are basic (local) objects in transformation group theory, i.e., a smooth G-action on a manifold M is locally linear.

Problem

How are (topological or algebraic) invariants of G-representations related if there exits an isovariant map between G-representations?

A similar problem can be considered for equivariant maps between representation spheres with G-fixed point free actions.

First we remark the following.

- If there is $x_0 \in SW^G$, then the constant map $c_{x_0} : V \to W$ or $SV \to SW$ is always equivariant. So, representation spheres with G-fixed point free actions are considered in the existence problem of equivariant maps.
- On the other hand, c_{x_0} is not isovariant unless V has trivial action. In isovariant case, it is meaningful to consider the existence problem of isovariant maps between representations.

Isovariant maps between representations or representation spheres

Proposition

The following are equivalent.

1. $\exists$ G-isov. $f : V \to W$.
2. $\exists$ G-isov. $f : V - V^G \to W - W^G$.
3. $\exists$ G-isov. $f : S(V - V^G) \to S(W - W^G)$.

Here $V - V^G$ *is the orthogonal complement of* V^G *as a* G*-subrepresentation in* V.

Corollary

If $V^G = W^G = 0$, *then*

$$\exists \ G\text{-isov. } f : V \to W \iff \exists \ G\text{-isov. } f : SV \to SW$$

Proof. Put $V^{\perp} = V - V^{G}$.

(1) $\Rightarrow$ (2) $\Rightarrow$ (3)

$$\overline{f} : V^{\perp} \xrightarrow{i} V \xrightarrow{f} W \xrightarrow{\mathrm{pr}} W^{\perp}.$$

$$\overline{\overline{f}} : S(V^{\perp}) \xrightarrow{j} V^{\perp} \setminus \{0\} \xrightarrow{\overline{f}} W^{\perp} \setminus \{0\} \xrightarrow{\mathrm{norm.}} S(W^{\perp})$$

(1) $\Leftarrow$ (2) $\Leftarrow$ (3)

$$g : S(V^{\perp}) \to S(W^{\perp})$$

$$\tilde{g} : V^{\perp} \to W^{\perp} \quad \text{radial extension}$$

$$h := \tilde{g} \oplus 0 : V = V^{\perp} \oplus V^{G} \to W^{\perp} \oplus W^{G} = W$$

Borsuk-Ulam type results in isovariant setting

A fundamental topological invariant is dimension. Borsuk-Ulam type theorems give some relations of dimensions. For example, the following is well known.

Theorem (Borsuk-Ulam theorem for free C_p-spheres)

Suppose that C_p acts freely on spheres S^n, S^m, p: prime. If there exists a C_p-map $f : S^n \to S^m$, then $n \leq m$.

Remark

If G acts freely on S^n, S^m, then the Borsuk-Ulam theorem still holds. This is clear if the action is restricted to a subgroup C_p.

Using this theorem, we obtain an isovariant version of the Borsuk-Ulam theorem.

Proposition

Let $G = C_p$, p: prime., or S^1. Let V and W be G-representations. If there exists a G-isovariant map $f : V \to W$ (or $f : SV \to SW$), then

$$\dim(V - V^G) \leq \dim(W - W^G).$$

Proof (Case 1: $G = C_p$)

Set $V^\perp = V - V^{C_p}$ and $W^\perp = W - W^{C_p}$. From f, we can construct a C_p-isovariant map $\overline{\overline{f}} : S(V^\perp) \to S(W^\perp)$ as follows.

$$\overline{f} : V^\perp \xrightarrow{i} V \xrightarrow{f} W \xrightarrow{\mathrm{pr}} W^\perp,$$

$$\overline{\overline{f}} : S(V^\perp) \xrightarrow{j} V^\perp \setminus \{0\} \xrightarrow{\overline{f}} W^\perp \setminus \{0\} \xrightarrow{\mathrm{norm.}} S(W^\perp).$$

Since C_p acts freely on $S(V^\perp)$ and $S(W^\perp)$, we have

$$\dim S(V^\perp) \leq \dim S(W^\perp).$$

Thus

$$\dim(V - V^{C_p}) \leq \dim(W - W^{C_p}).$$

Proof (Case 2: $G = S^1$)

In general a G-representation has finitely many conjugacy classes of isotropy subgroups.
We can take a sufficiently large prime p such that

$$V^{C_p} = V^{S^1} \text{ and } W^{C_p} = W^{S^1}.$$

Restricting the action, we have a C_p-isovariant map $\mathrm{res}_{C_p} f : V \to W$.
By case 1, we have $\dim(V - V^{C_p}) \leq \dim(W - W^{C_p})$ and this implies

$$\dim(V - V^{S^1}) \leq \dim(W - W^{S^1}).$$

More generally, by induction, the following is essentially proved by Wasserman.

