Topological Equivalence Relations on Representation Spaces

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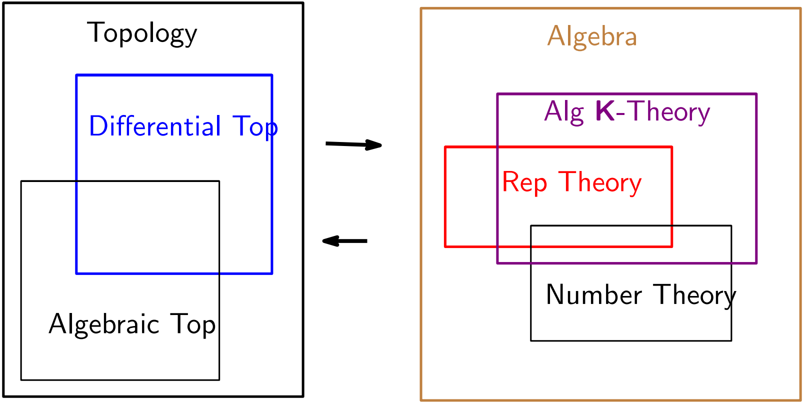
Overview



G-rep spaces

Families of Finite Groups

$\mathbf{p},\ \mathbf{q} = \mathbf{1}$ or prime

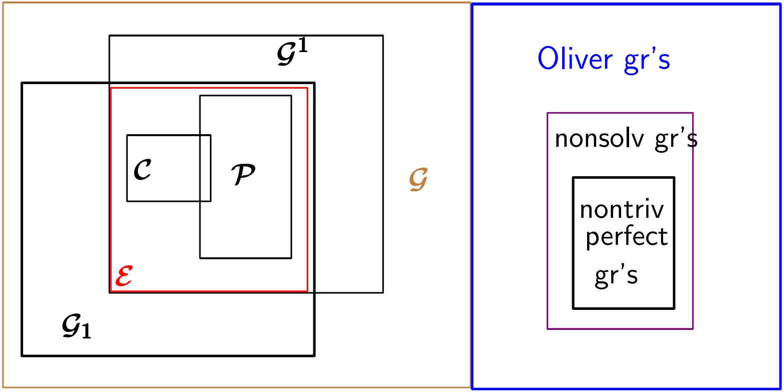
- $\mathcal{G}_{\mathbf{p}}^{\mathbf{q}} = \{\mathbf{G} \mid \exists\, \mathbf{P} \trianglelefteq \mathbf{H} \trianglelefteq \mathbf{G},\ |\mathbf{P}| = \mathbf{p}^{\mathbf{a}},\ \mathbf{H}/\mathbf{P}$ cyclic, $|\mathbf{G}/\mathbf{H}| = \mathbf{q}^{\mathbf{b}}\}$
- $\mathcal{G}_{\mathbf{1}} = \bigcup_{\mathbf{q}} \mathcal{G}_{\mathbf{1}}^{\mathbf{q}}$, hyperelementary group
- $\mathcal{G}^{\mathbf{1}} = \bigcup_{\mathbf{p}} \mathcal{G}_{\mathbf{p}}^{\mathbf{1}}$, hypoelementary group, mod $\mathcal{P}$ cyclic group
- $\mathcal{G} = \bigcup_{\mathbf{p},\mathbf{q}} \mathcal{G}_{\mathbf{p}}^{\mathbf{q}}$ mod $\mathcal{P}$ hyperelementary group

Notation $\mathcal{G}_{\mathbf{1}}(\mathbf{G}) = \{\mathbf{H} \mid \mathbf{H} \leq \mathbf{G},\ \mathbf{H} \in \mathcal{G}_{\mathbf{1}}\}$

$\mathcal{G}^{\mathbf{1}}(\mathbf{G}) = \{\mathbf{H} \mid \mathbf{H} \leq \mathbf{G},\ \mathbf{H} \in \mathcal{G}^{\mathbf{1}}\}$

Def $\mathbf{G}$ is Oliver group $\overset{\text{def}}{\iff} \mathbf{G} \notin \mathcal{G}$ (e.g. $\mathbf{C}_{\mathbf{p_1 p_2 p_3}} \times \mathbf{C}_{\mathbf{p_1 p_2 p_3}} \notin \mathcal{G}$)

Relation of Families of Finite Groups



$\mathcal{C}$: Cyclic groups

$\mathcal{P}$: Groups of prime power order

$\mathcal{E}$: Elementary groups

Equiv Relation $\sim_r$ and Set $\mathrm{RO}_r(G)$

Let $\sim_r$ be equiv. relation on family $\mathcal{M}$ of real **G**-modules

Assume $V \cong W$ ($V, W \in \mathcal{M}$) $\Longrightarrow V \sim_r W$

Def (Associated Set with $\sim_r$)

$$\mathrm{RO}_r(G) = \{[V] - [W] \in \mathrm{RO}(G) \mid V, W \in \mathcal{M}, V \sim_r W\}$$

Standard Relation

Def $\sim_r$ is called standard if

(1) $V_1 \sim_r W_1,\ V_2 \sim_r W_2 \Longrightarrow V_1 \oplus V_2 \sim_r W_1 \oplus W_2$

(2) $H < K,\ V \sim_r W \Longrightarrow \mathrm{ind}_H^K V \sim_r \mathrm{ind}_H^K W$

(3) $\varphi : H \to K$ homo, $V \sim_r W \Longrightarrow \varphi^* V \sim_r \varphi^* W$

Prop If $\sim_r$ is standard then $\mathrm{RO}_r(-)$ is Green submodule of $\mathrm{RO}(-)$ over Burnside ring functor $\Omega(-)$ and

$$\mathrm{Ind} : \bigoplus_{(H) \subset \mathcal{G}_1(G)} \mathrm{RO}_r(H) \longrightarrow \mathrm{RO}_r(G) \text{ is surjective}$$

$$\mathrm{Res} : \mathrm{RO}_r(G) \longrightarrow \bigoplus_{(H) \subset \mathcal{G}_1(G)} \mathrm{RO}_r(H) \text{ is injective}$$

Exotic Relation

Def $\sim_{\mathsf{r}}$ is called exotic if

(1) $\mathbf{ind}_{\mathsf{H}}^{\mathsf{K}}(\mathbf{RO}_{\mathsf{r}}(\mathbf{H})) \subset \mathbf{RO}_{\mathsf{r}}(\mathbf{K})$ for $\forall$ $(\mathbf{H}, \mathbf{K}) : \mathbf{H} < \mathbf{K}$,

