On the extension of torus actions on GKM manifolds

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(joint work w/ Mikiya Masuda)

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§1 Introduction

Definition (Hattori-Masuda)

Let M be a $2n$-dim mfd with n-dim torus T-action. Then, M (or (M, T)) is called a torus manifold if $M^T \neq \emptyset$.

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- $T^n \curvearrowright S^{2n} \subset \mathbb{C}^n \oplus \mathbb{R} \Rightarrow M^T = \{(0, 1), (0, -1)\}$.
- $T^n \curvearrowright \mathbb{C}P^n = (\mathbb{C}^{n+1} - \{O\})/\mathbb{C}^*$ (on last n coord) $\Rightarrow$ $M^T = \{[1 : 0 : \cdots : 0], \ldots, [0 : \cdots : 0 : 1]\}$.
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Definition (Guillemin-Holm-Zara)

M^{2m} (or (M^{2m}, T^n)) is called a GKM manifold if 1-skelton M_1/T has the structure of a graph, where $M_1 = \{x \in M \mid \dim T(x) \leq 1\}$.

Example (GKM mfds)

- A torus mfd (M, T) (if $n = m$) and some restricted T^k-action ($k < m$), e.g., $T^2 \curvearrowright \mathbb{C}P^3$ by $(t_1, t_2) \mapsto (t_1, t_2, t_1 t_2)$ is the restriction of $T^3 \curvearrowright \mathbb{C}P^3$ by $(t_1, t_2, t_3) \mapsto (t_1, t_2, t_3)$.

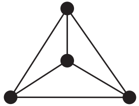

Figure: Graph of the torus mfd $(\mathbb{C}P^3, T^3)$ and the GKM mfd $(\mathbb{C}P^3, T^2)$.

- A homogeneous sp $(G/H, T)$, where $T \subset H \subset G$.

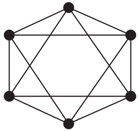

Figure: Graph of $(SU(4)/S(U(2) \times U(2)), T^3)$.

Motivational examples and Problems

Some T^n-actions are induced from G-actions or T^ℓ-actions ($n < \ell$)

Example

- $T^n \curvearrowright S^{2n} \subset \mathbb{C}^n \oplus \mathbb{R} \simeq \mathbb{R}^{2n+1}$ is induced from $SO(2n+1) \curvearrowright S^{2n}$.
- $T^n \curvearrowright \mathbb{C}P^n = (\mathbb{C}^{n+1} - \{O\})/\mathbb{C}^*$ is induced from $PU(n+1) \curvearrowright \mathbb{C}P^n$, where $PU(n+1) = SU(n+1)/\mathrm{center}$.
- $T^2 \curvearrowright \mathbb{C}P^3$ by $(t_1, t_2) \mapsto (t_1, t_2, t_1 t_2)$ is induced from the natural $T^3 \curvearrowright \mathbb{C}P^3$ by $(t_1, t_2, t_3) \mapsto (t_1, t_2, t_3)$.

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Problem

When does (M^{2m}, T^n) extend to (M^{2m}, G) (Prob1) or (M^{2m}, T^ℓ) (Prob2)?
Here, $(m \geq)\ell \geq n$ and G is a cpt Lie gr with $T^n \subset G$ (maximal).

Related works and main theorems

1. 1970 Demazure $\cdots$ $\mathrm{Aut}(X)$ of toric X.
2. 2007 Kuroki $\cdots$ cohomogeneity one (and homogeneous) torus mfds.
3. 2010 Masuda $\cdots$ symplectic toric, quasitoric.
4. 2012 Wiemeler $\cdots$ characterization of torus mfds with extended actions.

Remark

Works 2, 4 characterized extended actions directly.
Works 1, 3 characterized them by root systems of combinatorial objects (fan, polytope).

1. 2004 Shunji Takuma defines a combinatorial obstruction for the extension from (M^{2m}, T^n) to (M^{2m}, T^{n+1}) (unpublished).

