Equivariant unitary bordism and equivariant cohomology Chern numbers

(Joint work with Wei Wang)

Zhi Lü

School of Mathematical Sciences
Fudan University, Shanghai

November 13–15, 2014
The 41st symposium on transformation groups
Gamagori Civic Hall, Japan

To Professors Masuda,Morimoto,Yamaguchi

To Professors Masuda,Morimoto,Yamaguchi

Happy 60 birthday!

To Professors Masuda,Morimoto,Yamaguchi

Happy 60 birthday!

Happy family life!

To Professors Masuda,Morimoto,Yamaguchi

Happy 60 birthday!

Happy family life!

Happy mathematical life!

Outline

- Notations and background

Outline

- Notations and background
- Question

Outline

- Notations and background
- Question
- Main results

Outline

- Notations and background
- Question
- Main results
- Proofs

Unitary manifolds

Definition

Unitary manifolds

Definition

A **unitary manifold** M is a compact, oriented, smooth manifold whose tangent bundle admits a stably almost complex structure (i.e.,

$$J : TM \oplus \underline{\mathbb{R}}^l \longrightarrow TM \oplus \underline{\mathbb{R}}^l$$

such that $J^2 = -id$).

Unitary manifolds

Definition

A **unitary manifold** M is a compact, oriented, smooth manifold whose tangent bundle admits a stably almost complex structure (i.e.,

$$J : TM \oplus \underline{\mathbb{R}}^l \longrightarrow TM \oplus \underline{\mathbb{R}}^l$$

such that $J^2 = -id$).

Example: Quasi-toric manifolds are closed unitary manifolds.

Milnor and Novikov: classifying all closed manifolds up to unitary bordism.

$$\Omega_*^U = \{\text{all closed unitary manifolds}\}/\sim$$

where $\sim$: unitary bordism, which is defined by

$$M_1^n \sim M_2^n \Longleftrightarrow \exists W \text{ s. t. } \partial W = M_1^n \sqcup -M_2^n \text{ with same unitary structure}$$

Ω_*^U forms a ring with the following addition and multiplication

$$[M_1] + [M_2] = [M_1 \sqcup M_2]$$

$$[M] \cdot [N] = [M \times N]$$

Theorem (Milnor, Novikov)

- $[M] = 0$ in $\Omega_*^U \Longleftrightarrow$ all Chern numbers of M vanish.

Theorem (Milnor, Novikov)

- $[M] = 0$ in $\Omega_*^U \Longleftrightarrow$ all Chern numbers of M vanish.
- $\Omega_*^U = \mathbb{Z}[x_{2i} | i \geq 1]$, where x_{2i} can be represented by Milnor hypersurfaces.

Equivariant case

G: compact Lie group

Definition

A unitary G-manifold is a unitary manifold with a G-action preserving the unitary structure (i.e., there exists the following commutative diagram

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}}^l & \xrightarrow{J} & TM \oplus \underline{\mathbb{R}}^l \\ g \downarrow & & \downarrow g \\ TM \oplus \underline{\mathbb{R}}^l & \xrightarrow{J} & TM \oplus \underline{\mathbb{R}}^l \end{array}$$

where $J^2 = -id$ and $g \in G$.

$$\Omega_*^{U,G} = \{\text{all closed unitary } G\text{-manifolds}\}/\sim_G$$

where $\sim_G$: equivariant unitary bordism, defined by

$M_1 \sim_G M_2 \Longleftrightarrow \exists W$ s. t. $\partial W = M_1 \sqcup -M_2$ with same G-unitary stru.

$$\Omega_*^{U,G} = \{\text{all closed unitary } G\text{-manifolds}\}/\sim_G$$

where $\sim_G$: equivariant unitary bordism, defined by

$M_1 \sim_G M_2 \Longleftrightarrow \exists W$ s. t. $\partial W = M_1 \sqcup -M_2$ with same G-unitary stru.

$\Omega_*^{U,G}$ also forms a ring.

$$\Omega_*^{U,G} = \{\text{all closed unitary } G\text{-manifolds}\}/\sim_G$$

where $\sim_G$: equivariant unitary bordism, defined by

$M_1 \sim_G M_2 \Longleftrightarrow \exists W$ s. t. $\partial W = M_1 \sqcup -M_2$ with same G-unitary stru.

$\Omega_*^{U,G}$ also forms a ring.