(2) $\varphi^*(\mathbf{RO}_{\mathsf{r}}(\mathbf{K})) = \mathbf{RO}_{\mathsf{r}}(\mathbf{H})$ for $\forall$ $\varphi : \mathbf{H} \to \mathbf{K}$ iso, and

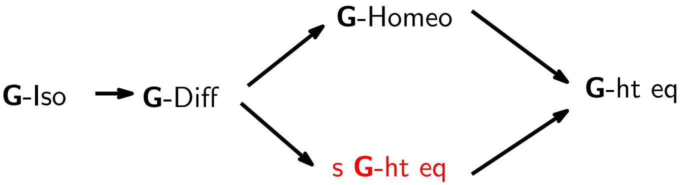
(3) $\mathbf{res}_{\mathsf{H}}^{\mathsf{K}}(\mathbf{RO}_{\mathsf{r}}(\mathbf{K})) \not\subset \mathbf{RO}_{\mathsf{r}}(\mathbf{H})$ for some $\mathbf{H} < \mathbf{K}$

In this case, Dress' hyperelementary induction theory is not helpful.

It would be interesting to study $\mathbf{RO}_{\mathsf{r}}(\mathbf{G})$ for Oliver groups $\mathbf{G}$

$$(\mathbf{G} \text{ is Oliver group} \overset{\text{def}}{\Longleftrightarrow} \mathbf{G} \notin \mathcal{G})$$

Relation between G-maps



s **G**-ht eq stands for simple **G**-homotopy equivalence

Standard Relations

Example Equiv. relations $\mathbf{V} \sim_{\mathbf{r}} \mathbf{W}$

1. $\mathbf{V} \sim_{\mathbf{d}} \mathbf{W}$: $\mathbf{S(V)} \cong_{\mathbf{G}\text{-diff}} \mathbf{S(W)}$
2. $\mathbf{V} \sim_{\mathbf{t}} \mathbf{W}$: $\mathbf{V} \cong_{\mathbf{G}\text{-homeo}} \mathbf{W}$

   $\sim_{\mathbf{t}}$ topological similar (Cappell-Shaneson)
3. $\mathbf{V} \sim_{\mathbf{s.h}} \mathbf{W}$: $\mathbf{S(V)} \sim_{\text{s.}\mathbf{G}\text{-ht}} \mathbf{S(W)}$ (simply **G**-homotopy equiv)
4. $\mathbf{V} \sim_{\mathbf{h}} \mathbf{W}$: $\mathbf{S(V)} \sim_{\mathbf{G}\text{-ht}} \mathbf{S(W)}$ (**G**-homotopy equiv)

   $\sim_{\mathbf{h}}$ homotopy equivalent (tom Dieck)
5. $\mathbf{V} \sim_{\mathbf{dim}} \mathbf{W}$: $\mathbf{\dim V^H = \dim W^H}$ for $\forall\ \mathbf{H} \leq \mathbf{G}$

Associated Modules

Example Submodules $\mathrm{RO}_{\mathbf{r}}(\mathbf{G}) \subset \mathrm{RO}(\mathbf{G})$

1. $\mathrm{RO}_{\mathbf{d}}(\mathbf{G}) = \{[\mathbf{V}] - [\mathbf{W}] \mid \mathbf{S}(\mathbf{V}) \cong_{\mathbf{G}\text{-diff}} \mathbf{S}(\mathbf{W})\}$
2. $\mathrm{RO}_{\mathbf{t}}(\mathbf{G}) = \{[\mathbf{V}] - [\mathbf{W}] \mid \mathbf{V} \cong_{\mathbf{G}\text{-homeo}} \mathbf{W}\}$ (C-S)
3. $\mathrm{RO}_{\mathbf{s.h}}(\mathbf{G}) = \{[\mathbf{V}] - [\mathbf{W}] \mid \mathbf{S}(\mathbf{V}) \sim_{\text{s.}\mathbf{G}\text{-ht}} \mathbf{S}(\mathbf{W})\}$
4. $\mathrm{RO}_{\mathbf{h}}(\mathbf{G}) = \{[\mathbf{V}] - [\mathbf{W}] \mid \mathbf{S}(\mathbf{V}) \sim_{\mathbf{G}\text{-ht}} \mathbf{S}(\mathbf{W})\}$ (tom Dieck)
5. $\mathrm{RO}_{\mathbf{dim}}(\mathbf{G}) = \{[\mathbf{V}] - [\mathbf{W}] \mid \dim \mathbf{V}^{\mathbf{H}} = \dim \mathbf{W}^{\mathbf{H}},\ \forall\ \mathbf{H} \leq \mathbf{G}\}$

Classification of Rep's 1

$\mathbf{G}$ finite group. $\mathbf{V}$, $\mathbf{W}$ real $\mathbf{G}$-rep. spaces (finite dim)

Fact

1. *(Obvious)* $\mathbf{V} \cong_{\mathbf{G}\text{-}\mathit{diff}} \mathbf{W} \iff \mathbf{V} \cong \mathbf{W}$

   $(\because\ \mathbf{f} : \mathbf{V} \to \mathbf{W}\ \mathit{diff} \Rightarrow \mathbf{V} \cong \mathbf{T_0(V)} \underset{\cong}{\overset{\mathbf{df}}{\to}} \mathbf{T_{f(0)}(W)} \cong \mathbf{W})$

2. *(Franz, de Rham)* $\mathbf{S(V)} \cong_{\mathbf{G}\text{-}\mathit{diff}} \mathbf{S(W)} \iff \mathbf{V} \cong \mathbf{W}$

   $\mathbf{RO_d(G) = 0}$

3. *(Illman)* $\mathbf{S(V)} \sim_{s\ \mathbf{G}\text{-}\mathit{ht}} \mathbf{S(W)} \iff \mathbf{V} \cong \mathbf{W}$

   $\mathbf{RO_{s.h}(G) = 0}$

4. *(Hsiang-Pardon, Madesen-Rothenberg)* $\mathbf{G}$ *odd order:*

   $\mathbf{V} \cong_{\mathbf{G}\text{-}\mathit{homeo}} \mathbf{W} \iff \mathbf{V} \cong \mathbf{W}$