GOAL

Define invariants in GKM (and torus) graphs and solve problems.

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Theorem 1 (K-Masuda, Wiemeler)

If $T \curvearrowright M$ (torus mfds) extends to $G \curvearrowright M$ and $G \approx G_1 \times \cdots \times G_\ell \times T'$, then G_i is locally isom to

- $SU(n_i + 1)$ *(type A_{n_i})*;
- $SO(2n_i + 1)$ *(type B_{n_i})*;
- $SO(2n_i)$ *(type D_{n_i})*.

Theorem 2 (K, a generalization of Takuma's work)

If $T^n \curvearrowright M^{2m}$ (almost cpx GKM mfds) extends to $T^\ell \curvearrowright M^{2m}$ for $n \leq \ell$, then the following holds:

- $\ell \leq \mathrm{rk}\mathcal{O}(c_{(\Gamma_M, \mathcal{A}_M)})$,

where $\mathcal{O}(c_{(\Gamma_M, \mathcal{A}_M)})$ is the free $\mathbb{Z}$-module induced from (M, T^n).

§2 Torus graph [MMP] and GKM graph [GZ]

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be an <u>m-valent graph</u>, i.e., $\#E_p(\Gamma) = m$ for all $p \in V(\Gamma)$.

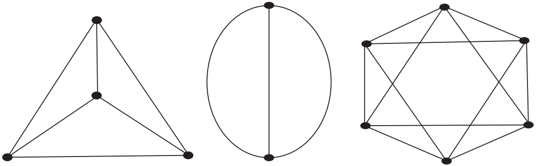

Figure: Two 3-valent graphs and one 4-valent graph.

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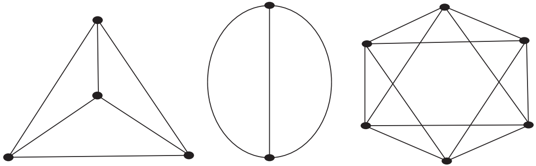


Figure: Two 3-valent graphs and one 4-valent graph.

Definition

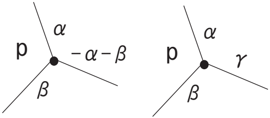
A GKM graph (torus graph) is a labelled graph $(\Gamma, \mathcal{A})$, where a label $\mathcal{A} : E(\Gamma) \to H^2(BT^n) \simeq \mathbb{Z}^n$ for $1 \leq n \leq m$ ($n=m$) satisfies the following conditions:

Axial function $\mathcal{A}$ (I)

$\mathcal{A} : E(\Gamma) \to H^2(BT^n) \simeq \mathbb{Z}^n$ (called axial function) satisfies the following 3 conditions:

(1) $\mathcal{A}(pq) = -\mathcal{A}(qp)$ ($\mathcal{A}(pq) = \pm\mathcal{A}(qp)$)

(2) $\{\mathcal{A}(e) \mid e \in E_p(\Gamma)\}$ spans $\mathbb{Z}^n$ and pairwise linearly indep.



where $H^2(BT^3) = \langle \alpha, \beta, \gamma \rangle$.

Axial function $\mathcal{A}$ (II) and Examples

(3) $\forall pq \in E(\Gamma)$, $\exists$ a bijection $\nabla_{pq} : E_p(\Gamma) \to E_q(\Gamma)$ which satisfies $\forall e \in E_p(\Gamma)$, $\exists c_{pq}(e) \in \mathbb{Z}$ s.t. $\mathcal{A}(\nabla_{pq}(e)) - \mathcal{A}(e) = c_{pq}(e)\mathcal{A}(pq)$.
($\nabla = \{\nabla_e \mid e \in E(\Gamma)\}$ is called a connection)