Theorem (Isovariant Borsuk-Ulam theorem)

Let G be a solvable compact Lie group. If there exists a G-isovariant map $f : V \to W$ (or $f : SV \to SW$), then

$$\dim(V - V^G) \leq \dim(W - W^G).$$

Remark

A solvable compact Lie group G is characterized as the existence of a composition series

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_r = G$$

such that G_i/G_{i-1} is a cyclic group of prime order or S^1.

Comments on equivariant case

Borsuk-Ulam type theorems in equivariant case have been studied by many people. For example the following result is deduced from their studies.

Theorem

Let $G = C_p^k$ or T^k. Suppose that G acts G-fixed point freely on S^n and S^m. If there exits a G-map $f : S^n \to S^m$, then $n \leq m$.

On the other hand, Waner gave a counterexample for a cyclic group not of prime power order. Furthermore, Bartsch proved that, for finite group G, a Borsuk-Ulam type theorem holds (in a weak sense) iff G is of prime power order.

Example — Waner's counterexample

Let $G = C_n$ be a cyclic group of order n and let c be a generator of C_n. Let U_k ($= \mathbb{C}$) denote the (unitary) irreducible representation of C_n on which c acts by $c \cdot z = \xi_n^k z$, where $0 \leq k \leq n-1$ and $z \in U_k$ and $\xi_n = \exp(2\pi\sqrt{-1}/n)$.

Proposition

Assume that n is divided by distinct primes p and q. Then for any positive integer k, there exists a C_n-map

$$f : S(U_1^k \oplus U_p \oplus U_q) \to S(U_p \oplus U_q).$$

Remark

By the isovariant Borsuk-Ulam theorem, the above C_n-map f is never isovariant.

What about non-solvable groups?

Definition

A compact Lie group G is called a Borsuk-Ulam group (BUG) if the isovariant Borsuk-Ulam theorem holds for G-representations.

Wasserman conjectures that all finite groups are BUGs. In fact, a counterexample is not known at present.

But there are some partial results.

Remark

If we permit a non-linear action on a Euclidean space or a sphere, there is a counterexample when G is non-solvable.

An odd order group is solvable by the Feit-Thompson theorem and so it is a BUG. Using other deep results in finite group theory, we can find new families of BUGs which include non-solvable groups.

Let G_p denote a p-Sylow subgroup of a finite group G.

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Let G_p denote a p-Sylow subgroup of a finite group G.

Theorem (N–Ushitaki)

A finite group G satisfying one of the following conditions is a BUG.

1. G_2 *is cyclic. (In this case, G is solvable.)*
2. $G_2 = D_{2^s}$*: dihedral group of order* 2^s, $s \geq 2$, *where* $D_4 = C_2 \times C_2$, *e.g.* $PSL(2, p^r)$.
3. $G_2 = Q_{2^s}$*: generalized quaternion group of order* 2^s, $s \geq 3$, *e.g.* $SL(2, p^r)$.
4. G_2 *is abelian and* G_p *is cyclic for each odd prime* p, *e.g. Janko group* J_1 *whose order is* $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$.

Compact Lie group case

Unfortunately, (non-trivial) connected BUGs are not known except a torus T^n.

However, a weaker version of the isovariant Borsuk-Ulam theorem holds.

Theorem (Weak isovariant Borsuk-Ulam theorem)

For an arbitrary compact Lie group G (not necessarily connected), there exists a constant $0 < c \leq 1$ such that if there exists a G-isovariant map $f : V \to W$, then the inequality

$$c \dim(V - V^G) \leq \dim(W - W^G)$$

holds.

Definition

Let c_G be the maximum of a constant c as in the above theorem.

Clearly $c_G = 1$ if and only if G is a BUG. Hence if G is solvable, then $c_G = 1$.

Example

If $G = SO(3)$ or $SU(2)$, then $\frac{4}{5} \leq c_G \leq 1$.

(II) Bi-isovariantly equivalent representations

The next topic is about bi-isovariantly equivalent representations. We would like to consider relation to the dimension function of a representation.