Remark

Complicated!!!

$$\Omega_*^{U,G} = \{\text{all closed unitary } G\text{-manifolds}\}/\sim_G$$

where $\sim_G$: equivariant unitary bordism, defined by

$M_1 \sim_G M_2 \Longleftrightarrow \exists W$ s. t. $\partial W = M_1 \sqcup -M_2$ with same G-unitary stru.

$\Omega_*^{U,G}$ also forms a ring.

Remark

Complicated!!!
The ring structure of $\Omega_*^{U,G}$ is still open for arbitrary G

§2 Question

Natural question

What is the complete invariant of $\sim_G$?

§2 Question

Natural question

What is the complete invariant of $\sim_G$?

Theorem (tom Dieck, 1971)

Let $G = T^k \times \mathbb{Z}_m$. Then $[M]_G = 0$ in $\Omega_*^{U,G} \Longleftrightarrow$ all equivariant K-theoretic Chern numbers of M vanish.

§2 Question

Natural question

What is the complete invariant of $\sim_G$?

Theorem (tom Dieck, 1971)

Let $G = T^k \times \mathbb{Z}_m$. Then $[M]_G = 0$ in $\Omega_*^{U,G} \Longleftrightarrow$ all equivariant K-theoretic Chern numbers of M vanish.

Theorem (Guillemin–Ginzburg–Karshon, 2002)

Let $G = T^k$. Then a closed unitary T^k-manifold M with only isolated fixed points represents the zero element in $\Omega_*^{U,T^k} \Longleftrightarrow$ all equivariant cohomology Chern numbers of M vanish.

Without the restriction of isolated fixed-points, Guillemin–Ginzburg–Karshon posed

Conjecture (Guillemin–Ginzburg–Karshon, 2002)

$[M]_{T^k} = 0$ in $\Omega_*^{U,T^k} \Longleftrightarrow$ all equivariant cohomology Chern numbers of M vanish.

Without the restriction of isolated fixed-points, Guillemin–Ginzburg–Karshon posed

Conjecture (Guillemin–Ginzburg–Karshon, 2002)

$[M]_{T^k} = 0$ in $\Omega_*^{U,T^k} \iff$ all equivariant cohomology Chern numbers of M vanish.

Remark

In their book, Guillemin–Ginzburg–Karshon discussed the problem of calculating the ring $\mathcal{H}_*^{T^k}$ of equivariant Hamiltonian bordism classes of all unitary Hamiltonian T^k-manifolds.

Without the restriction of isolated fixed-points, Guillemin–Ginzburg–Karshon posed

Conjecture (Guillemin–Ginzburg–Karshon, 2002)

$[M]_{T^k} = 0$ in $\Omega_*^{U,T^k} \Longleftrightarrow$ all equivariant cohomology Chern numbers of M vanish.

Remark

In their book, Guillemin–Ginzburg–Karshon discussed the problem of calculating the ring $\mathcal{H}_*^{T^k}$ of equivariant Hamiltonian bordism classes of all unitary Hamiltonian T^k-manifolds. They designed three series of questions, the first one of which is stated as follows:

Without the restriction of isolated fixed-points, Guillemin–Ginzburg–Karshon posed

Conjecture (Guillemin–Ginzburg–Karshon, 2002)

$[M]_{T^k} = 0$ in $\Omega_*^{U,T^k} \Longleftrightarrow$ all equivariant cohomology Chern numbers of M vanish.

Remark

In their book, Guillemin–Ginzburg–Karshon discussed the problem of calculating the ring $\mathcal{H}_*^{T^k}$ of equivariant Hamiltonian bordism classes of all unitary Hamiltonian T^k-manifolds. They designed three series of questions, the first one of which is stated as follows:
Do mixed equivariant characteristic numbers form a full system of invariants of Hamiltonian bordism?

Without the restriction of isolated fixed-points, Guillemin–Ginzburg–Karshon posed

Conjecture (Guillemin–Ginzburg–Karshon, 2002)

$[M]_{T^k} = 0$ in $\Omega_*^{U,T^k} \Longleftrightarrow$ all equivariant cohomology Chern numbers of M vanish.