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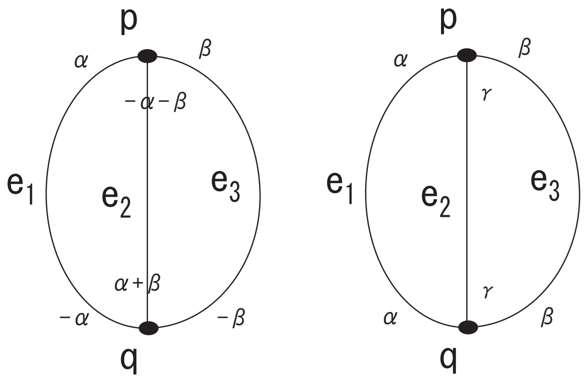
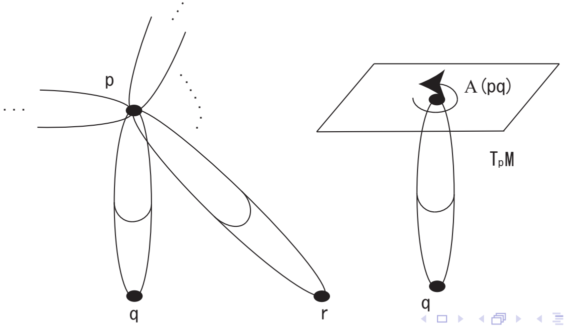


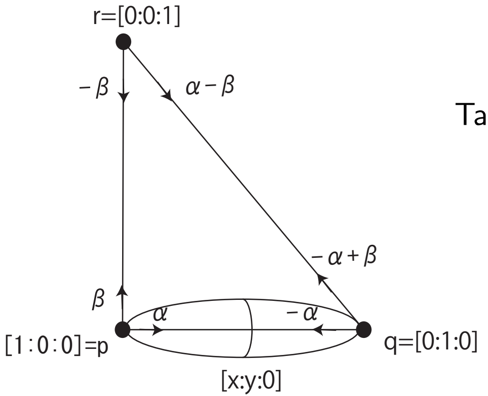
Examples

The GKM (torus) graph $(\Gamma_M, \mathcal{A}_M)$ of GKM (torus) mfd (M, T) can be defined by

1. $V(\Gamma_M)$ is M^T;
2. $E(\Gamma_M)$ is invariant S^2's;
3. $\mathcal{A}_{\mathcal{M}} : E(\Gamma_M) \to H^2(BT)$ is tangential representation on T_pM for all $p \in M^T$.



GKM (torus) graph of $T^2 \curvearrowright \mathbb{C}P^2$ by $[x:y:z] \mapsto [x:t_1y:t_2z]$.



Tangential rep.'s are

$$T_pM \simeq V(\alpha) \oplus V(\beta);$$
$$T_qM \simeq V(-\alpha) \oplus V(\beta - \alpha);$$
$$T_rM \simeq V(-\beta) \oplus V(-\beta + \alpha).$$

Remark

$(\Gamma_M, \mathcal{A}_M)$ *induced from a torus mfd* (M^{2n}, T^n) *is torus graph,*
$(\Gamma_M, \mathcal{A}_M)$ *induced from an (almost cpx) GKM mfd* (M^{2m}, T^n) *is GKM graph.*

§3 1st main results –Root systems of torus graphs–

Let $(\Gamma, \mathcal{A})$ be a GKM (torus) graph. Equivariant cohomology $H_T^*(\Gamma, \mathcal{A})$ is defined by

$$\{f : V(\Gamma) \to H^*(BT) \mid f(p) - f(q) \equiv 0 \mod \mathcal{A}(pq)\}.$$

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FACT 1

$H_T^*(\Gamma, \mathcal{A})$ has the $H^*(BT)$-alg. structure by $\pi^* : H^*(BT) \to H_T^*(\Gamma, \mathcal{A})$ s.t. $\pi^*(\alpha) = \alpha$ (constant map).