Definition

G-representations V and W are bi-isovariantly equivalent if there exist G-isovariant maps $f : V \to W$ and $g : W \to V$. In the case we write $V \rightleftharpoons_G W$.

Definition

Let $S(G)$ be the set of closed subgroups of G. For a G-representation V, the dimension function $\mathrm{Dim}\, V : S(G) \to \mathbb{Z}$ is defined by

$$(\mathrm{Dim}\, V)(H) = \dim V^H$$

for $H \in S(G)$.

Theorem (N-Ushitaki)

Let G be an arbitrary compact Lie group. If $V \rightleftharpoons_G W$, then

$$Dim(V - V^G) = Dim(W - W^G).$$

Recall that $V - V^G$ is the orthogonal complement of V^G in V.

Theorem (N-Ushitaki)

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Proof (outline)

Case 1: G a finite group.

Applying the isovariant Borsuk-Ulam theorem to a C-isovariant map $res_C f : V \to W$ and $res_C g : W \to V$, we have

$$\dim V - \dim V^C = \dim W - \dim W^C$$

for all cyclic subgroups C of G.

Proof (continued)

For any subgroup H, set

$$d(H) = |H|(\dim W - \dim W^H - \dim V + \dim V^H).$$

Using character theory, we have

$$d(H) = \sum_{C \in \mathrm{Cy}(H)} \left(\sum_{C \leq D \in \mathrm{Cy}(H)} \mu(C, D) \right) d(C),$$

where μ is the Möbius function on $Cy(H)$ the set of cyclic subgroups of H. Since $d(C) = 0$, we see that $d(H) = 0$ for any $H \leq G$.

Proof (continued)

In particular,

$$\dim W - \dim V = \dim W^H - \dim V^H = \dim W^G - \dim V^G.$$

So we have

$$\dim(V - V^G)^H = \dim(W - W^G)^H.$$

Proof (continued)

Case 2: G a compact Lie group.

A theorem of Traczyk below says that $V - V^{G_0} \cong W - W^{G_0}$ as G-representations. Hence their dimension functions coincide.
On the other hand V^{G_0} and W^{G_0} are regarded as G/G_0-representations and $V^{G_0} \rightleftharpoons_{G/G_0} W^{G_0}$. By case 1, $V^{G_0} - V^G$ and $W^{G_0} - W^G$ have the same dimension function. Thus we see

$$\mathrm{Dim}\,(V - V^G) = \mathrm{Dim}\,(W - W^G).$$

Theorem (Traczyk)

If $\dim V^C = \dim W^C$ *for every (finite) cyclic subgroup* C*, then* $V - V^{G_0} \cong W - W^{G_0}$*, where* G_0 *is the identity component.*

Corollary

Waner studied the existence problem of equivariant maps from a G-representation sphere SV to its subrepresentation sphere SW (where $V^G = 0$ and G is finite) :

$$f : SV \to SW \subset SV.$$

In this case, he gave a necessary and sufficient condition for the existence of an equivariant map in terms of the Burnside ring.

Using this result, one can find a counterexample of Borsuk-Ulam theorem for C_{pq} as mentioned before.

Let us consider an isovariant version of Waner's setting. We can see the following.

Corollary

Let G be a compact Lie group and assume that $V^G = 0$ (for simplicity).
If there is an isovariant map $f : V \to U \subset V$ (or $f : SV \to SU \subset SV$), then $U = V$, i.e, there is no isovariant map to a proper subrepresentation.

Proof

Since the inclusion $i : U \to V$ is isovariant, V and U are bi-isovariantly equivalent. Hence we have $\mathrm{Dim}\, V = \mathrm{Dim}\, U$ and in particular $\dim V = \dim U$. Hence $V = U$.

Does the converse hold?

Let us consider the converse of the above theorem, i.e., when $\mathrm{Dim}\,(V - V^G) = \mathrm{Dim}\,(W - W^G)$, do there exist isovariant maps bi-directionally? In abelian case, we can see the following.

Theorem

If G is an abelian compact Lie group, then the converse holds. Thus

$$V \rightleftharpoons_G W \iff Dim\,(V - V^G) = Dim\,(W - W^G).$$

Proof (Outline of $\Leftarrow$)

- Decompose $V = \oplus_{K \leq G} V(K)$ and $W = \oplus_{K \leq G} W(K)$, where $V(K)$ [resp. $W(K)$] is the direct sum of irreducible representations in V [resp. W] with kernel K.
- Show that if $K \neq G$, then $\operatorname{Dim} V(K) = \operatorname{Dim} W(K)$ and that G/K is finite cyclic or S^1 if $V(K) \neq 0$ or $W(K) \neq 0$.
- So the problem is reduced to the cyclic or S^1 case, but in this case, one can easily construct isovariant maps.