Remark

In their book, Guillemin–Ginzburg–Karshon discussed the problem of calculating the ring $\mathcal{H}_*^{T^k}$ of equivariant Hamiltonian bordism classes of all unitary Hamiltonian T^k-manifolds. They designed three series of questions, the first one of which is stated as follows:
Do mixed equivariant characteristic numbers form a full system of invariants of Hamiltonian bordism?
Then Guillemin - Ginzburg - Karshon constructed a monomorphism

$$\mathcal{H}_*^{T^k} \longrightarrow \Omega_{*+2}^{U,T^k}$$

Main results

Theorem A (Lü-Wang)

$[M]_{T^k} = 0$ in $\Omega_*^{U,T^k} \Longleftrightarrow$ all equivariant cohomology Chern numbers of M vanish.

Main results

Theorem A (Lü-Wang)

$[M]_{T^k} = 0$ in $\Omega_*^{U,T^k} \Longleftrightarrow$ all equivariant cohomology Chern numbers of M vanish.

Corollary

Mixed equivariant characteristic numbers form a full system of invariants of Hamiltonian bordism.

Main results

Using the equivariant Riemann–Roch relation of Atiyah–Hizebruch type, we also obtain

Theorem B (Lü–Wang)

Let $[M]_{T^k} \in \Omega_*^{U,T^k}$. Then All equivariant cohomology Chern numbers of M vanish $\Longleftrightarrow$ all equivariant K-theoretic Chern numbers of M vanish.

Main results

Using the equivariant Riemann–Roch relation of Atiyah–Hizebruch type, we also obtain

Theorem B (Lü–Wang)

Let $[M]_{T^k} \in \Omega_*^{U,T^k}$. Then All equivariant cohomology Chern numbers of M vanish $\Longleftrightarrow$ all equivariant K-theoretic Chern numbers of M vanish.

Remark

With a different way, we actually obtain the tom Dieck's Theorem in the case where G is a torus.

Proof of Theorem A

Key points

Proof of Theorem A

Key points

- Kronecker pairing between bordism and cobordism
- Universal toric genus

Kronecker pairing between bordism and cobordism

Notions-homotopic bordism and cobordism

- $$MU_*(X) = \lim_{r \longrightarrow \infty} [S^{2r+*}, X_+ \wedge MU(r)]$$

Kronecker pairing between bordism and cobordism

Notions-homotopic bordism and cobordism

- $$MU_*(X) = \lim_{r \longrightarrow \infty} [S^{2r+*}, X_+ \wedge MU(r)]$$
- $$MU^*(X) = \lim_{r \longrightarrow \infty} [S^{2r+*} \wedge X_+, MU(r)]$$

where $X_+ = X \cup \{pt\}$, $MU(r)$: Thom space of universal complex r-dim. vector bundle over $BU(r)$.

Kronecker pairing between bordism and cobordism

Notions-homotopic bordism and cobordism

- $$MU_*(X) = \lim_{r \longrightarrow \infty} [S^{2r+*}, X_+ \wedge MU(r)]$$
- $$MU^*(X) = \lim_{r \longrightarrow \infty} [S^{2r+*} \wedge X_+, MU(r)]$$

  where $X_+ = X \cup \{pt\}$, $MU(r)$: Thom space of universal complex r-dim. vector bundle over $BU(r)$.

Remark. By Thom-Pontryagin construction, $MU_*(X) \cong \Omega_*^U(X)$,

Kronecker pairing between bordism and cobordism

Notions-homotopic bordism and cobordism

- $$MU_*(X) = \lim_{r \longrightarrow \infty} [S^{2r+*}, X_+ \wedge MU(r)]$$
- $$MU^*(X) = \lim_{r \longrightarrow \infty} [S^{2r+*} \wedge X_+, MU(r)]$$

  where $X_+ = X \cup \{pt\}$, $MU(r)$: Thom space of universal complex r-dim. vector bundle over $BU(r)$.

Remark. By Thom-Pontryagin construction, $MU_*(X) \cong \Omega_*^U(X)$, where $\Omega_*^U(X)$ is formed by the bordism classes of singular manifolds $f : M \longrightarrow X$ for M: unitary manifold

Kronecker pairing between bordism and cobordism

Quillen's geometric interpretation of elements in $MU^*(X)$

Each element $\alpha \in MU^{\pm n}$ can be represented by an oriented complex map $f : M \longrightarrow X$,

Kronecker pairing between bordism and cobordism

Quillen's geometric interpretation of elements in $MU^*(X)$

Each element $\alpha \in MU^{\pm n}$ can be represented by an oriented complex map $f : M \longrightarrow X$, where X is a smooth manifold and $\dim X - \dim M = \pm n$.