Remark (Cappell-Shaneson) Case $\mathbf{G} = \mathbf{C_{4q}}$:

$\exists$ $\mathbf{V}$, $\mathbf{W}$ such that $\mathbf{V} \cong_{\mathbf{G}\text{-homeo}} \mathbf{W}$ but $\mathbf{V} \ncong \mathbf{W}$

Classification of Rep's 2

$$\mathrm{RO}_t(G) = \{[V] - [W] \in \mathrm{RO}(G) \mid V \cong_{G\text{-homeo}} W\}$$

$$TO(G) = \mathrm{RO}(G)/\mathrm{RO}_t(G)$$

Fact (Cappell-Shaneson-Steinberger-West)

$G = C_{4q} = \langle t \mid t^{4q} = e\rangle$ *with* q *odd. Then*

$$\mathrm{TO}(G) = \mathrm{RO}(G/C_2) \oplus A \oplus B, \text{ where}$$

$$A = \langle t^i \mid i \text{ odd}, 1 \leq i \leq q\rangle_{\mathbb{Z}}$$

$$B = \langle t^d - t^{d+2q} \mid d|q\ ,\ d \neq q\rangle_{\mathbb{Z}_2}$$

Relations $\sim_h$ and $\sim_{dim}$

V, W real G-modules.

- $V \sim_h W \overset{\text{def}}{\Longleftrightarrow} S(V) \sim_{G\text{-ht}} S(W)$

$$\mathrm{RO}_h(G) = \{[V] - [W] \in \mathrm{RO}(G) \mid V \sim_h W\}$$

$$\mathrm{JO}(G) = \mathrm{RO}(G)/\mathrm{RO}_h(G)$$

- $V \sim_{dim} W \overset{\text{def}}{\Longleftrightarrow} \dim V^H = \dim W^H$ for $\forall\, H \le G$

$$V \sim_h W \Longrightarrow V \sim_{dim} W$$

$$\mathrm{RO}_{dim}(G) = \{[V] - [W] \in \mathrm{RO}(G) \mid V \sim_{dim} W\}$$

$$\supset \mathrm{RO}_h(G)$$

Fact $\mathrm{JO}(G) \cong \mathrm{RO}(G)/\mathrm{RO}_{dim}(G) \oplus \mathrm{RO}_{dim}(G)/\mathrm{RO}_h(G)$

Galois group Γ

- $\Gamma = \mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\zeta_n))$, $n = |G|$, Galois group
- $\mathbb{Z}[\Gamma]$ acts on $\mathrm{RO}(G)$

$$\mathfrak{I} = \mathrm{Ker}[\mathbb{Z}[\Gamma] \to \mathbb{Z}] \quad \text{Augmentation Ideal}$$

Fact (G.N. Lie-Wasserman) $\mathrm{RO}_{\mathrm{dim}}(G) = \mathfrak{I}\mathrm{RO}(G)$

Fact (tom Dieck)

1. For arbitrary G, $\mathfrak{I}^2\mathrm{RO}(G) \subset \mathrm{RO}_h(G)$

$$([V] - [W] \in \mathfrak{I}^2\mathrm{RO}(G) \Longrightarrow V \oplus U \sim_h W \oplus U)$$

2. For G abelian or of prime power order, $\mathfrak{I}^2\mathrm{RO}(G) = \mathrm{RO}_h(G)$

Reidemeister Torsion

$$C_n = \langle t \mid t^n = e \rangle, \quad \mathbb{Q}[C_n] = N \oplus (\Sigma),$$

$$N = \mathrm{Ker}[\mathbb{Q}[C_n] \to \mathbb{Q}],$$

$$\Sigma = e + t + t^2 + \cdots + t^{n-1},$$

$$\zeta_n = \exp(2\pi i/n),$$

$\mathbb{C}(r)$ 1-dim complex C_n-module; $(t, z) \longmapsto \zeta_n^r z.$

$V = \mathbb{C}(r_1) \oplus \mathbb{C}(r_2) \oplus \cdots \oplus \mathbb{C}(r_m)$ complex C_n-module

Def (J. Milnor) Reidemeister torsion $\Delta(V)$ in N (or $\mathbb{Q}[C_n]/(\Sigma)$)

$$\Delta(V) = (t^{r_1} - 1)(t^{r_2} - 1) \cdots (t^{r_m} - 1)$$

Franz Independence Lemma

Fact (Franz)

For $\phi(\mathbf{n})/2$ *units* $\mathbf{t^r - 1 \in U(N)}$, *where* $\mathbf{1 \leq r < n/2}$ *with* $\mathbf{(r, n) = 1}$, *there can be no (non-trivial) relation of the form*

$$\prod_{\mathbf{r}} \mathbf{(t^r - 1)^{a_r} = \pm u \quad (u \in C_n).}$$

Franz, de Rham Theorem

Fact (Franz, de Rham)

$\mathbf{V} = \mathbb{C}(\mathbf{r_1}) \oplus \cdots \oplus \mathbb{C}(\mathbf{r_m})$, $\mathbf{W} = \mathbb{C}(\mathbf{s_1}) \oplus \cdots \oplus \mathbb{C}(\mathbf{s_p})$

free complex $\mathbf{C_n}$*-modules*

1. $\mathbf{S(V)} \sim_{\mathbf{C_n}\text{-}ht\ eq} \mathbf{S(W)} \iff$

   $\mathbf{m} = \mathbf{p}$ *and* $\prod \mathbf{r_k} \equiv \pm \prod \mathbf{s_j} \mod \mathbf{n}$

2. $\mathbf{S(V)} \sim_{s\ \mathbf{C_n}\text{-}ht\ eq} \mathbf{S(W)}$

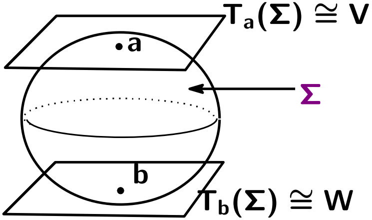
   $\iff \mathbf{\Delta(V)} = \pm \mathbf{u}\mathbf{\Delta(W)}$ *for some* $\mathbf{u} \in \mathbf{C_n}$

   $\stackrel{\text{Franz}}{\iff} \mathbf{V} \cong \mathbf{W}$

Definition of Smith Equivalence

$V \sim_{\mathrm{Sm}} W$ (Smith equivalent) $\overset{\text{def}}{\iff}$ $\exists$ ht sphere X s.t.

$$X^G = \{a, b\},\ T_a(X) \cong V, \text{ and } T_b(X) \cong W$$



- Such X is called Smith sphere for V and W

Remark $\sim_{\mathrm{Sm}}$ is equiv relation on $\{V \mid V^G = 0\}$

X Smith sphere $\not\Longrightarrow$ $\mathrm{res}^G_H X$ Smith sphere

Necessary Condition

Thm (Sanchez) Suppose $|\mathbf{G}|$ is power of odd prime

Then

$$\mathbf{V} \sim_{\mathrm{Sm}} \mathbf{W} \Longrightarrow \mathbf{V} \cong \mathbf{W}$$

Cor $\mathbf{G}$ finite group, $\mathbf{p}$ odd prime.