On the other hand, in non-abelian case, the converse does not necessarily hold. We give a simple example.

Let $D_{2n} = \langle a, b \mid a^n = b^2 = 1, bab^{-1} = a^{-1} \rangle$, $n \geq 3$, and set $C_n = \langle a \rangle$, $D_2^{(i)} = \langle a^i b \rangle \cong C_2$ for $0 \leq i < n$.

Consider the (real) 2-dimensional D_{2n}-representation $V_k = \mathbb{C}$ defined by $az = \xi_n^k z$, $\xi_n = \exp \frac{2\pi\sqrt{-1}}{n}$ and $bz = \bar{z}$ for $z \in V_k$. Suppose that k is a positive integer less than $n/2$ and prime to n.

Then C_n acts freely on $V_k \smallsetminus \{0\}$ and all V_k have the same dimension function; indeed,

$$\dim V_k^H = \begin{cases} 2 & H = 1 \\ 1 & H = D_2^{(i)} \\ 0 & \textit{otherwise}. \end{cases}$$

We can see the following.

Proposition

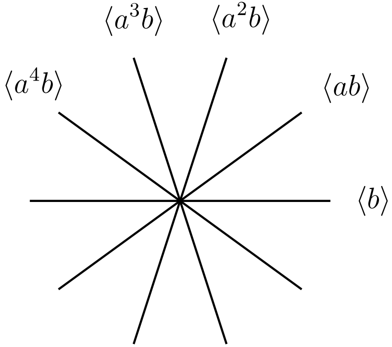
Suppose that $n \geq 5$ and $n \neq 6$ and $0 < k, l < n/2$ are integers prime to n. Then if $k \neq l$, then there does not exist a D_{2n}-isovariant map from V_k to V_l.

Remark

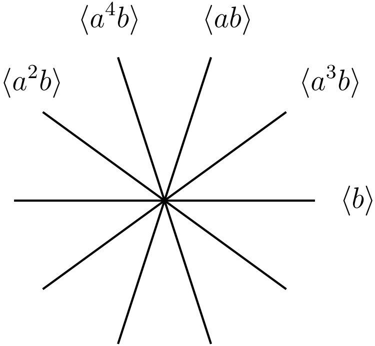
There exists a D_{2n}-map $f : SV_k \to SV_l$.

Sketch of proof

We illustrate it when $n = 5$, $k = 1$ and $l = 2$.



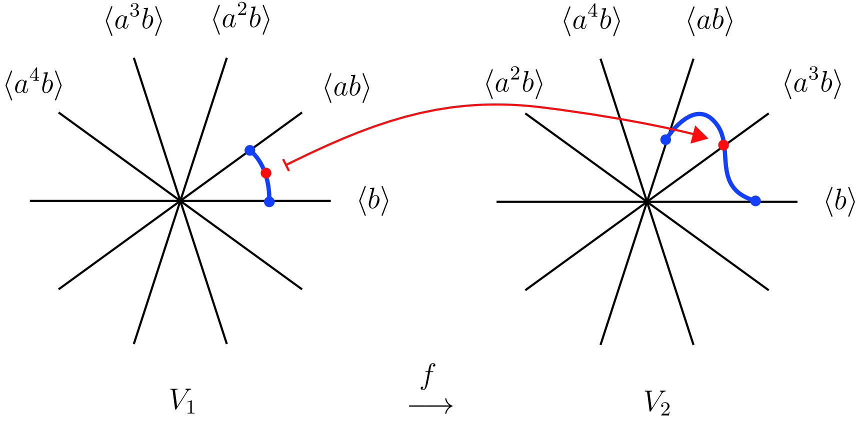
V_1



V_2

Sketch of proof

We illustrate it when $n = 5$, $k = 1$ and $l = 2$.



Bi-isovariant rigidity

By a further argument, we have

Proposition

Let V and W be 2-dimensional D_{2n}-representations, $n \geq 3$. Then

$$V \rightleftharpoons_{D_{2n}} W \iff V - V^{D_{2n}} \cong W - W^{D_{2n}}$$

We call this property the bi-isovariant rigidity.