Kronecker pairing between bordism and cobordism

Quillen's geometric interpretation of elements in $MU^*(X)$

Each element $\alpha \in MU^{\pm n}$ can be represented by an oriented complex map $f : M \longrightarrow X$, where X is a smooth manifold and $\dim X - \dim M = \pm n$.

If n is even, f is a composition of

$$M \hookrightarrow E \longrightarrow X$$

such that the normal bundle of M in E admits a complex structure,

Kronecker pairing between bordism and cobordism

Quillen's geometric interpretation of elements in $MU^*(X)$

Each element $\alpha \in MU^{\pm n}$ can be represented by an oriented complex map $f : M \longrightarrow X$, where X is a smooth manifold and $\dim X - \dim M = \pm n$.

If n is even, f is a composition of

$$M \hookrightarrow E \longrightarrow X$$

such that the normal bundle of M in E admits a complex structure, where $E \longrightarrow X$ is a complex vector bundle.

Kronecker pairing between bordism and cobordism

Quillen's geometric interpretation of elements in $MU^*(X)$

Each element $\alpha \in MU^{\pm n}$ can be represented by an oriented complex map $f : M \longrightarrow X$, where X is a smooth manifold and $\dim X - \dim M = \pm n$.

If n is even, f is a composition of

$$M \hookrightarrow E \longrightarrow X$$

such that the normal bundle of M in E admits a complex structure, where $E \longrightarrow X$ is a complex vector bundle.

If n is odd, E is replaced by $E \times \mathbb{R}$.

Kronecker pairing between bordism and cobordism

Kronecker pairing

$$\langle , \rangle : MU^{\pm n}(X) \otimes MU_m(X) \longrightarrow MU_{m \mp n}.$$

Kronecker pairing between bordism and cobordism

Kronecker pairing

$$\langle , \rangle : MU^{\pm n}(X) \otimes MU_m(X) \longrightarrow MU_{m \mp n}.$$

For example, let X be a smooth manifold.

Kronecker pairing between bordism and cobordism

Kronecker pairing

$$\langle , \rangle : MU^{\pm n}(X) \otimes MU_m(X) \longrightarrow MU_{m \mp n}.$$

For example, let X be a smooth manifold.
$\alpha \in MU^{-n}(X)$ is represented by a smooth fiber bundle $E \longrightarrow X$ with $\dim E - \dim X = n$.

Kronecker pairing between bordism and cobordism

Kronecker pairing

$$\langle , \rangle : MU^{\pm n}(X) \otimes MU_m(X) \longrightarrow MU_{m \mp n}.$$

For example, let X be a smooth manifold.
$\alpha \in MU^{-n}(X)$ is represented by a smooth fiber bundle $E \longrightarrow X$ with $\dim E - \dim X = n$.
$\beta \in MU_m(X)$ is represented by a smooth map $f : M \longrightarrow X$

Kronecker pairing between bordism and cobordism

Kronecker pairing

$$\langle , \rangle : MU^{\pm n}(X) \otimes MU_m(X) \longrightarrow MU_{m \mp n}.$$

For example, let X be a smooth manifold.
$\alpha \in MU^{-n}(X)$ is represented by a smooth fiber bundle $E \longrightarrow X$ with $\dim E - \dim X = n$.
$\beta \in MU_m(X)$ is represented by a smooth map $f : M \longrightarrow X$
Then $\langle \alpha, \beta \rangle$ is the bordism class of the pull-back $\widetilde{f}^*(E)$

$$\begin{array}{ccc} \widetilde{f}^*(E) & \xrightarrow{\widetilde{f}} & E \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

Universal toric genus

$$\Phi : \Omega_*^{U,T^k} \longrightarrow MU^*(BT^k)$$

Universal toric genus

$$\Phi : \Omega_*^{U,T^k} \longrightarrow MU^*(BT^k)$$

- Defined by tom Dieck
- Φ is a monomorphism (due to Hanke and Löffler)

Universal toric genus

$$\Phi : \Omega_*^{U,T^k} \longrightarrow MU^*(BT^k)$$

- Defined by tom Dieck
- Φ is a monomorphism (due to Hanke and Löffler)
- Re-defined by Buchstaber–Ray–Panov in a geometric way as follows:

$$[M]_{T^k} \longmapsto [\pi : ET^k \times_{T^k} M \longrightarrow BT^k]$$

Proof of Theorem A

Proof of Theorem A

Take $[M]_{T^k} \in \Omega_n^{U,T^k}$, and $[f : N \longrightarrow BT^k] \in MU_*(BT^k)$,