Then

$$\mathbf{V} \sim_{\mathrm{Sm}} \mathbf{W} \Longrightarrow \mathrm{res}^{\mathbf{G}}_{\mathbf{G_p}} \mathbf{V} \cong \mathrm{res}^{\mathbf{G}}_{\mathbf{G_p}} \mathbf{W}$$

G-Signature

$\mathbf{X}$ conn. ori. closed mfd., $\mathbf{dim = 2k}$ (even), with smooth $\mathbf{G}$-action

Suppose each $\mathbf{g} \in \mathbf{G}$ preserves the orient. of $\mathbf{X}$

Bilinear form $\mathbf{H^k(X;\mathbb{R}) \times H^k(X;\mathbb{R}) \longrightarrow \mathbb{R}}$

$$\mathbf{(a,b) \longmapsto (a \cup b)[X]}$$

This form is $\mathbf{(-1)^k}$-symmetric, $\mathbf{G}$-invariant.

$$\mathbf{Sign(G,X)} = \begin{cases} \rho_+ - \rho_- & \mathbf{(k\ even)} \quad \in \mathbf{RO(G)} \\ \rho - \rho^* & \mathbf{(k\ odd)} \quad \in \mathbf{R(G)} \end{cases}$$

For $\mathbf{g} \in \mathbf{G}$, $\mathbf{Sign(g,X)}$ stands for $\mathbf{Sign(G,X)(g)} \in \mathbb{C}$

Algebraic Number $\nu(-)$

$$C_n = \langle t \mid t^n = e \rangle$$

$\mathbb{C}(r)$ complex C_n-module (forgetting the action, $\mathbb{C}(r) = \mathbb{C}$)

$$C_n \times \mathbb{C}(r) \to \mathbb{C}(r);\ (t, z) \longmapsto \zeta_n^r z$$

$$\text{where } \zeta_n = \exp\left(\frac{2r\pi i}{n}\right)$$

$U \cong \mathbb{C}(r_1) \oplus \cdots \oplus \mathbb{C}(r_k)$ (as ori. $\mathbb{R}$ C_n-modules) s.t. $U^{C_n} = 0$

$$\nu(U) \overset{\text{def}}{=} \prod_j \frac{\zeta_n^{-r_j} - \zeta_n^{r_j}}{(1 - \zeta_n^{-r_j})(1 - \zeta_n^{r_j})}$$

Atiyah-Bott's Theorem

Fact (Atiyah-Bott)

$\mathbf{X}$ *conn. ori. closed mfd.*, $\mathbf{dim = 2k}$ *(even), with smooth* $\mathbf{G}$*-action*

For $\mathbf{g} \in \mathbf{G}$ *s.t.* $|\mathbf{X}^{\mathbf{g}}| < \infty$, *(reg* $\mathbf{ord(g)}$ *as* $\mathbf{n}$ *and* $\mathbf{g}$ *as* $\mathbf{t}$ *above),*

$$\mathbf{Sign(g, X)} \;=\; \sum_{\mathbf{x} \in \mathbf{X}^{\mathbf{g}}} \nu(\mathbf{T_x(X)})$$

Remark If $\mathbf{H^k(X; \mathbb{R}) = 0}$ then $\mathbf{Sign(g, X) = 0}$

Franz-Bass Independence Lemma

Fact (Franz-Bass)

Suppose $\mathbf{b_s}$ *(*$\mathbf{s} \in \mathbb{Z}_\mathbf{n}$*) are integers such that* $\mathbf{b_{-s}} = \mathbf{b_s}$ *and such that for each* $\mathbf{n}$*-th root of unity* ξ*,*

$$\prod_{\mathbf{s}\in\mathbb{Z}_\mathbf{n}} \mathbf{e}(\xi^\mathbf{s})^{\mathbf{b_s}} = \mathbf{1}, \ \textit{where} \ \ \mathbf{e}(\xi^\mathbf{s}) = \begin{cases} \mathbf{1} - \xi^\mathbf{s}, & \xi^\mathbf{s} \neq \mathbf{1} \\ \mathbf{1}, & \xi^\mathbf{s} = \mathbf{1}. \end{cases}$$

Then $\mathbf{b_s} = \mathbf{O}$ *for all* $\mathbf{s} \in \mathbb{Z}_\mathbf{n} \setminus \{\mathbf{0}\}$*.*

Definition of Smith Set

Def $\mathrm{Sm}(\mathbf{G}) \overset{\text{def}}{=} \{[\mathbf{V}] - [\mathbf{W}] \in \mathrm{RO}(\mathbf{G}) \mid \mathbf{V} \sim_{\mathrm{Sm}} \mathbf{W}\}$

This set is called Smith set

We have studied certain subsets of $\mathrm{Sm}(\mathbf{G})$, because

Fact (Bredon-Petrie-Cappell-Shaneson) $\exists \mathbf{N} \triangleleft \mathbf{G}$ s.t. $\mathbf{G}/\mathbf{N} \cong \mathbf{C_8}$

$\Longrightarrow \mathrm{Sm}(\mathbf{G})$ is not additively closed

Definition of Subset $\mathsf{R}_{\mathcal{B}}^{\mathcal{A}}$ of $\mathrm{RO}(\mathsf{G})$

Let R be subset of $\mathrm{RO}(\mathsf{G})$, and $\mathcal{A}$, $\mathcal{B}$ sets of subgr's of G