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Theorem (Goresky-Kottwictz-MacPherson, Masuda-Panov)

If $H^{odd}(M) = 0$, then $H_T^*(M) \simeq H_T^*(\Gamma_M, \mathcal{A}_M)$.
($\mathbb{Z}$-coeff for torus mfds, $\mathbb{Q}$-coeff for GKM mfds)

Let $(\Gamma, \mathcal{A})$ be a torus graph.

FACT 2 [Maeda-Masuda-Panov]

$H^2_T(\Gamma, \mathcal{A}) \simeq \oplus_{K\subset\Gamma}\mathbb{Z}\tau_K$.

Here, K runs through all $(n-1)$-valent torus subgraphs.
Thom class $\tau_K : V(\Gamma) \to H^2(BT) \in H^*_T(\Gamma, \mathcal{A})$ is defined by the normal axial fcts of K.

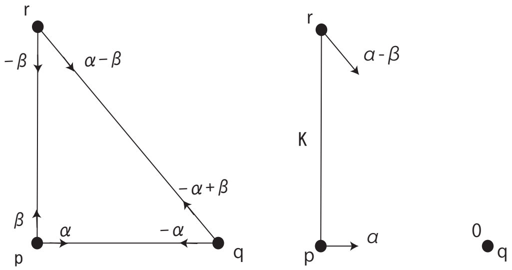

Figure: $\tau_K(p) = \alpha$, $\tau_K(q) = 0$, $\tau_K(r) = \alpha - \beta$.

Root system (review)

Definition (Root system)

Let $R \subset \mathbb{R}^n$ be a set of vectors s.t.

- R spans $\mathbb{R}^n$;
- $\alpha,\ k\alpha \in R\ (k \in \mathbb{R}) \Rightarrow k = \pm 1$;
- $\alpha,\ \beta \in R \Rightarrow r_\alpha(\beta) \in R$ (r_α is the reflection along α);
- $r_\alpha(\beta) = \beta - a_{\beta,\alpha}\alpha \Rightarrow a_{\beta,\alpha} \in \mathbb{Z}$.

Example

For $T \subset G$, $T \curvearrowright \mathfrak{g} = \mathfrak{t} \oplus \oplus_{i=1}^m V(\alpha_i)$.
Then, $R(G) = \{\pm\alpha_i\}$ is the root system (of G).

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- So, $W(G) \curvearrowright H_T^2(M) \simeq H_T^2(\Gamma_M, \mathcal{A}_M) \simeq \oplus_{i=1}^m \mathbb{Z}\tau_i$ preserves τ_i's up to sign.
- Moreover, $W(G) \curvearrowright H^2(M)$ trivial (so G is connected).

Lemma (FACT 3)

Let $\alpha \in R(G) \subset \mathfrak{t}^* \simeq H^2(BT)$ *and* $r_\alpha \in W(G)$ *be its reflection. Then,* $r_\alpha : \oplus_{i=1}^m \mathbb{Z}\tau_i \to \oplus_{i=1}^m \mathbb{Z}\tau_i$ *is one of the followings:*

1. $r_\alpha(\tau_i) = -\tau_i,\ r_\alpha(\tau_k) = \tau_k$ *for* $k \neq i$;
2. $r_\alpha(\tau_i) = \tau_j,\ r_\alpha(\tau_k) = \tau_k$ *for* $k \neq i, j$;
3. $r_\alpha(\tau_i) = -\tau_j,\ r_\alpha(\tau_k) = \tau_k$ *for* $k \neq i, j$.

Moreover, $\varphi^*(\alpha)$ *is one of the followings (respectively):*

1. $\pm\tau_i$;
2. $\pm(\tau_i - \tau_j)$;
3. $\pm(\tau_i + \tau_j)$.

where $\varphi : ET \times_T M \to BT$.