Proposition

1. *Let G be a compact Lie group and G_0 the identity component.*

$$V \rightleftharpoons_G W \iff \begin{cases} (a)\ V^{G_0} \rightleftharpoons_{G/G_0} W^{G_0}, \\ (b)\ V - V^{G_0} \cong W - W^{G_0} \text{ as } G\text{-reps.} \end{cases}$$

2. *In particular if G is connected, then the bi-isovariant rigidity holds:*

$$V \rightleftharpoons_G W \iff V - V^G \cong W - W^G.$$

Proof

(1) ($\Rightarrow$) By G_0-fixing, we have (a). By our theorem $\mathrm{Dim}\,(V - V^G) = \mathrm{Dim}\,(W - W^G)$ and $\mathrm{Dim}\,(V^{G_0} - V^G) = \mathrm{Dim}\,(W^{G_0} - W^G)$. Hence we have $\mathrm{Dim}\,(V - V^{G_0}) = \mathrm{Dim}\,(W - W^{G_0})$. By Traczyk's theorem, we have (b).
($\Leftarrow$) Straightforward.

Corollary

If G/G_0 is abelian, then

$$V \rightleftharpoons_G W \iff \begin{cases} (a)\ Dim(V^{G_0} - V^G) = Dim(W^{G_0} - W^G) \\ (b)\ V - V^{G_0} \cong W - W^{G_0} \text{ as } G\text{-reps.} \end{cases}$$

Other examples of bi-isovariant rigidity

Let G be a finite group. As a result of representation theory, G-representations with the same dimension function are characterized as follows.

Proposition

Let $V = V_1 \oplus \cdots \oplus V_r$ and $W = W_1 \oplus \cdots \oplus W_s$ be irreducible decompositions of G-representations V and W respectively. Then $\mathrm{Dim}\, V = \mathrm{Dim}\, W$ if and only if $r = s$ and every irreducible summand V_i is Galois conjugate to $W_{\sigma(i)}$ for some permutation σ of $\{1, 2, \ldots, r\}$.

(Rem: This result is found in papers of Lee-Wasserman and tom Dieck.)

Definition

Let n be the exponent of G and ξ_n a primitive n^{th} root of unity. We say that V and W are Galois conjugate if there exists a field automorphism ψ on the cyclotomic field $\mathbb{Q}(\xi_n)$ such that $\psi(\chi_V(g)) = \chi_W(g)$ for every $g \in G$, where χ_V denotes the character of a G-representation V.

Then the Galois group $\Gamma := Gal(\mathbb{Q}(\xi_n)/\mathbb{Q}) \cong (\mathbb{Z}/n)^*$ acts on the set $\mathrm{Irr}(G, \mathbb{R})$ of real irreducible G-representations.

Remark

Since complex conjugate c in Γ, which corresponds to $-1 \in (\mathbb{Z}/n)^*$, acts trivially on $\mathrm{Irr}(G, \mathbb{R})$. Hence $\Gamma/\langle c \rangle \cong (\mathbb{Z}/n)^*/\pm 1$ acts on $\mathrm{Irr}(G, \mathbb{R})$.

Theorem

Let G be a compact Lie group. Suppose that $\Gamma/\langle c\rangle$ acts trivially on $Irr(G/G_0, \mathbb{R})$. Then bi-isovariant rigidity holds for G-representations.

Proof

By previous propositions, if $V \rightleftharpoons_G W$, then we have $V - V^{G_0} \cong W - W^{G_0}$ and $V^{G_0} - V^G \cong W^{G_0} - W^G$. Hence $V - V^G \cong W - W^G$. The converse is trivial.

Corollary

If the characters of V^{G_0} and W^{G_0} are integer-valued, then $V \rightleftharpoons_G W \iff V - V^G \cong W - W^G$.

Example

If G/G_0 satisfies one of the following, then bi-isovariant rigidity holds for G-representations.

1. $G/G_0 = S_n$ the symmetric group. (Indeed, any S_n-representation is rational.) More generally, if G/G_0 is isomorphic to the Weyl group of some compact Lie group, then bi-isovariant rigidity holds.
2. $G/G_0 = C_2^k \times C_3^l$, $C_2^k \times C_4^l$, Q_8, etc. (Indeed, $\Gamma/\langle c \rangle$ itself is trivial.)