- $\mathsf{R}_{\mathcal{B}}^{\mathcal{A}} \stackrel{\text{def}}{=} \{\mathsf{x} = [\mathsf{V}] - [\mathsf{W}] \in \mathsf{R} \mid (1) \text{ and } (2) \text{ below}\}$

  (1) $\mathsf{V}^{\mathsf{L}} = \mathbf{0} = \mathsf{W}^{\mathsf{L}}$ for $\forall\ \mathsf{L} \in \mathcal{A}$ and

  (2) $\mathrm{res}_{\mathsf{P}}^{\mathsf{G}}\mathsf{V} \cong \mathrm{res}_{\mathsf{P}}^{\mathsf{G}}\mathsf{W}$ for $\forall\ \mathsf{P} \in \mathcal{B}$

- $\mathsf{R}_{\mathcal{B}} \stackrel{\text{def}}{=} \mathsf{R}_{\mathcal{B}}^{\emptyset}$, $\quad \mathsf{R}^{\mathcal{A}} \stackrel{\text{def}}{=} \mathsf{R}_{\emptyset}^{\mathcal{A}}$ $\quad$ ($\emptyset$ is empty set)

Definition of Primary Smith Set

$$\mathcal{P}(\mathbf{G}) \stackrel{\text{def}}{=} \{\mathbf{P} \mid \mathbf{P} \leq \mathbf{G},\ |\mathbf{P}| \text{ is prime power}\}$$

$$\mathrm{Sm}(\mathbf{G})_{\mathcal{P}(\mathbf{G})} \stackrel{\text{def}}{=} \{[\mathbf{V}] - [\mathbf{W}] \in \mathrm{Sm}(\mathbf{G}) \mid \mathrm{res}^{\mathbf{G}}_{\mathbf{G_p}} \mathbf{V} \cong \mathrm{res}^{\mathbf{G}}_{\mathbf{G_p}} \mathbf{W},\ \forall \mathbf{p}\}$$

$$\stackrel{\text{Sanchez}}{=} \{[\mathbf{V}] - [\mathbf{W}] \in \mathrm{Sm}(\mathbf{G}) \mid \mathrm{res}^{\mathbf{G}}_{\mathbf{G_2}} \mathbf{V} \cong \mathrm{res}^{\mathbf{G}}_{\mathbf{G_2}} \mathbf{W}\}$$

This set is called primary Smith set

Fact (Bredon-Petrie-Randhall-Qi-M)

$$\mathrm{Sm}(\mathbf{G}) \smallsetminus \mathrm{Sm}(\mathbf{G})_{\mathcal{P}(\mathbf{G})} \text{ is finite set}$$

Remark on $\mathrm{res}^G_H : \mathrm{RO}(G) \to \mathrm{RO}(H)$

Prop The 'subfunctors' $F(-) = \mathrm{Sm}(-)$, $\mathrm{Sm}(-)_{\mathcal{P}(-)}$ of $\mathrm{RO}(-)$ are exotic.

Example $G = C_{30} \times C_{30}$. $\Longrightarrow$

(i) $\mathrm{Sm}(H) = \mathrm{Sm}(H)_{\mathcal{P}(H)}$ for any $H \leq G$.

(ii) $\mathrm{res}^G_H(\mathrm{Sm}(G)) \not\subset \mathrm{Sm}(H)$ for $H = C_{15}$.

Additivity in Smith Set

Fact (Bredon-Petrie-Cappell-Shaneson) $\exists \mathbf{N} \triangleleft \mathbf{G}$ s.t. $\mathbf{G}/\mathbf{N} \cong \mathbf{C_8}$

$\Longrightarrow \mathbf{Sm(G)}$ is not additively closed

Thm $\mathbf{G}$ Oliver group s.t. $\mathbf{G_2} \triangleleft \mathbf{G}$ and

$\exists\ \mathbf{N} \triangleleft \mathbf{G}$ with $\mathbf{G}/\mathbf{N} \cong \mathbf{C_{pqr}}$

for some distinct odd primes $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$

$\Longrightarrow \mathbf{Sm(G)}_{\mathcal{P}(\mathbf{G})}$ is not additively closed

Our Problem

Prob

Find relatively large subset $\mathbf{A(G)}$ of $\mathrm{Sm}(\mathbf{G})_{\mathcal{P}(\mathbf{G})}$ having following properties:

1. $\mathbf{A(G)}$ is additively closed in $\mathrm{RO}(\mathbf{G})$.
2. If $\mathbf{G}$ is gap group then $\mathrm{RO}(\mathbf{G})^{\mathcal{L}(\mathbf{G})}_{\mathcal{P}(\mathbf{G})} \subset \mathbf{A(G)}$.
3. For many gropus $\mathbf{G}$, $\mathbf{A(G)} \setminus \mathrm{RO}(\mathbf{G})^{\mathcal{L}(\mathbf{G})}_{\mathcal{P}(\mathbf{G})} \neq \emptyset$.

$$\mathbf{G}^{\{p\}} = \bigcap_{\mathbf{L} \trianglelefteq \mathbf{G}} \mathbf{L} : \ |\mathbf{G}/\mathbf{L}| \text{ is power of } \mathbf{p}$$

$$\mathcal{L}(\mathbf{G}) = \{\mathbf{H} \leq \mathbf{G} \mid \mathbf{H} \supset \mathbf{G}^{\{p\}} \text{ for some } \mathbf{p}\}$$

Gap and Weak Gap Conditions

$\mathbf{M}$ $\mathbf{G}$-mfd, $\mathbf{P} < \mathbf{H} \leq \mathbf{G}$

- $\mathbf{M}$ satisfies gap condition for $(\mathbf{P}, \mathbf{H}) \overset{\text{def}}{\iff}$

(gap) $$\dim \mathbf{M^P}_\mathbf{i} > 2 \dim \mathbf{M^H}_\mathbf{j} \quad (\mathbf{M^P}_\mathbf{i} \supset \mathbf{M^H}_\mathbf{j})$$

- $\mathbf{M}$ satisfies weak gap condition for $(\mathbf{P}, \mathbf{H}) \overset{\text{def}}{\iff}$

(w-gap) $$\dim \mathbf{M^P}_\mathbf{i} \geq 2 \dim \mathbf{M^H}_\mathbf{j} \quad (\mathbf{M^P}_\mathbf{i} \supset \mathbf{M^H}_\mathbf{j})$$