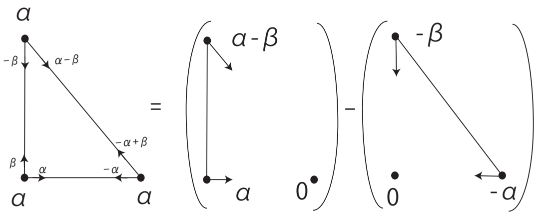
Root systems of torus graphs

Definition (Root systems of $(\Gamma, \mathcal{A})$)

We call the following set, say $R(\Gamma, \mathcal{A})$, a root system of a torus graph $(\Gamma, \mathcal{A})$: $\{\alpha \in H^2(BT) \mid \pi^*(\alpha) = \pm\tau_i, \pm(\tau_i - \tau_j), \text{ or } \pm(\tau_i + \tau_j)\}$.

Example

$R(\Gamma_{\mathbb{C}P^2}, \mathcal{A}_{\mathbb{C}P^2}) = \{\pm\alpha, \pm\beta, \pm(\alpha - \beta)\}$.



Main theorem 1

Let Φ be an irreducible subsystem of $R(\Gamma, \mathcal{A})$ and Δ be its basis.

Theorem (K-Masuda)

Φ is of type A, B or D. More precisely,

1. Φ *is of type B* $\iff \exists \alpha \in \Phi$ *s.t.* $\pi^*(\alpha) = \tau_i$,
2. Φ *is of type D* $\iff \nexists \alpha \in \Phi$ *s.t.* $\pi^*(\alpha) = \tau_i$; *moreover,* $\exists \alpha, \beta \in \Delta$ *s.t.* $\pi^*(\alpha) = \tau_i - \tau_j$ *and* $\pi^*(\beta) = \tau_i + \tau_j$;
3. Φ *is of type A* $\iff$ *otherwise.*

Remark

We also have $R(G) \subset R(\Gamma_M, \mathcal{A}_M)$ *if there is a torus manifold* (M, T) *with an extended action* (M, G).

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Problem (Combinatorial interpretation of $(M^{2m}, T^n) \Rightarrow (M^{2m}, T^\ell)$)

When does $\mathcal{A} : E(\Gamma) \to H^2(BT^n)$ ((m, n)-type) extend to $\widetilde{\mathcal{A}} : E(\Gamma) \to H^2(BT^\ell)$ ((m, ℓ)-type)?
Here, $n \leq \ell \leq m$.

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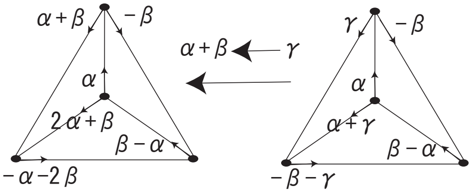
Example



Key fact

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FACT 4 [Takuma]

The integer $c_{pq}(e)$ of the condition (3) does NOT change! Namely, $\forall e \in E_p(\Gamma)$, $\exists c_{pq}(e) \in \mathbb{Z}$ s.t.

$$\mathcal{A}(\nabla_{pq}(e)) - \mathcal{A}(e) = c_{pq}(e)\mathcal{A}(pq) \text{ for } (\Gamma, \mathcal{A}),$$

$$\widetilde{\mathcal{A}}(\nabla_{pq}(e)) - \widetilde{\mathcal{A}}(e) = c_{pq}(e)\widetilde{\mathcal{A}}(pq) \text{ for } (\Gamma, \widetilde{\mathcal{A}}).$$

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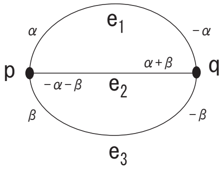
Thus, the map

$$c_{(\Gamma,\mathcal{A})} : E(\Gamma) \to \mathbb{Z}^m \quad \text{s.t.} \quad c_{(\Gamma,\mathcal{A})}(pq) = (c_{pq}(e_1), \ldots, c_{pq}(e_m))$$

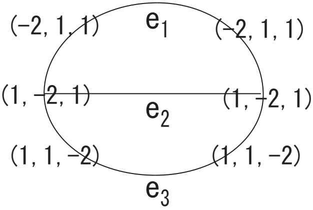
is invariant under the extension! (where $E_p(\Gamma) = \{e_1, \ldots, e_m\}$)

Example ($c_{(\Gamma,\mathcal{A})} : E(\Gamma) \to \mathbb{Z}^n$)

Let $(\Gamma, \mathcal{A})$ be the following $(3, 2)$-type GKM graph.