Proof of Theorem A

Take $[M]_{T^k} \in \Omega_n^{U,T^k}$, and $[f : N \longrightarrow BT^k] \in MU_*(BT^k)$, consider

$$\begin{array}{ccc} \widetilde{f}^*(ET^k \times_{T^k} M) & \xrightarrow{\widetilde{f}} & ET^k \times_{T^k} M \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{f} & BT^k \end{array}$$

Proof of Theorem A

Take $[M]_{T^k} \in \Omega_n^{U,T^k}$, and $[f : N \longrightarrow BT^k] \in MU_*(BT^k)$, consider

$$\begin{array}{ccc} \widetilde{f}^*(ET^k \times_{T^k} M) & \xrightarrow{\widetilde{f}} & ET^k \times_{T^k} M \\ \pi' \big\downarrow & & \pi \big\downarrow \\ N & \xrightarrow{f} & BT^k \end{array}$$

By universal toric genus and Kronecker pairing,

$$\langle \Phi([M]_{T^k}), [f : N \longrightarrow BT^k] \rangle = [\widetilde{f}^*(ET^k \times_{T^k} M)] \in MU_* = \Omega_*^U$$

Proof of Theorem A

Take $[M]_{T^k} \in \Omega_n^{U,T^k}$, and $[f : N \longrightarrow BT^k] \in MU_*(BT^k)$, consider

$$\begin{array}{ccc} \widetilde{f}^*(ET^k \times_{T^k} M) & \xrightarrow{\widetilde{f}} & ET^k \times_{T^k} M \\ \pi' \big\downarrow & & \pi \big\downarrow \\ N & \xrightarrow{f} & BT^k \end{array}$$

By universal toric genus and Kronecker pairing,

$$\langle \Phi([M]_{T^k}), [f : N \longrightarrow BT^k] \rangle = [\widetilde{f}^*(ET^k \times_{T^k} M)] \in MU_* = \Omega_*^U$$

Remark: $\widetilde{f}^*(ET^k \times_{T^k} M)$ is a closed unitary manifold of dimension=dim $M +$ dim N.

Proof of Theorem A

Proof of Theorem A

Step I:

Proof of Theorem A

Step I: Suppose that all equivariant cohomology Chern numbers of M vanish.

Proof of Theorem A

Step I: Suppose that all equivariant cohomology Chern numbers of M vanish.

$\Longrightarrow$ for any $f : N \longrightarrow BT^k$,

$$\langle \Phi([M]_{T^k}), [f : N \longrightarrow BT^k] \rangle = [\widetilde{f}^*(ET^k \times_{T^k} M)] = 0 \in MU_* = \Omega_*^U$$

Proof of Theorem A

Step I: Suppose that all equivariant cohomology Chern numbers of M vanish.

$\Longrightarrow$ for any $f : N \longrightarrow BT^k$,

$$\langle \Phi([M]_{T^k}), [f : N \longrightarrow BT^k] \rangle = [\widetilde{f}^*(ET^k \times_{T^k} M)] = 0 \in MU_* = \Omega_*^U$$

$\Longrightarrow \Phi([M]_{T^k}) = 0$

Proof of Theorem A

Step I: Suppose that all equivariant cohomology Chern numbers of M vanish.

$\Longrightarrow$ for any $f : N \longrightarrow BT^k$,

$$\langle \Phi([M]_{T^k}), [f : N \longrightarrow BT^k] \rangle = [\widetilde{f}^*(ET^k \times_{T^k} M)] = 0 \in MU_* = \Omega_*^U$$

$\Longrightarrow \Phi([M]_{T^k}) = 0$

$\Longrightarrow [M]_{T^k} = 0$ since Φ is injective.

Proof of Theorem A

Proof of Theorem A

Step II:

Proof of Theorem A

Step II: Suppose that $[M]_{T^k} = 0$ in Ω_*^{U,T^k}.

Proof of Theorem A

Step II: Suppose that $[M]_{T^k} = 0$ in Ω_*^{U,T^k}.