Here $\mathbf{M^P}_\mathbf{i}$ are conn. comp's of $\mathbf{M^P}$

Gap and Weak Gap Groups

- $\mathcal{L}(G) = \{L \le G \mid L \supset G^{\{p\}}$ for some $p\}$
- Real G-module V is $\mathcal{L}(G)$-free $\overset{\text{def}}{\Longleftrightarrow}$ $V^L = 0$ for $\forall\, L \in \mathcal{L}(G)$

G is gap group $\overset{\text{def}}{\Longleftrightarrow}$ $\exists\ \mathcal{L}(G)$-free real G-module V satisfying

(gap) $\dim V^P > 2 \dim V^H$ for $\forall\ (P, H) : P \in \mathcal{P}(G),\ H > P$

G is weak gap group $\overset{\text{def}}{\Longleftrightarrow}$ If $\mathcal{L}(G)$-free real G-modules A, B

satisfy $\mathrm{res}^G_P A \cong \mathrm{res}^G_P B$ for $\forall\ P \in \mathcal{P}(G)$ then

$\exists\ \mathcal{L}(G)$-free V s.t. $A \oplus V$ and $B \oplus V$ satisfy

(w-gap) $\dim W^P \ge 2 \dim W^H$ for $\forall\ (P, H) : P \in \mathcal{P}(G),\ H > P$

where $W = A \oplus V,\ B \oplus V$

Subsets of Smith Set

$$\begin{array}{ccccc}
\mathbf{Sm(G)}^{\mathcal{L}(G)}_{\mathcal{P}(G)} & \longrightarrow & \mathbf{Sm(G)}_{\mathcal{P}(G)} & \longrightarrow & \mathbf{Sm(G)} \\
\overset{?}{=}\downarrow & & \downarrow & & \downarrow \\
\mathbf{RO(G)}^{\mathcal{L}(G)}_{\mathcal{P}(G)} & \longrightarrow & \mathbf{RO(G)}_{\mathcal{P}(G)} & \longrightarrow & \mathbf{RO(G)}
\end{array}$$

Fact (Pawałowski-Solomon) **G** gap Oliver group $\Longrightarrow$

$$\mathbf{Sm(G)}^{\mathcal{L}(G)}_{\mathcal{P}(G)} = \mathbf{RO(G)}^{\mathcal{L}(G)}_{\mathcal{P}(G)} \quad \text{(subgr of } \mathbf{RO(G)}\text{)}$$

Fact (M.) **G** weak gap Oliver group $\Longrightarrow$

$$\mathbf{Sm(G)}^{\mathcal{L}(G)}_{\mathcal{P}(G)} = \mathbf{RO(G)}^{\mathcal{L}(G)}_{\mathcal{P}(G)} \quad \text{(subgr of } \mathbf{RO(G)}\text{)}$$

Remark on Oliver Groups

Fact (Smith-Lefschetz) If $\mathbf{G} \in \mathcal{G}_{\mathbf{p}}^{\mathbf{q}}$: $\mathbf{P} \trianglelefteq \mathbf{H} \triangleleft \mathbf{G}$ s.t.

$$|\mathbf{P}| = \mathbf{p^a},\ \mathbf{H}/\mathbf{P} \text{ cyclic},\ |\mathbf{G}/\mathbf{H}| = \mathbf{q^b}$$

and if $\mathbf{G}$ acts smoothly on disk $\mathbf{D}$

then $\chi(\mathbf{D^G}) \equiv \mathbf{1} \mod \mathbf{q}$ (Thus $\mathbf{D^G} \neq \emptyset$)

Fact $\mathbf{G} \notin \mathcal{G} \overset{\text{Oliver}}{\Longleftrightarrow} \exists$ smooth $\mathbf{G}$-action on disk $\mathbf{D}$ s.t. $\mathbf{D^G} = \emptyset$

$\overset{\text{Oliver}}{\Longleftrightarrow} \exists$ smooth $\mathbf{G}$-action on disk $\mathbf{D}$ s.t. $|\mathbf{D^G}| = \mathbf{2}$

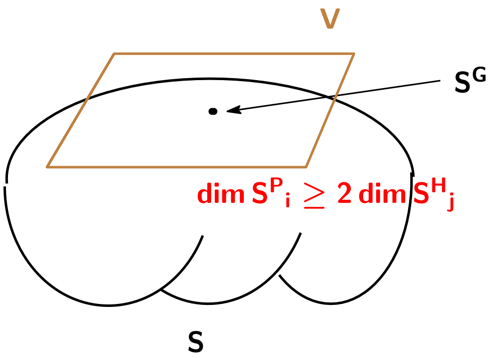
$\overset{\text{Laitinen-M}}{\Longleftrightarrow} \exists$ smooth $\mathbf{G}$-action on sphere $\mathbf{S}$ s.t. $|\mathbf{S^G}| = \mathbf{1}$

One Fixed Point Action

Def $\mathcal{V}_{\mathfrak{D}}(\mathbf{G})$ the family of real **G**-modules **V** possessing smooth **G**-actions on ht spheres **S** satisfying

(1) (w-gap) for $(\mathbf{P}, \mathbf{H})$: $\mathbf{P} \in \mathcal{P}(\mathbf{G})$, $\mathbf{P} < \mathbf{H} \leq \mathbf{G}$,

(2) $\mathbf{S}^{\mathbf{G}} = \{\mathbf{a}\}$, and (3) $\mathbf{T}_{\mathbf{a}}(\mathbf{S}) \cong \mathbf{V}$



Stable Property of $\mathcal{V}_{\mathfrak{D}}(G)$

G Oliver group, $\mathbb{R}[G]$ regular representation

Def $\mathbb{R}[G]_{\mathcal{L}(G)} = (\mathbb{R}[G] - \mathbb{R}) - \bigoplus_p (\mathbb{R}[G/G^{\{p\}}] - \mathbb{R})$

- $\mathbb{R}[G]_{\mathcal{L}(G)}$ satisfies $\mathcal{P}(G)$-weak gap condition.