Then, the map $c_{(\Gamma,\mathcal{A})} : E(\Gamma) \to \mathbb{Z}^3$ is as follows:



Main Theorem 2

IDEA and definition

By defining a sheaf of $(\Gamma, \mathcal{A})$ from $c_{(\Gamma,\mathcal{A})} : E(\Gamma) \to \mathbb{Z}^m$ and taking its (modified) global sections (in the sence of Braden-MacPherson), we define the following $\mathbb{Z}$-module from $c_{(\Gamma,\mathcal{A})} : E(\Gamma) \to \mathbb{Z}^m$:

$$\mathcal{O}(c_{(\Gamma,\mathcal{A})}) = \{f : V(\Gamma) \to \mathbb{Z}^m \mid \nabla_{pq}(f_p) - f_q = f_q(qp)c_{(\Gamma,\mathcal{A})}(qp)\}$$

where $f(p) = f_p \in \mathbb{Z}^m = \mathbb{Z}E_p(\Gamma)$ and $f_q(qp) \in \mathbb{Z}$ is an integer corresponding to the edge qp.

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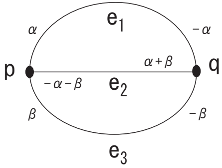
Theorem (Obstruction)

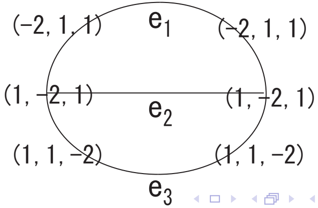
1. *$\mathcal{O}(c_{(\Gamma,\mathcal{A})})$ is a free $\mathbb{Z}$-module with $n \leq \mathrm{rk}\mathcal{O}(c_{(\Gamma,\mathcal{A})}) \leq m$;*
2. *$\exists$ an (m, ℓ)-type extension $\overset{iff}{\Longleftrightarrow}$ $\ell \leq \mathrm{rk}\mathcal{O}(c_{(\Gamma,\mathcal{A})})$.*

Application of $\mathcal{O}(c_{(\Gamma,\mathcal{A})})$ to solve Prob2

Let $(\Gamma, \mathcal{A})$ be the following $(3,2)$-type GKM graph induced from $(G_2/SU(3)(\simeq S^6), T^2)$.



Then, the map $c_{(\Gamma,\mathcal{A})} : E(\Gamma) \to \mathbb{Z}^3$ is as follows:



So, we have

$$\mathcal{O}(c_{(\Gamma,\mathcal{A})}) = \{f : \{p,q\} \to \mathbb{Z}^3 \mid \nabla_{e_i}(f_p) - f_q = f_q(\overline{e_i})c_{(\Gamma,\mathcal{A})}(\overline{e_i})\}$$
$$= \{(f_p, f_q) = ((x,y,z),(-x,-y,-z)) \mid x+y+z=0\} \simeq \mathbb{Z}^2.$$

Therefore, $\mathrm{rk}\mathcal{O}(c_{(\Gamma,\mathcal{A})}) = 2(< 3)$.

$\therefore\ \nexists\ (3,3)$-extensions!

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$$= \{(f_p, f_q) = ((x, y, z), (-x, -y, -z)) \mid x + y + z = 0\} \simeq \mathbb{Z}^2.$$

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$\therefore \nexists$ $(3, 3)$-extensions!

Corollary

The GKM manifold (S^6, T^2) does not extend to a torus manifold (S^6, T^3).

Thank you for your attention

Happy 60th Birthday, Professors Mikiya Masuda, Masaharu Morimoto and Kohei Yamaguchi!