$\Longrightarrow \Phi([M]_{T^k}) = [\pi : ET^k \times_{T^k} M \longrightarrow BT^k] = 0$ in $MU_*(BT^k)$

Proof of Theorem A

Step II: Suppose that $[M]_{T^k} = 0$ in Ω_*^{U,T^k}.

$\Longrightarrow \Phi([M]_{T^k}) = [\pi : ET^k \times_{T^k} M \longrightarrow BT^k] = 0$ in $MU_*(BT^k)$

Consider $\pi_!(c_\omega^{T^k}(M)) = c_\omega^{T^k}(M)[M] \in H^*(BT^k) = \mathbb{Z}[x_1, ..., x_k]$,

Proof of Theorem A

Step II: Suppose that $[M]_{T^k} = 0$ in Ω_*^{U,T^k}.

$\Longrightarrow \Phi([M]_{T^k}) = [\pi : ET^k \times_{T^k} M \longrightarrow BT^k] = 0$ in $MU_*(BT^k)$

Consider $\pi_!(c_\omega^{T^k}(M)) = c_\omega^{T^k}(M)[M] \in H^*(BT^k) = \mathbb{Z}[x_1, ..., x_k]$, where $\omega = (i_1, ..., i_r)$ is a partition.

Proof of Theorem A

Step II: Suppose that $[M]_{T^k} = 0$ in Ω_*^{U,T^k}.

$\Longrightarrow \Phi([M]_{T^k}) = [\pi : ET^k \times_{T^k} M \longrightarrow BT^k] = 0$ in $MU_*(BT^k)$

Consider $\pi_!(c_\omega^{T^k}(M)) = c_\omega^{T^k}(M)[M] \in H^*(BT^k) = \mathbb{Z}[x_1, ..., x_k]$, where $\omega = (i_1, ..., i_r)$ is a partition.

If dim M is odd, then $\pi_!(c_\omega^{T^k}(M)) = 0$.

Proof of Theorem A

Step II: Suppose that $[M]_{T^k} = 0$ in Ω_*^{U,T^k}.

$\Longrightarrow \Phi([M]_{T^k}) = [\pi : ET^k \times_{T^k} M \longrightarrow BT^k] = 0$ in $MU_*(BT^k)$

Consider $\pi_!(c_\omega^{T^k}(M)) = c_\omega^{T^k}(M)[M] \in H^*(BT^k) = \mathbb{Z}[x_1, ..., x_k]$, where $\omega = (i_1, ..., i_r)$ is a partition.

If dim M is odd, then $\pi_!(c_\omega^{T^k}(M)) = 0$.

If dim $M = 2m$, then $\pi_!(c_\omega^{T^k}(M)) \in H^{2|\omega|-2n}(BT^k)$

Proof of Theorem A

Step II: Suppose that $[M]_{T^k} = 0$ in Ω_*^{U,T^k}.

$\Longrightarrow \Phi([M]_{T^k}) = [\pi : ET^k \times_{T^k} M \longrightarrow BT^k] = 0$ in $MU_*(BT^k)$

Consider $\pi_!(c_\omega^{T^k}(M)) = c_\omega^{T^k}(M)[M] \in H^*(BT^k) = \mathbb{Z}[x_1, ..., x_k]$, where $\omega = (i_1, ..., i_r)$ is a partition.

If dim M is odd, then $\pi_!(c_\omega^{T^k}(M)) = 0$.

If dim $M = 2m$, then $\pi_!(c_\omega^{T^k}(M)) \in H^{2|\omega|-2n}(BT^k)$

Note: Clearly if $|\omega| < m$, then $\pi_!(c_\omega^{T^k}(M)) = 0$.

Proof of Theorem A

Step II:(continued)

Proof of Theorem A

Step II:(continued)

We perform an induction on $|\omega| - m \geq 0$.

Proof of Theorem A

Step II:(continued)

We perform an induction on $|\omega| - m \geq 0$.

Easy to check that $\pi_!(c_\omega^{T^k}(M)) = 0$ if $|\omega| = m$.

Proof of Theorem A

Step II:(continued)

We perform an induction on $|\omega| - m \geq 0$.

Easy to check that $\pi_!(c_\omega^{T^k}(M)) = 0$ if $|\omega| = m$.

Assume inductively that $\pi_!(c_\omega^{T^k}(M)) = 0$ if $|\omega| - m \leq \ell$.