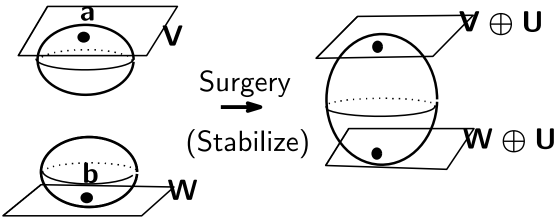
Thm $V \in \mathcal{V}_{\mathfrak{D}}(G) \Longrightarrow \forall\, n \geq 3,\ V \oplus \mathbb{R}[G]^n_{\mathcal{L}(G)} \in \mathcal{V}_{\mathfrak{D}}(G)$

- S ht sphere, $a \in S^G$, $T_a(S) \supset \mathbb{R}[G]_{\mathcal{L}(G)}$

$$\Longrightarrow S^P \text{ is connected } \ (\forall\, P \in \mathcal{P}(G))$$

Primitive Idea

{one fixed pt actions on ht spheres} $\Longrightarrow$ {Smith spheres}



Additive Subgroup

Def $\mathrm{RO}_{\mathfrak{D}}(\mathsf{G}) \overset{\text{def}}{=} \{[\mathsf{V}] - [\mathsf{W}] \mid \mathsf{V},\ \mathsf{W} \in \mathcal{V}_{\mathfrak{D}}(\mathsf{G})\}$

Thm **G** Oliver group

(1) $\mathrm{RO}_{\mathfrak{D}}(\mathsf{G})_{\mathcal{P}(\mathsf{G})} \subset \mathrm{Sm}(\mathsf{G})_{\mathcal{P}(\mathsf{G})}$

(2) $\mathrm{RO}_{\mathfrak{D}}(\mathsf{G})_{\mathcal{P}(\mathsf{G})}$ is subgroup of $\mathrm{RO}(\mathsf{G})$

(3) If **G** is weak gap group then $\mathrm{RO}(\mathsf{G})^{\mathcal{L}(\mathsf{G})}_{\mathcal{P}(\mathsf{G})} \subset \mathrm{RO}_{\mathfrak{D}}(\mathsf{G})_{\mathcal{P}(\mathsf{G})}$

where $\mathcal{L}(\mathsf{G}) = \{\mathsf{L} \leq \mathsf{G} \mid \mathsf{L} \supset \mathsf{G}^{\{\mathsf{p}\}}$ for some $\mathsf{p}\}$

Remark If $\mathsf{A},\ \mathsf{B} \subset \mathrm{RO}_{\mathfrak{D}}(\mathsf{G})_{\mathcal{P}(\mathsf{G})} \Longrightarrow$

$$\langle \mathsf{A} \rangle + \langle \mathsf{B} \rangle \subset \mathrm{RO}_{\mathfrak{D}}(\mathsf{G})_{\mathcal{P}(\mathsf{G})} \subset \mathrm{Sm}(\mathsf{G})_{\mathcal{P}(\mathsf{G})}$$

Nontriviality of $\mathrm{RO}_{\mathfrak{D}}(G)_{\mathcal{P}(G)}$ I

Prop **G** Oliver group

If **G** is nonsolvable, or nilpotent, or of odd order

then $\mathrm{RO}_{\mathfrak{D}}(G)_{\mathcal{P}(G)} \neq 0 \iff \mathrm{Sm}(G)_{\mathcal{P}(G)} \neq 0$

Fact (Pawałowski-Solomon) If **G** is gap Oliver group and $\mathrm{RO}(G)_{\mathcal{P}(G)}^{\{G^{\mathrm{nil}}\}} \neq 0$, where $G^{\mathrm{nil}} = \bigcap_N N$: $N \trianglelefteq G$, G/N is nilpotent, then $\mathrm{Sm}(G)_{\mathcal{P}(G)}^{\mathcal{L}(G)} \neq 0$

Thm **G** Oliver group. If G^{nil} contains two elements **a** and **b** such that $(a)^{\pm} \neq (b)^{\pm}$ in **G** then

$$\mathrm{Sm}(G)_{\mathcal{P}(G)}^{\mathcal{L}(G)} \supset \mathrm{RO}_{\mathfrak{D}}(G)_{\mathcal{P}(G)}^{\mathcal{L}(G)} \neq 0$$

Inclusions

Suppose $\mathbf{G}$ is Oliver group.

$$\mathrm{Sm}(\mathbf{G})^{\mathcal{L}(\mathbf{G})}_{\mathcal{P}(\mathbf{G})} \subset \mathrm{RO}(\mathbf{G})^{\mathcal{L}(\mathbf{G})}_{\mathcal{P}(\mathbf{G})} \overset{\text{w-gap}}{\subset}$$

$$\mathrm{RO}_{\mathfrak{D}}(\mathbf{G})_{\mathcal{P}(\mathbf{G})} \subset \mathrm{Sm}(\mathbf{G})_{\mathcal{P}(\mathbf{G})} \subset \mathrm{RO}(\mathbf{G})^{\{\mathbf{G}^{\cap 2}\}}_{\mathcal{P}(\mathbf{G})}$$

- $\mathcal{L}(\mathbf{G}) = \{\mathbf{L} \leq \mathbf{G} \mid \mathbf{L} \supset \mathbf{G}^{\{\mathbf{p}\}} \text{ for some } \mathbf{p}\}$
- $\mathbf{G}^{\cap 2} = \bigcap_{\mathbf{H}} \mathbf{H} : \mathbf{H} \leq \mathbf{G} \text{ with } |\mathbf{G} : \mathbf{H}| \leq 2$

Claim If $\mathrm{RO}(\mathbf{G})^{\{\mathbf{G}^{\cap 2}\}}_{\mathcal{P}(\mathbf{G})} \subset \mathrm{RO}_{\mathfrak{D}}(\mathbf{G})_{\mathcal{P}(\mathbf{G})}$ then

$$\mathrm{RO}_{\mathfrak{D}}(\mathbf{G})_{\mathcal{P}(\mathbf{G})} = \mathrm{Sm}(\mathbf{G})_{\mathcal{P}(\mathbf{G})} = \mathrm{RO}(\mathbf{G})^{\{\mathbf{G}^{\cap 2}\}}_{\mathcal{P}(\mathbf{G})}$$

Case $G_2 \lhd G$

Thm

G *Oliver group with* $G_2 \lhd G$ *and* $G/N \cong C_{pqr}$ *for distinct odd primes* p, q, r

$$\Longrightarrow \mathrm{RO}_{\mathfrak{D}}(G)_{\mathcal{P}(G)} \neq \mathrm{Sm}(G)_{\mathcal{P}(G)}$$

Case $G_2 \not\trianglelefteq G$

- $G^{\cap 2} \overset{\text{def}}{=} \bigcap_L \; L : \; |G : L| = 2$
- $G^{\{p\}} \overset{\text{def}}{=} \bigcap_{L \trianglelefteq G} \; L : \; |G/L|$ is power of p
- $G^{\text{nil}} \overset{\text{def}}{=} \bigcap_p \; G^{\{p\}} \; : \; p$ prime

Thm G Oliver group s.t. $G_2 \not\trianglelefteq G$.