Proof of Theorem A

Step II:(continued)

We perform an induction on $|\omega| - m \geq 0$.

Easy to check that $\pi_!(c_\omega^{T^k}(M)) = 0$ if $|\omega| = m$.

Assume inductively that $\pi_!(c_\omega^{T^k}(M)) = 0$ if $|\omega| - m \leq \ell$. When $|\omega| - m = \ell + 1$,

Proof of Theorem A

Step II:(continued)

We perform an induction on $|\omega| - m \geq 0$.

Easy to check that $\pi_!(c_\omega^{T^k}(M)) = 0$ if $|\omega| = m$.

Assume inductively that $\pi_!(c_\omega^{T^k}(M)) = 0$ if $|\omega| - m \leq \ell$. When $|\omega| - m = \ell + 1$, write

$$\pi_!(c_\omega^{T^k}(M)) = \sum_J n_J x^J$$

Proof of Theorem A

Step II:(continued)

We perform an induction on $|\omega| - m \geq 0$.

Easy to check that $\pi_!(c_\omega^{T^k}(M)) = 0$ if $|\omega| = m$.

Assume inductively that $\pi_!(c_\omega^{T^k}(M)) = 0$ if $|\omega| - m \leq \ell$. When $|\omega| - m = \ell + 1$, write

$$\pi_!(c_\omega^{T^k}(M)) = \sum_J n_J x^J$$

where $J = (j_1, ..., j_k)$ with $|J| = |\omega| - m$,

Proof of Theorem A

Step II:(continued)

We perform an induction on $|\omega| - m \geq 0$.

Easy to check that $\pi_!(c_\omega^{T^k}(M)) = 0$ if $|\omega| = m$.

Assume inductively that $\pi_!(c_\omega^{T^k}(M)) = 0$ if $|\omega| - m \leq \ell$. When $|\omega| - m = \ell + 1$, write

$$\pi_!(c_\omega^{T^k}(M)) = \sum_J n_J x^J$$

where $J = (j_1, ..., j_k)$ with $|J| = |\omega| - m$, and $x^J = x_1^{j_1} \cdots x_k^{j_k}$.

Proof of Theorem A

Step II:(continued)

We perform an induction on $|\omega| - m \geq 0$.

Easy to check that $\pi_!(c_\omega^{T^k}(M)) = 0$ if $|\omega| = m$.

Assume inductively that $\pi_!(c_\omega^{T^k}(M)) = 0$ if $|\omega| - m \leq \ell$. When $|\omega| - m = \ell + 1$, write

$$\pi_!(c_\omega^{T^k}(M)) = \sum_J n_J x^J$$

where $J = (j_1, ..., j_k)$ with $|J| = |\omega| - m$, and $x^J = x_1^{j_1} \cdots x_k^{j_k}$.

For each J,

Proof of Theorem A

Step II:(continued)

We perform an induction on $|\omega| - m \geq 0$.

Easy to check that $\pi_!(c_\omega^{T^k}(M)) = 0$ if $|\omega| = m$.

Assume inductively that $\pi_!(c_\omega^{T^k}(M)) = 0$ if $|\omega| - m \leq \ell$. When $|\omega| - m = \ell + 1$, write

$$\pi_!(c_\omega^{T^k}(M)) = \sum_J n_J x^J$$

where $J = (j_1, ..., j_k)$ with $|J| = |\omega| - m$, and $x^J = x_1^{j_1} \cdots x_k^{j_k}$.

For each J, choose $N = \mathbb{C}P^{j_1} \times \cdots \times \mathbb{C}P^{j_k}$,

Proof of Theorem A

Step II:(continued)

We perform an induction on $|\omega| - m \geq 0$.

Easy to check that $\pi_!(c_\omega^{T^k}(M)) = 0$ if $|\omega| = m$.

Assume inductively that $\pi_!(c_\omega^{T^k}(M)) = 0$ if $|\omega| - m \leq \ell$. When $|\omega| - m = \ell + 1$, write

$$\pi_!(c_\omega^{T^k}(M)) = \sum_J n_J x^J$$

where $J = (j_1, ..., j_k)$ with $|J| = |\omega| - m$, and $x^J = x_1^{j_1} \cdots x_k^{j_k}$.

For each J, choose $N = \mathbb{C}P^{j_1} \times \cdots \times \mathbb{C}P^{j_k}$, we can obtain that $n_J = 0$.

Thank You!