1. If G is gap group and $\exists$ real G-modules V, W s.t. $V^{G^{\text{nil}}} = 0 = W^{G^{\text{nil}}}$ and $\mathrm{res}^G_{G_p}(\mathbb{R} \oplus V) \cong \mathrm{res}^G_{G_p} W$ for $\forall$ p then $\mathrm{RO}_{\mathfrak{D}}(G)_{\mathcal{P}(G)} = \mathrm{RO}(G)^{\{G^{\cap 2}\}}_{\mathcal{P}(G)} = \mathrm{Sm}(G)_{\mathcal{P}(G)}$

   T. Sumi also proved $=$

2. If $G = G^{\{2\}}$ $(= G^{\cap 2})$ and $\exists$ real G-modules V, W s.t. $V^{G^{\text{nil}}} = 0 = W^{G^{\text{nil}}}$ and $\mathrm{res}^G_{G_2}(\mathbb{R} \oplus V) \cong \mathrm{res}^G_{G_2} W$ then $\mathrm{RO}_{\mathfrak{D}}(G)_{\mathcal{P}(G)} = \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)} = \mathrm{Sm}(G)_{\mathcal{P}(G)}$

Induction $\mathrm{RO}(H) \to \mathrm{RO}(G)$

Thm $\mathrm{ind}_H^G(\mathrm{RO}_{\mathfrak{D}}(H)_{\mathcal{P}(H)}) \subset \mathrm{RO}_{\mathfrak{D}}(G)_{\mathcal{P}(G)}$

Thm **G** Oliver group s.t. $G_2 \not\trianglelefteq G$,

1. $G^{\mathrm{nil}} \subset H \subset K \subset G$, **K** gap group

   If $\exists$ real **H**-modules **V**, **W** s.t.

   $V^{G^{\mathrm{nil}}} = 0 = W^{G^{\mathrm{nil}}}$ and $\mathrm{res}^H_{H_p}(\mathbb{R} \oplus V) \cong \mathrm{res}^H_{H_p} W \ \forall\ p$

   then $\mathrm{ind}_H^G(\mathrm{RO}(H)^{\{H^{\cap 2}\}}_{\mathcal{P}(H)}) \subset \mathrm{RO}_{\mathfrak{D}}(G)_{\mathcal{P}(G)}$

2. $G^{\mathrm{nil}} \subset H \subset G^{\{2\}}$

   If $\exists$ real **H**-modules **V**, **W** s.t.

   $V^{G^{\mathrm{nil}}} = 0 = W^{G^{\mathrm{nil}}}$ and $\mathrm{res}^H_{H_2}(\mathbb{R} \oplus V) \cong \mathrm{res}^H_{H_2} W$

   then $\mathrm{ind}_H^G(\mathrm{RO}(H)^{\{H\}}_{\mathcal{P}(H)}) \subset \mathrm{RO}_{\mathfrak{D}}(G)_{\mathcal{P}(G)}$

Nontriviality of $\mathrm{RO}_{\mathfrak{D}}(G)_{\mathcal{P}(G)}$ II

- $r(G) \stackrel{\text{def}}{=} \#(\{(g)^{\pm} \mid g \in G \text{ not of prime power order}\})$

Prop G perfect group $\neq \{e\}$. Then

$$\mathrm{RO}_{\mathfrak{D}}(G)_{\mathcal{P}(G)} = \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)}$$

(By Laitinen-Pawałowski, $\mathrm{rank} = \max(r(G) - 1, 0)$)

Prop If G is Oliver group with $N \triangleleft G$ s.t. $G/N \cong C_{pq}$ ($p \neq q$ odd primes) then

$$\mathrm{RO}_{\mathfrak{D}}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} = \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \stackrel{\text{P-S}}{\neq} 0$$

Prop If G is Oliver group of odd order then

$$\mathrm{RO}_{\mathfrak{D}}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} = \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \stackrel{\text{P-S}}{\neq} 0$$

(P-S stands for Pawałowski-Solomon)

Nontriviality of $\mathrm{RO}_{\mathfrak{D}}(\mathbf{G})_{\mathcal{P}(\mathbf{G})}$ III

Thm $\mathbf{G}$ Oliver group s.t. $\mathbf{G}^{\mathrm{nil}}{}_{\mathbf{2}} \ntrianglelefteq \mathbf{G}^{\mathrm{nil}}$

$\mathbf{H}$ subgroup s.t. $\mathbf{H} > \mathbf{G}^{\mathrm{nil}}$ and $\mathbf{H}/\mathbf{G}^{\mathrm{nil}} \cong \mathbf{C}_{\mathbf{p}}$ ($\mathbf{p}$ odd prime)

If $\exists$ $\mathbf{V}$, $\mathbf{W}$ real $\mathbf{H}$-modules s.t. $\mathbf{V}^{\mathbf{H}} = \mathbf{0}$, $\mathbf{V}^{\mathbf{G}^{\mathrm{nil}}} \neq \mathbf{0}$, $\mathbf{W}^{\mathbf{G}^{\mathrm{nil}}} = \mathbf{0}$,

and $\mathbf{res}^{\mathbf{H}}_{\mathbf{P}}\mathbf{V} \cong \mathbf{res}^{\mathbf{H}}_{\mathbf{P}}\mathbf{W}$ for $\forall$ $\mathbf{P} \in \mathcal{P}(\mathbf{H})$,

then $\mathrm{RO}_{\mathfrak{D}}(\mathbf{G})_{\mathcal{P}(\mathbf{G})} \smallsetminus \mathrm{RO}(\mathbf{G})^{\{\mathbf{G}^{\{p\}}\}}_{\mathcal{P}(\mathbf{G})} \neq \mathbf{0}$

Thank You Very Much!